Ordering homotopy string links over surfaces and a presentation for the homotopy generalized string links over surfaces

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"Life is a play that does not allow testing. So, sing, cry, dance, laugh, live and love intensely, before the curtain closes and the play ends with no applauses."

Charles Chaplin

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To finalize, I am going to cite a phrase by Chico Xavier:

“I would like to thank to all the difficulties that I had until now: because of them, I could move on in life, evolved myself. Sometimes, the easiness does not allow us to move on.”

Abstract

In this work, we prove that the set of link-homotopy classes of generalized string links over a closed, connected and orientable surface M of genus $g \geq 1$ form a group, denoted by $\widehat{B_n}(M)$ and we find a presentation for it. Moreover, we prove that its normal subgroup $\widehat{PB}_n(M)$, namely, the homotopy string links over M, is bi-orderable. These results extend results proved by Juan Gonzalez–Meneses in [GM], [GM2] and Ekaterina Yurasovskaya in [Y], respectively. Also, we obtain an exact sequence for link-homotopy braid groups, which is an extension of [Go, Theorem 1].

Contents

Introduction

In 1925, Artin introduced the study of braid groups, which is closely related to the study of knots and links. Artin obtained an important result, namely, the presentation theorem for braid groups, which gives a presentation for the braid groups over the unit disk, a result which allows us to recognize this group through its generators and the relations between its elements. Since this early result, the theory of braids had developed in many directions with works of Alexander, Goldberg, Markov, Birman, Goldsmith, Rolfsen, González–Meneses, Paris and others. The basic theory can be found in [B].

Nowadays, two research areas in development are braids groups on surfaces and link-homotopy braid groups over surfaces. The presentation theorem obtained by González–Meneses [GM] for the braid groups on surfaces is a generalization of Artin's presentation for the braid groups over the disk. In his recent work, González–Meneses found the smallest presentation for braid groups over surfaces. In particular, he produced a presentation for the braid groups over closed and orientable surfaces of genus $g \geq 1$ and over non orientable surfaces of genus $g \geq 2$.

Recently, Rolfsen, Dynnikov, Dehornoy and Wiest, demonstrated topological reasons for the existence of a left-ordering of the braid groups over the disk, i.e., there is a strict total ordering of the braids that is invariant under multiplication from the left. They also showed the pure braid groups over the unit disk are bi-orderable, i.e., there is a left and right invariant strict total ordering for this group, for details see [R]. Later, González–Meneses in [GM2], proved that the pure braid groups over orientable surfaces, with genus $g \geq 1$, are bi-orderable too. For $n \geq 3$, the pure braid group of the sphere $PB_n(\mathbb{S}^2)$ has torsion, so it is not bi-orderable.

Along these same lines, we can study generalized string links, which informally are generalizations of braids. The difference is that for the former, we consider embedded strands up to link-homotopy, whereas for the latter, we consider embedded strands up to isotopy. We say that generalized string links are generalizations of braids since a link homotopy always involves a finite numbers of isotopies and crossing changes (see [Mil] and [HL]).

One of the properties of link-homotopy is to allow that each string in a generalized string link has self intersection, i.e., to allow that it has a finite number of self crossings (crossing changes). The generalized string links with this last equivalence relation, with the operation of concatenation, become a group, called the homotopy generalized string links over surfaces. Ekaterina Yurasovskaya in her PhD thesis [Y] (supervised by Dale Rolfsen), obtained a presentation in terms of generators and relations of the group of link-homotopy classes of string links based on a closed orientable surface. Furthermore, she also proved that the group of homotopy string links over the disk is bi-orderable. Note that when we say only "string links" we refer the pure case of generalized string links.

In this work, we prove that the set of link-homotopy classes of generalized string links over a closed, connected and orientable surface M of genus $g \geq 1$ form a group, denoted by $\widehat{B_n}(M)$ and we find a presentation for it. Moreover, we prove that its normal subgroup $\widehat{PB}_n(M)$, namely, the group of link-homotopy string links over M, is bi-orderable. These results extend results proved by Juan Gonzalez–Meneses in [GM], [GM2] and Ekaterina Yurasovskaya in [Y], respectively. Also, we obtain an exact sequence for link-homotopy braid groups, which is an extension of [Go, Theorem 1].

Specifically, in Chapter 1, will be introduced the braid theory: the construction of braid groups over surfaces, its presentations and the orderability theory for these groups. Moreover, will be discussed tools for find presentations and orderability theory of certain groups which will be very important to prove our aims. This chapter was extracted from the following references: [GM], [DDRW], [Y] and [GM2].

In Chapter 2 will be presented the theory of homotopy string links over surfaces. More specifically, will be given its definitions, notations and its constructions as a group and a presentation for it. Also, will be studied the orderability theory for certain quotients of free groups, namely, reduced free groups. This chapter was extracted from Ekaterina Yurasovskaya's PhD thesis in [Y]. The basics references used in this chapter are [DDRW], [Mil], [HL], [Mag] and [F].

In Chapters 3 and 4 will be obtained the main results, goals of this thesis. In Chapter 3, will be extended the results of [GM2] and [Y], proving that the homotopy string links over surfaces, namely $\widehat{PB}_n(M)$, is bi-orderable. For this, will be used some techniques introduced by [GM3]. Moreover, will be obtained an exact sequence for link-homotopy braid groups, which is an extension of [Go, Theorem 1]. Finally, in Chapter 4, will be defined the generalized string links over surfaces, an extension of the definition of string links over surfaces. Will be shown that the set $\widehat{B_n}(M)$ of link-homotopy classes of generalized string links over a closed, connected and orientable surface M of genus $g \geq 1$ becomes a group (with $\widehat{PB}_n(M) \subset \widehat{B_n}(M)$ as a normal subgroup) and will be given a presentation for it. This result generalizes results proved by [GM].

All the figures in this thesis were extracted of [Y], [GM] , [GM2] and [GM3].

Chapter 1

Braid groups

In this chapter, we will give a brief introduction to the surface braid theory and we introduce presentation and ordering theory for surface braid groups that will be useful as tools for the results of this thesis. The basic references used in this chapter are: [DDRW], [GM], [GM2] and [GM3].

1.1 Surface braid groups

This section was extracted from *New Presentations of Surface Braid Groups* by Juan González–Meneses in [GM] and *Homotopy String Links over Surfaces* by Ekaterina Yurasovskaya in [Y].

Definition 1.1.1. [GM, p. 431] Let M be a closed (compact without boundary) surface, not necessarily orientable, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a set of n distinct points of M. A geometric braid over M based at $\mathcal{P}$ is an n-tuple $\gamma = (\gamma_1, \dots, \gamma_n)$ of paths, $\gamma_i : [0,1] \to M$, such that:

(1) $\gamma_i(0) = P_i$, for all $i = 1, \dots, n$,

(2) $\gamma_i(1) \in \mathcal{P}$, for all $i = 1, \dots, n$,

(3) $\{\gamma_1(t), \dots, \gamma_n(t)\}$ are n distinct points of M, for all $t \in [0,1]$.

For all $i = 1, \dots, n$, we will call γ_i the i-th string of γ.

Remark 1.1.2. *In the Definition 1.1.1 is equivalent to consider each strand γ_i from $[0,1]$ to $M \times [0,1]$ defined by $\gamma_i(t) = (\gamma_i(t), t)$ in the cylinder $M \times [0,1]$. By this way, we see that there is not intersection in different heights t_1, t_2 in $[0,1]$ and thus, the condition (3) ensures that different strands never intersect.*

" Two geometric braids based at $\mathcal{P}$ are said to be equivalent if there exists a homotopy which deforms one of them into the other, provided that at anytime we always have a geometric braid based at $\mathcal{P}$. We can naturally define the product of two braids as induced by the usual product of paths: for every $i = 1, \ldots, n$, we compose the string of the first braid which ends at P_i, with the i-th string of the second braid. This product is clearly well defined, and it endows the set of equivalence classes of braids with a group structure. This group is called the *braid group on n strings over M* based at $\mathcal{P}$, and it is denoted by $B_n(M, \mathcal{P})$. It does not depend, up to isomorphism, on the choice of $\mathcal{P}$, but only on the number of strings, so we may write $B_n(M)$ instead of $B_n(M, \mathcal{P})$.

A braid $\gamma = (\gamma_1, \ldots, \gamma_n)$ is said to be pure if $\gamma_i(1) = P_i$, for all $i = 1, \ldots, n$, that is, if all its strings are loops. The set of equivalence classes of pure braids forms a normal subgroup of $B_n(M, \mathcal{P})$ called *pure braid group on n strings over M* based at $\mathcal{P}$, and denoted $PB_n(M, \mathcal{P})$. Again, we may write $PB_n(M)$ since it does not depend on the choice of $\mathcal{P}$. "

Remark 1.1.3. *In* [GM], *González–Meneses gives the presentations of $B_n(M)$ and $PB_n(M)$, when M is a closed surface not necessarily orientable. For our purposes, we will consider just the case which M is a closed, orientable surface of genus $g \geq 1$. For more details, see* [GM].

Recall that M is a closed and orientable surface of genus $g \geq 1$. First, let us obtain a geometrical representation of a braid over M: we represent M as a polygon L of $4g$ sides, identified in the way of Figure 1.1:

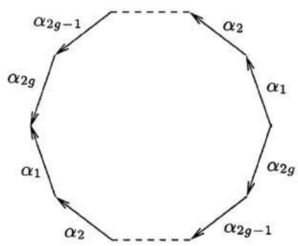


Figure 1.1: The polygon L representing M [GM].

“ We could now take the cylinder $L \times I$, where I is the closed unit interval $[0, 1]$, and represent a braid γ over M as it is usually done for the open disc, that is, in $L \times \{t\}$ we draw the n points $\gamma_1(t), \ldots, \gamma_n(t)$ but in this case, a string could “go through a wall” of the cylinder and appear from the other side. Hence, if we look at the cylinder from the usual viewpoint, it would not be clear which are “crossed walls”. ”

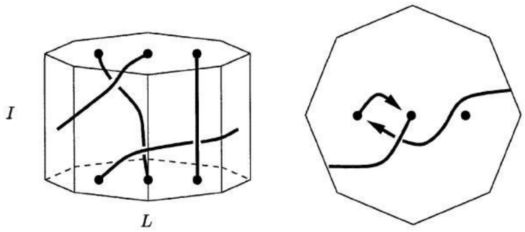


Figure 1.2: Two different ways to see a braid on a surface [GM].

“ Now, let us define the generators of $B_n(M)$. We choose the n base points along the horizontal diameter of L, as shown in the Figure 1.3. Now, given r, $1 \leq r \leq 2g$, we define the braid $a_{1,r}$ as follows: its only nontrivial string is the first one, which goes through the r-th wall. Just for notation, the first string will go upwards if r is odd, and downwards otherwise.

We also define, for all $i = 1, \ldots, n-1$, the braid σ_i as in the same figure. Note that $\sigma_1, \ldots, \sigma_{n-1}$ are the classical generators of the braid group $B_n(\mathbb{D})$ of the disc. We will see later that the set $\{a_{1,1}, \ldots, a_{1,2g}, \sigma_1, \ldots, \sigma_{n-1}\}$ is a set of generators of $B_n(M)$. ”

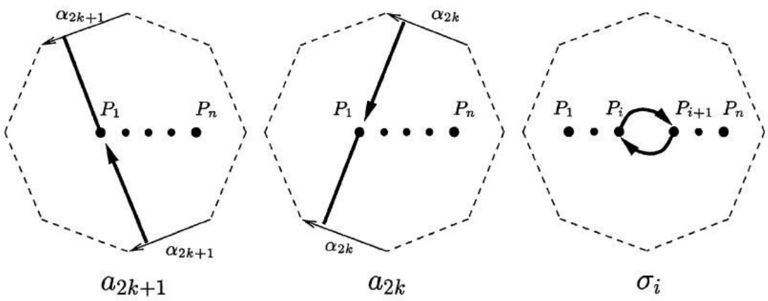


Figure 1.3: Elements of $B_n(M)$, where $a_r = a_{1,r}$ [GM].

“ We observe that the classical relations in $B_n(\mathbb{D})$:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \le i \le n-2,$$

still hold in $B_n(M)$. Also, if $i \in \{2, \ldots, n-1\}$ and $r \in \{1, \ldots, 2g\}$, then the non-trivial strings of σ_i and the one of $a_{1,r}$ may be taken to be disjoint. This implies that these two braids commute. Hence we have:

$$a_{1,r}\sigma_i = \sigma_i a_{1,r} \ , \quad 1 \le r \le 2g; \quad i \ge 2. \text{”}$$

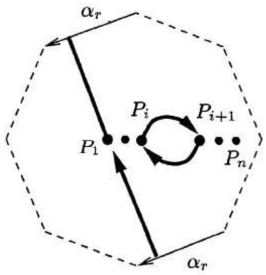


Figure 1.4: The braid $a_{1,r}\sigma_i$ [GM].

“ Now, in order to find more relations between the set of generators, we do the following construction: denote by s_r the first string of $a_{1,r}$, for all $r = 1, \ldots, 2g$, and consider all the paths $s_1, \ldots, s_{2g}$. We can “cut” the polygon L along them, and “glue” the pieces along the paths $a_{1,1}, \ldots, a_{1,2g}$. We obtain another polygon of $4g$ sides which are labeled by $s_1, \ldots, s_{2g}$ (see in the following figure the case of a surface of genus 2, the general case is analogous). We will call this new polygon the *P_1-polygon of M*, since all of its vertices are identified to P_1, while L will be called the *initial polygon*. We obtain in this way a new representation of the surface M. ”

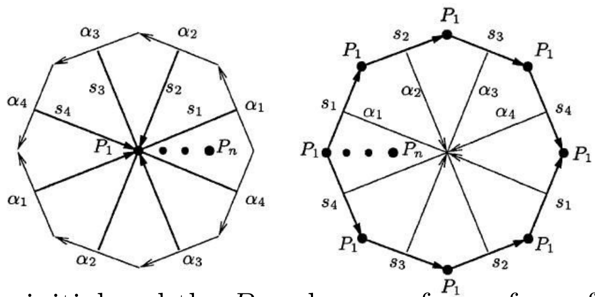


Figure 1.5: The initial and the P_1-polygons of a surface of genus 2 [GM].

“We will use the P_1-polygon to show three more relations in $B_n(M)$. For instance, consider the product of braids $a_{1,1} \cdots a_{1,2g} a_{1,1}^{-1} \cdots a_{1,2g}^{-1}$. If we look at P_1-polygon, we see that it is equivalent to the braid on Figure 1.6. Also, this one can be seen in the initial polygon as a braid that does not go through the walls, namely, an element of $B_n(\mathbb{D})$, the braid group of the disk. Then, we can easily see that it is equivalent to the braid $\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1$. So we have:

$$a_{1,1}\cdots a_{1,2g}a_{1,1}^{-1}\cdots a_{1,2g}^{-1} = \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1.\text{''}$$

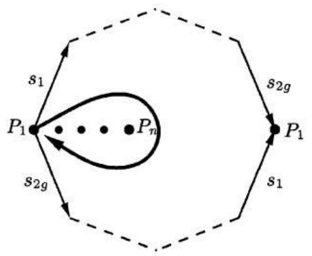


Figure 1.6: The braid $a_{1,1}\cdots a_{1,2g}a_{1,1}^{-1}\cdots a_{1,2g}^{-1}$ [GM].

“ Now, for each $r = 1, \ldots, 2g-1$, we define the braid:

$$A_{2,r} = \sigma_1(a_{1,1}\cdots a_{1,r-1}a_{1,r+1}^{-1}\cdots a_{1,2g}^{-1})\sigma_1^{-1}.$$

We will use the P_1-polygon to see how it looks like. In the left hand side of Figure 1.7, we can see a braid which is equivalent to $A_{2,r}$ (if r is odd, the other case being analogous). If we “cut” and “glue” to see this braid in the P_1-polygon, we obtain the situation of the right hand side of Figure 1.7. That is, $A_{2,r}$ can be seen as a braid whose only nontrivial string is the second one, which goes upwards and crosses once the r-th wall s_r. ”

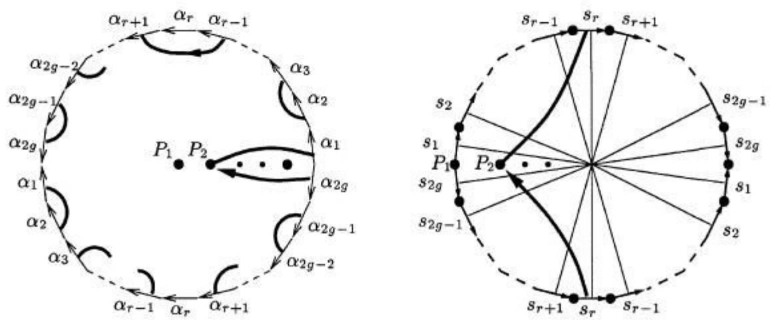


Figure 1.7: The braid $A_{2,r}$ in the initial polygon and in the P_1-polygon [GM].

“ Note that, unlike the case of $a_{1,r}$, $A_{2,r}$ always points upwards in the P_1-polygon, no matter the parity of r. Therefore we have seen that the braid $A_{2,r}$ can be represented by a geometric braid, whose only non trivial string can be taken disjoint from all the paths

s_t, $t \neq r$. This implies that

$$a_{1,t}A_{2,r} = A_{2,r}a_{1,t}, \ \ 1 \leq t \leq 2g, \ \ 1 \leq r \leq 2g-1; \ \ t \neq r.$$

Now, we finish our set of relations by considering the commutator of the braids $(a_{1,1} \cdots a_{1,r})$ and $A_{2,r}$, for all $r = 1, \dots, 2g-1$. In Figure 1.8 we can see a sketch of the homotopy which starts with this commutator and deforms it to a braid equivalent to σ_1^2. Therefore, we obtain the relation:

$$(a_{1,1} \cdots a_{1,r})A_{2,r} = \sigma_1^2 A_{2,r}(a_{1,1} \cdots a_{1,r}), \ 1 \leq r \leq 2g-1. \text{ ”}$$

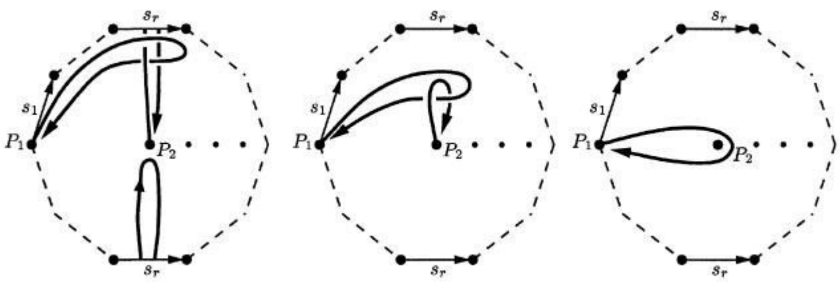


Figure 1.8: The braid $[a_{1,1} \cdots a_{1,r}, A_{2,r}]$ [GM].

The following theorem follows from all of the previous results statements.

<u>Theorem</u> 1.1.4. [GM, Theorem 2.1] *If M is a closed, orientable surface of genus $g \geq 1$, then $B_n(M)$ admits the following presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, a_{1,1}, \dots, a_{1,2g}$.
- *Relations:*

(R1) $\sigma_i\sigma_j = \sigma_j\sigma_i$, $\quad |i-j| \geq 2$;

(R2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $\quad 1 \leq i \leq n-2$;

(R3) $a_{1,1} \cdots a_{1,2g} a_{1,1}^{-1} \cdots a_{1,2g}^{-1} = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1$,

(R4) $a_{1,r}A_{2,s} = A_{2,s}a_{1,r}$, $\quad 1 \leq r \leq 2g$; $1 \leq s \leq 2g-1$; $r \neq s$;

(R5) $(a_{1,1} \cdots a_{1,r})A_{2,r} = \sigma_1^2 A_{2,r}(a_{1,1} \cdots a_{1,r})$, $\quad 1 \leq r \leq 2g-1$;

(R6) $a_{1,r}\sigma_i = \sigma_i a_{1,r}$, $\quad 1 \leq r \leq 2g$; $i \geq 2$.

Now, to give the presentation of $PB_n(M)$, let us know the generators and the relations for this group.

For the generators of $PB_n(M)$, consider:

1. Let $a_{i,r}$ be the braid such that the i-th string goes through the r-th wall. This string will go upwards if r is odd, and downwards otherwise. The other strings are trivial.

2. Let $T_{i,j} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i$ the braid that starts in P_i, goes around P_j from the back and turns back to P_i passing in front the points $P_j, \ldots, P_{i+1}$. The other strings are trivial.

The braids defined previously are given in Figure 1.9:

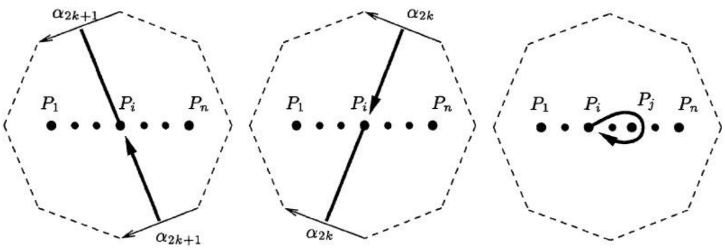


Figure 1.9: The braids $a_{i,r}$ and $T_{i,j}$ [GM].

“ We will denote by $s_{i,r}$ the i-th string of $a_{i,r}$. One can easily show that for any i, the set of paths $\{s_{i,1}, \ldots, s_{i,2g}\}$ generates $\pi_1(M)$. Now, for any $i \in \{2, \ldots, n\}$ we can define the P_i-polygon as we defined the P_1-polygon: we cut L along $s_{i,1}, \ldots, s_{i,2g}$ and glue along $\alpha_{i,1}, \ldots, \alpha_{i,2g}$. We define, for $2 \leq j \leq n$ and $1 \leq r \leq 2g-1$, the braid:

$$A_{j,r} = a_{j,1} \cdots a_{j,r-1} a_{j,r+1}^{-1} \cdots a_{j,2g}^{-1}.$$

Like in the representation of $A_{2,r}$ in the P_1-polygon, $A_{j,r}$ can be represented in the P_i-polygon (for $1 \leq i < j$), as the braid on Figure 1.10, whose only nontrivial string is the j-th one, which goes upwards and crosses once the r-th wall $s_{i,r}$. Note that this representation does not depend on i, but it is only valid when $i < j$. ”

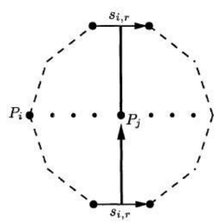


Figure 1.10: $A_{j,r}$ in the P_i-polygon [GM].

Now, let us enunciate the presentation of $PB_n(M)$. For more details about the relations, see [GM].

Theorem 1.1.5. [GM, Theorem 4.2] *$PB_n(M)$ admits the following presentation:*

Generators: $\{a_{i,r};\ 1 \leq i \leq n,\ 1 \leq r \leq 2g\} \cup \{T_{j,k};\ 1 \leq j < k \leq n\}$.

Relations:

(PR1) $a_{n,1}^{-1}a_{n,2}^{-1}\cdots a_{n,2g}^{-1}a_{n,1}a_{n,2}\cdots a_{n,2g} = \prod_{i=1}^{n-1} T_{i,n-1}^{-1}T_{i,n}$;

(PR2) $a_{i,r}A_{j,s} = A_{j,s}a_{i,r}$, $\quad 1 \leq i < j \leq n; 1 \leq r \leq 2g; 1 \leq s \leq 2g-1; r \neq s$;

(PR3) $(a_{i,1}\cdots a_{i,r})A_{j,r}(a_{i,r}^{-1}\cdots a_{i,1}^{-1})A_{j,r}^{-1} = T_{i,j}T_{i,j-1}^{-1}$, $\quad 1 \leq i < j \leq n; 1 \leq r \leq 2g-1$;

(PR4) $T_{i,j}T_{k,l} = T_{k,l}T_{i,j}$, $\quad 1 \leq i < j < k < l \leq n$ *or* $1 \leq i < k < l \leq j \leq n$;

(PR5) $T_{k,l}T_{i,j}T_{k,l}^{-1} = T_{i,k-1}T_{i,k}^{-1}T_{i,j}T_{i,l}^{-1}T_{i,k}T_{i,k-1}^{-1}T_{i,l}$, $\quad 1 \leq i < k \leq j < l \leq n$;

(PR6) $a_{i,r}T_{j,k} = T_{j,k}a_{i,r}$, $\quad 1 \leq i < j < k \leq n$ *or* $1 \leq j < k < i \leq n; 1 \leq r \leq 2g$;

(PR7) $a_{i,r}(a_{j,2g}^{-1}\cdots a_{j,1}^{-1}T_{j,k}a_{j,2g}\cdots a_{j,1}) = (a_{j,2g}^{-1}\cdots a_{j,1}^{-1}T_{j,k}a_{j,2g}\cdots a_{j,1})a_{i,r}$, $\quad 1 \leq j < i \leq k \leq n$;

(PR8) $T_{j,n} = \left(\prod_{i=1}^{j-1} a_{i,2g}^{-1}\cdots a_{i,1}^{-1}T_{i,j-1}T_{i,j}^{-1}a_{i,1}\cdots a_{i,2g}\right) a_{j,1}\cdots a_{j,2g}a_{j,1}^{-1}\cdots a_{j,2g}^{-1}$.

Where $A_{j,s} = a_{j,1}\cdots a_{j,s-1}a_{j,s+1}^{-1}\cdots a_{j,2g}^{-1}$.

Remark 1.1.6. *In the next two sections will be shown the tools that we will use to find the presentation of the group defined in Chapter* 4. *These are the same tools used in* [*GM*] *to find the presentation for the surface braid groups, namely, Theorems 1.1.4 and 1.1.5.*

1.2 Methods for finding presentation of groups

This section was extracted from *New Presentations of Surface Braid Groups* by González–Meneses in [GM].

" Consider an exact sequence of groups A, $\widetilde{G}$, G:

$$1 \longrightarrow A \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \longrightarrow 1 \tag{1.2.1}$$

Suppose that the groups A and G admit presentations $\langle X; R_A\rangle$ and $\langle Y; R_G\rangle$ respectively, where X and Y are sets of generators, while R_A and R_G are sets of relations. The following well-known procedure outlines a method for putting together a presentation of $\widetilde{G}$:

Generators of $\widetilde{G}$: Let $\widetilde{X} = \{\tilde{x} = i(x);\ \ x \in X\}$ be the images of the generators X of A under the homomorphism i. Now, given $y \in Y$, let $\tilde{y}$ denote a chosen pre-image of y under p, i.e., $p(\tilde{y}) = y$. Define $\widetilde{Y} = \{\tilde{y};\ \ y \in Y\}$ the set of all such pre-images. Then $\widetilde{X} \cup \widetilde{Y}$ constitute a set of generators for $\widetilde{G}$.

Relations: There are three types of relations in $\widetilde{G}$:

Type 1: Relations of the form $\widetilde{R}_A = \{\tilde{r}_A;\ \ r_A \in R_A\}$; where $\widetilde{R}_A$ is the set of words in $\widetilde{X}$ obtained from R_A by replacing each x by $\tilde{x}$. Thus each $\tilde{r}_A$ is an image under the injective homomorphism i of a relation r_A in $\widetilde{G}$.

Type 2: Let $\tilde{r}_G$ be a word obtained from a relation r_G in R_G by replacing each y by its chosen pre-image $\tilde{y}$. We see that p maps $\tilde{r}_G$ in $\widetilde{G}$ to relation r_G in G, therefore $\tilde{r}_G$ lies in the $\ker(p)$. Since the sequence (1.2.1) is exact, we know that $\ker(p)$ equals the image $i(A)$ of A under the homomorphism i. Thus $\tilde{r}_G = w_r$, where w_r is a word in $\widetilde{X}$. We thus have a second set of relations:

$$\widetilde{R}_G = \{\tilde{r}_G = w_r;\ \ r_G \in R_G\}.$$

Type 3: Choose any $\tilde{y}$ from the set $\widetilde{Y}$ of chosen pre-images of the generator set Y under p. The image of A under i is a normal subgroup of $\widetilde{G}$, therefore each conjugate of generator $\tilde{x}$ again belongs to $i(A)$. Thus $\tilde{y}\tilde{x}\tilde{y}^{-1}$ can be written as a word w_x over the generators $\widetilde{X}$ of the kernel. We put

$$\widetilde{C} = \{\tilde{y}\tilde{x}\tilde{y}^{-1} = w_x;\ \ x \in X,\ \ \tilde{y} \in \widetilde{Y}\}.\text{''}$$

Proposition 1.2.1. [J, pp.138–140] *With the previous notation, the group $\widetilde{G}$ has the presentation*

$$\langle \widetilde{X}, \widetilde{Y};\ \ \widetilde{R}_A, \widetilde{R}_G, \widetilde{C}\rangle.$$

The following result is an alternative group presentation, which is equivalent to the group presentation of Proposition 1.2.1.

Lemma 1.2.2. [GM, Lemma 4.1] *Let $\mathbb{F}(2g)$ be the free group freely generated by $\{x_1, \ldots, x_{2g}\}$. Set $X_r = x_1 \cdots x_{r-1} x_{r+1}^{-1} \cdots x_{2g}^{-1}$. Then $\{X_1, \ldots, X_{2g}\}$ is a free system of generators of $\mathbb{F}(2g)$.*

Proof: " The following are the formula of the change of generators:

$$x_k = (X_1X_2^{-1}\cdots X_{k-2}X_{k-1}^{-1})(X_{k+1}X_{k+2}^{-1}\cdots X_{2g-1}X_{2g}^{-1}),\ \ if\ k\ is\ odd.$$

$$x_k^{-1} = (X_1X_2^{-1}\cdots X_{k-2}^{-1}X_{k-1})(X_{k+1}^{-1}X_{k+2}\cdots X_{2g-1}^{-1}X_{2g}),\ \ if\ k\ is\ even.\text{''}$$

■

1.3 Ordering braids

This section was extracted from *Ordering Braids* by D. Rolfsen, P. Dehornoy, I. Dynnikov and B. Wiest in [DDRW].

1.3.1 Ordering a group

Definition 1.3.1. [DDRW, p. 11] A *strict ordering of a set* Ω is a binary relation $<$ that satisfies the following conditions:

(antireflexive): $x < x$ never holds, for all $x \in \Omega$;

(transitive): $x < y$ and $y < z$ implies $x < z$, for all $x, y, z \in \Omega$.

A strict ordering of Ω is called *linear or total* if, for all $x, x' \in \Omega$, one of the following holds

$$x = x',\ \ x <\ x'\ \text{ or }\ x' < x.$$

Definition 1.3.2. [DDRW, p. 12]

(i) A *left-invariant ordering or left-ordering* of a group G is a strict linear ordering $<$ of G that satisfies:

$$g < h \Rightarrow fg < fh,\ \text{ for all }\ f, g, h \in G.$$

A group G is *left-orderable* if there exists at least one left-invariant ordering of G.

(ii) A *right-invariant ordering or right-ordering* of a group G is a strict linear ordering $<$ of G that satisfies:

$$g < h \Rightarrow gf < hf, \text{ for all } \; f, g, h \in G.$$

A group G is *right-orderable* if there exists at least one right-invariant ordering of G.

(iii) A *bi-invariant ordering or bi-ordering* of a group G is a strict linear ordering $<$ of G that satisfies:

$$g < h \Rightarrow fg < fh \text{ and } gf < hf, \text{ for all } \; f, g, h \in G.$$

A group G is *bi-orderable* if there exists at least one bi-invariant ordering of G.

<u>Lemma</u> 1.3.3. [DDRW, Lemma 1.3, p. 12] *Assume that G is a group and $<$ is a left-invariant ordering of G. Define $g \tilde{<} h$ to mean $g^{-1} < h^{-1}$. Then $\tilde{<}$ is a right-ordering of G.*

<u>Definition</u> 1.3.4. [DDRW, p. 12] A subset P of a group G is called a *positive cone* on G if it is closed under multiplication and $G \setminus \{1\}$ is the disjoint union of P and $P^{-1} = \{p^{-1};\; p \in P\}$. In symbols:

(i) $P \cdot P = P$

(ii) $G \setminus \{1\} = P \sqcup P^{-1}$.

<u>Lemma</u> 1.3.5. [DDRW, Lemma 1.5, p. 12]

(i) *Assume that $<$ is a left-ordering of a group G. Then the set $P = \{x \in G;\; x > 1\}$ is a positive cone on G, and $g < h$ is equivalent to $g^{-1}h \in P$.*

(ii) *Assume that P is a positive cone on a group G. Then, the relation $gh^{-1} \in P$ is a left-ordering of G and P is then the set of all elements of G that are larger than 1.*

<u>Remark</u> 1.3.6. *Note in the second condition of Lemma 1.3.5 that if P is a positive cone on a group G, then the order $<$ defined previously is a left-ordering for the group G.*

Lemma 1.3.7. [DDRW, Lemma 3.1, p. 269] *Assume we have an exact sequence of groups:*

$$1 \longrightarrow N \xrightarrow{\subseteq} G \xrightarrow{p} H \longrightarrow 1 ,$$

and, moreover, $<_N$ is a left-invariant ordering of N and $<_H$ is a left-invariant ordering of H. For $g, g' \in G$, declare that $g < g'$ is true if we have either $p(g) <_H p(g')$ or else $p(g) = p(g')$ and $1 <_N g^{-1}g'$. Then:

(i) *The relation $<$ is a left-invariant ordering of G.*

(ii) *If $<_N$ and $<_H$ are bi-invariant orderings, then $<$ is a bi-invariant ordering of G, if and only if, conjugation of N by G is order-preserving, i.e., $f <_N f'$ implies $gfg^{-1} <_N gf'g^{-1}$, for all $f, f' \in N$ and $g \in G$.*

Proof: See [MR]. ∎

1.3.2 Ordering free groups

" Let $\mathbb{F}$ be a free group on the generator set $\{x_1, \dots, x_n\}$. Let us prove that $\mathbb{F}$ is bi-orderable.

We denote by $\mathbb{Z}\langle\langle X_1, \dots, X_n\rangle\rangle$ the ring of formal power series in n non-commuting indeterminates X_i. Such series are infinite sums of monomials, each of which is a word on the letters X_i, so they have the generic form:

$$f = \sum_{W \in \{X_1, \dots, X_n\}^*} f_W W, \quad \text{for } f_W \in \mathbb{Z}.$$

where $\{X_1, \dots, X_n\}^*$ denotes the set of all finite length words on the alphabet $\{X_1, \dots, X_n\}$. The length of the word W is called the degree of the monomial $f_W W$. As we consider n non-commutative variables, there exist n^d monomials of degree d.

Addition of $\mathbb{Z}\langle\langle X_1, \dots, X_n\rangle\rangle$ is defined by summing the coefficients, while multiplication is given by:

$$\left(\sum f_W W\right)\left(\sum g_W W\right) = \sum_W \left(\sum_{UV=W} f_U g_V\right) W.$$

We use $O(X^k)$ to denote the ideal of $\mathbb{Z}\langle\langle X_1, \dots, X_n\rangle\rangle$ made of the series involving only monomials of degree $\geq k$. ”

Definition 1.3.8. [DDRW, Definition 2.7, p. 267] Assume that $\mathbb{F}$ is a free group and $\{x_1, \dots, x_n\}$ is a basis of $\mathbb{F}$. *The Magnus expansion* of $\mathbb{F}$ relative to $(x_1, \dots, x_n)$ is the map $\mu : \mathbb{F} \to \mathbb{Z}\langle\langle X_1, \dots, X_n\rangle\rangle$ given by $\mu(x_i) = 1 + X_i$ and $\mu(x_i^{-1}) = 1 - X_i + X_i^2 - \cdots$.

Example 1.3.9. [DDRW, Example 2.8, p. 267] *For $w = x_1^{-1}x_2x_1$, we find:*

$$\begin{aligned}\mu(w) &= (1 - X_1 + X_1^2 - X_1^3 + \cdots)(1 + X_2)(1 + X_1)\\ &= 1 + X_2 - X_1X_2 + X_2X_1 + X_1^2X_2 - X_1X_2X_1 \bmod O(X^4).\end{aligned}$$

Proposition 1.3.10. [DDRW, Proposition 2.9, p. 267] *Assume that $\mathbb{F}$ is a free group, and μ is a Magnus expansion of $\mathbb{F}$.*

(i) *The map μ is an injective map of $\mathbb{F}$ into $1 + O(X)$.*

(ii) *For each nonnegative k, the Magnus image of the k-th term in the lower central series of $\mathbb{F}$ is included in $1 + O(X^{k+1})$.*

“ We can use Magnus expansions to order free groups. First, we order $\mathbb{Z}\langle\langle X_1, \dots, X_n\rangle\rangle$ as follows: for each d, the natural ordering $X_1 < \cdots < X_n$ induces a lexicographical ordering on monomials of total degree d. We therefore have a natural increasing enumeration of these monomials. For instance, for $n = d = 2$, the increasing enumeration of the degree 2 monomials is the sequence $(X_1^2, X_1X_2, X_2X_1, X_2^2)$. ”

Definition 1.3.11. [DDRW, Definition 2.11, p. 268]

(i) For $d \geq 0$ and $f \in \mathbb{Z}\langle\langle X_1, \dots, X_n\rangle\rangle$, say $f = \sum f_W W$, we denote by $C_d(f)$ the sequence $(f_{W_1}, \dots, f_{W_N})$, where $W_1, \dots, W_N$ is the increasing enumeration of all degree d monomials. We denote by $c_d(f)$ the sum of all coefficients $f_{W_i} \in C_d(f)$.

(ii) For $f, g \in \mathbb{Z}\langle\langle X_1, \dots, X_n\rangle\rangle$, we declare that $f <^{SumLex} g$ is true if there exists d such that the sequences $C_{d'}(f)$ and $C_{d'}(g)$ coincide for $d' < d$, and

- we have $c_d(f) < c_d(g)$, or

- we have $c_d(f) = c_d(g)$ and the sequence $C_d(f)$ is lexicographically smaller than the sequence $C_d(g)$, i.e., there is an index k such that the first $(k-1)$ entries are the same, and the k-th entry in $C_d(f)$ is smaller than the k-th entry in $C_d(g)$.

" The previous comparison procedure is a variant of the so-called DegLex-ordering, where one first considers the degree, and then, a lexicographical ordering inside entries of a given degree. The specificity here is that we give priority to the sum of all coefficients corresponding to a given degree before starting the lexicographic comparison, which explains our terminology. "

Example 1.3.12. [DDRW, Example 2.12, p. 268] *Let us compare the series f of the Example 1.3.9 with the polynomial $g = 1 + X_2$. In degree 0, there is only the constant monomial, and we find $C_0(f) = C_0(g) = (1)$. In degree 1, the increasing enumeration of the two monomials is X_1, X_2, and we find $C_1(f) = C_1(g) = (0, 1)$. In degree 2, the increasing enumeration of the four monomials is $X_1^2, X_1X_2, X_2X_1, X_2^2$, and we have now $C_2(f) = (0, -1, 1, 0)$, and $C_2(g) = (0, 0, 0, 0)$. We find $c_2(f) = c_2(g) = 0$, so we compare the sequences $C_2(f)$ and $C_2(g)$ starting from the left. The second entry of f is smaller than that of g, so $f <^{SumLex} g$ is true.*

Lemma 1.3.13. [DDRW, Lemma 2.13, p. 268] *The relation $<^{SumLex}$ is a linear ordering of*

$$\mathbb{Z}\langle\langle X_1, \ldots, X_n\rangle\rangle$$

that is invariant under addition, and under multiplication on either side by an element of the multiplicative subgroup $1 + O(X)$.

" Note that the ordering $<^{SumLex}$ on $\mathbb{Z}\langle\langle X_1, \ldots, X_n\rangle\rangle$ is not invariant under an arbitrary multiplication, typically by -1.

Using the Magnus expansion, we define an ordering of every finitely generated free group with a prescribed basis-naturally called the *Magnus ordering.* "

Definition 1.3.14. [DDRW, Definition 2.14, p. 269] Assume that $\mathbb{F}$ is a free group and $\{x_1, \ldots, x_n\}$ is a basis of $\mathbb{F}$. For w, w' in $\mathbb{F}$, we declare that $w <_\mu w'$ is true if we have $\mu(w) <^{SumLex} \mu(w')$, where μ is the Magnus expansion relative to $\{x_1, \ldots, x_n\}$.

Proposition 1.3.15. [DDRW, Proposition 2.15, p. 269] *For each finite rank free group $\mathbb{F}$ and each basis $\{x_1, \dots, x_n\}$ of $\mathbb{F}$, the Magnus ordering of $\mathbb{F}$ relative to $\{x_1, \dots, x_n\}$ is a linear ordering that is invariant under multiplication on both sides.*

Proof: " By Proposition 1.3.10(i), the Magnus expansion is injective, so the relation $<_\mu$ is a linear ordering on $\mathbb{F}$. Its invariance under multiplication on both sides follows from Lemma 1.3.13, since, by construction, the image of $\mathbb{F}$ under the Magnus expansion is included in the multiplicative subgroup $1 + O(X)$. " ■

Example 1.3.16. [DDRW, Example 2.16, p. 269] *Let us compare x_2 and $x_1^{-1}x_2x_1$. The Magnus expansions are*

$$\mu(x_2) = 1 + X_2 \ \ \textit{and} \ \ \mu(x_1^{-1}x_2x_1) = 1 + X_2 - X_1X_2 + X_2X_1 \bmod O(X^2).$$

These series have been compared in Example 1.3.12: the latter is $<^{SumLex}$ smaller than the former. So we have $x_1^{-1}x_2x_1 <_\mu x_2$.

1.3.3 Ordering semi-direct products

This section was extracted from *Homotopy String Links over Surfaces* by Ekaterina Yurasovskaya in [Y].

" Let H and Q be bi-ordered groups. The direct product $H \times Q$ is bi-orderable, under the lexicographic ordering:

$$hq < h'q', \text{ if and only if, } q < q'; \text{ or } q = q' \text{ and } h < h'.$$

Remark 1.3.17. *To accommodate semi-direct product written as $\rtimes$, the lexicographic ordering becomes "eastern", with comparison starting on the right. The terms lexicographic ordering refers to the "eastern" lexicographic ordering as defined above.*

However, a semi-direct product $H \rtimes Q$ is not necessarily bi-ordered under lexicographic ordering, as follows. "

Example 1.3.18. [Y, Example 8.9] *The fundamental group G of the Klein bottle can be written as a semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$ of two infinite cyclic groups. Now write the* *presentation of this group in terms of the generators and relations:*

$$G = \langle x, y;\ \ yxy^{-1} = x^{-1} \rangle.$$

Consider $x \neq 1$ and suppose that x is positive. If G is bi-ordered, then yxy^{-1} is also positive, which is a contradiction to the fact that x^{-1} must be negative. If we assume that $x \neq 1$ is negative, then x^{-1} must be positive, and we have the same contradiction.

“ Given elements $h \in H$, and $q \in Q$, we let h^q denote h under the action of q. Recall that the multiplication in a semi-direct product $H \rtimes Q$ is given by the formula $(h, q)(h', q') = (hh'^q, qq')$, where $h^q = qhq^{-1}$. The following result provides a necessary and sufficient condition for lexicographic ordering of a semi-direct product to be a bi-order. ”

Lemma 1.3.19. [Y, Lemma 8.10] *Let H and Q be bi-ordered groups. Then the lexicographic order on $H \rtimes Q$ is a bi-ordering, if and only if, the action of Q on H preserves the order on H. Equivalently, $q(P_H)q^{-1} \subset P_H$, for all $q \in Q$.*

1.3.4 Ordering surface pure braid groups

This section contains the proof that $PB_n(M)$ is bi-orderable, when M is a closed, connected and orientable surface of genus $g \geq 1$, and was extracted from the work *Ordering Pure Braid Groups on Compact, Connected Surfaces* by Juan González–Meneses in [GM2].

“ Given a pure braid $b = (b_1, \ldots, b_n) \in PB_n(M)$, we can consider, for all $i = 1, \ldots, n$, the loop μ_i in M constructed as follows: take the i-th string b_i (which is a path in $M \times [0, 1]$); μ_i is the projection of b_i over the first coordinate (i.e., over M). Since $b \in PB_n(M)$, μ_i is a loop in M based at P_i for all $i = 1, \ldots, n$, which represents an element of $\pi_1(M, P_i) \simeq \pi_1(M)$. This defines an epimorphism $\theta_n : PB_n(M) \to \pi_1(M)^n$, which sends $(b_1, \ldots, b_n)$ to $(\mu_1, \ldots, \mu_n)$. ”

The well definition of this epimorphism was proved by Birman in [B].

“ Define $\mathbb{K}_n = \ker(\theta_n)$. One has the exact sequence:

$$1 \longrightarrow \mathbb{K}_n \xrightarrow{\subseteq} PB_n(M) \xrightarrow{\theta_n} \pi_1(M)^n \longrightarrow 1\ .$$

In order to use Lemma 1.3.7 to prove that $PB_n(M)$ is bi-orderable, we need to show that $\mathbb{K}_n$ and $\pi_1(M)^n$ are bi-orderable. Then we prove that the conjugation of $\mathbb{K}_n$ by the elements of $PB_n(M)$ is order-preserving. ”

<u>Theorem</u> 1.3.20. [KR] The Magnus Order on a free group $\mathbb{F}$ is preserved under any $\Phi \in Aut(\mathbb{F})$ which induces the identity on the abelianization $H_1(\mathbb{F}) = \frac{\mathbb{F}}{[\mathbb{F},\mathbb{F}]}$.

“ Let ψ be a permutation of the set $\{X_i\}_{i\in I}$ and consider its extension $\Psi \in Aut(\mathbb{F})$. One has the following: ”

<u>Theorem</u> 1.3.21. [GM2, Theorem 2.2] If ψ preserves the order on $\{X_i\}_{i\in I}$, then Ψ preserves the Magnus Order on $\mathbb{F}$.

<u>Theorem</u> 1.3.22. [Ba] If M is a closed, orientable surface, then $\pi_1(M)$ is a bi-orderable group. Thus, $\pi_1(M)^n$ is bi-orderable.

The structure of $\mathbb{K}_n$

This section was extracted from *Vassiliev Invariants for Braids on Surfaces* by González–Meneses and Luis Paris in [GM3].

Consider the “forgetting” homomorphism: $\rho : PB_n(M) \longrightarrow PB_{n-1}(M)$ defined by $\rho(\beta) = \rho(\beta_1, \dots, \beta_n) = (\beta_2, \dots, \beta_n)$ (for details, see [B]).

“ Since $\mathbb{K}_n$ is a subgroup of $PB_n(M)$, we can consider the image of $\mathbb{K}_n$ by ρ. By definition, it is equal to $\mathbb{K}_{n-1}$. If we denote $\mathbb{F}_n = \ker\rho \cap \mathbb{K}_n$, we obtain the following commutative diagram, where all rows and columns are exact:

$$\begin{tikzcd}
 & 1 & 1 & 1 & \\
1 \arrow[r] & \pi_1(M,P_1) \arrow[u] \arrow[r] & \pi_1(M)^n \arrow[u] \arrow[r] & \pi_1(M)^{n-1} \arrow[u] \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(M\setminus\mathcal{P}_{n-1},P_1) \arrow[u] \arrow[r] & PB_n(M) \arrow[u, "\theta_n"] \arrow[r, "\rho"] & PB_{n-1}(M) \arrow[u, "\theta_{n-1}"] \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{F}_n \arrow[u] \arrow[r] & \mathbb{K}_n \arrow[u] \arrow[r, "\rho"] & \mathbb{K}_{n-1} \arrow[u] \arrow[r] & 1 \\
 & 1 \arrow[u] & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd} \tag{1.3.1}$$

Notice that $\mathbb{F}_n$ is a free group, since it is a subgroup of $\pi_1(M \setminus \mathcal{P}_{n-1}, P_1)$, which is a free group. "

Gonzalez–Meneses shows that there exists a homomorphism $\sigma : \mathbb{K}_{n-1} \to \mathbb{K}_n$ which is a section of ρ and that $\mathbb{K}_{n-1}$ acts trivially on the abelianization of $\mathbb{F}_n$. For this, he first finds a free set of generators for $\mathbb{F}_n$.

" Let $\Omega = \{w_1, \dots, w_{2g}\}$ be a set of $2g$ letters. It is well known that a presentation for $\pi_1(M)$ is

$$\langle \Omega;\ \ w_1 w_2 \cdots w_{2g} w_1^{-1} w_2^{-1} \cdots w_{2g}^{-1} = 1 \rangle.$$

For every element $\gamma \in \pi_1(M)$ we choose a unique word $\tilde{\gamma}$ over $\Omega \cup \Omega^{-1}$ which represents γ. We call this word the *normal form of* γ. Normal forms are chosen in such a way that they are prefix-closed (namely, if $w_1 w_2$ is a normal form, then w_1 is also a normal form). For every word w over $\Omega \cup \Omega^{-1}$, we will denote by $w_{(i)}$ the word over $\{a_{i,1}^{\pm 1}, \dots, a_{i,2g}^{\pm 1}\}$ obtained from w by replacing $w_j^{\pm 1}$ by $a_{i,j}^{\pm 1}$, for all $j = 1, \dots, 2g$. "

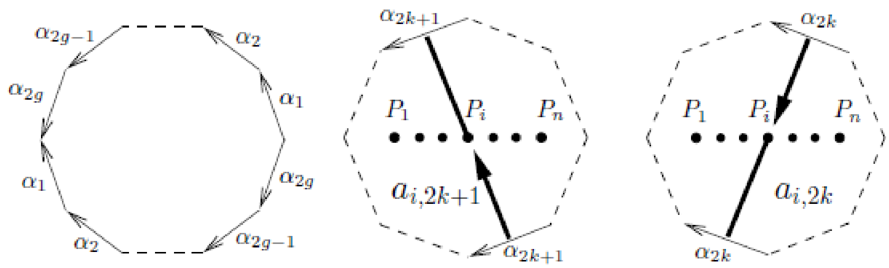


Figure 1.11: The polygon representing M and the braids $a_{i,2k+1}$ and $a_{i,2k}$ [GM].

Consider, for $1 \le i < j \le n$, the braid $t_{i,j} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1}^2 \sigma_{j-2}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-1}$ drawn in Figure 1.12.

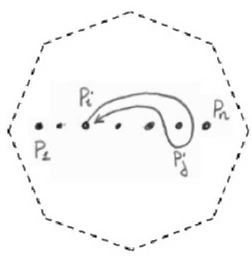


Figure 1.12: The braid $t_{i,j}$ [Y].

Lemma 1.3.23. [GM3, Lemma 2.5] *The following set is a free system of generators for* *the free group $\mathbb{F}_n$:*

$$\{\tilde{\gamma}_{(1)} t_{1,j} \tilde{\gamma}_{(1)}^{-1};\ \ 2 \leq j \leq n,\ \ \gamma \in \pi_1(M)\}.$$

Lemma 1.3.24. [GM3, Lemma 2.6] *There is a homomorphism $\sigma : \mathbb{K}_{n-1} \to \mathbb{K}_n$ which is a section of $\rho : \mathbb{K}_n \to \mathbb{K}_{n-1}$.*

“ Now, $\mathbb{K}_{n-1}$ acts on $\mathbb{F}_n$ in the following way: given $b \in \mathbb{K}_{n-1}$, the action induced by b sends $f \in \mathbb{F}_n$ to $\sigma(b) f \sigma(b)^{-1}$. This action induces an action of $\mathbb{K}_{n-1}$ on the abelianization $\frac{\mathbb{F}_n}{[\mathbb{F}_n, \mathbb{F}_n]}$ of $\mathbb{F}_n$ (here $[\mathbb{F}_n, \mathbb{F}_n]$ denotes the commutator subgroup of $\mathbb{F}_n$). ”

Lemma 1.3.25. [GM3, Lemma 2.7] *The action of $\mathbb{K}_{n-1}$ on the abelianization of $\mathbb{F}_n$ is trivial.*

For the next result, we recall that from [GM2]: we are representing M as a polygon of $4g$ sides, identified in the way of the Figure 1.1.

“ For all $i = 1, \ldots, n$ and all $r = 1, \ldots, 2g$, we are defining the braid $a_{i,r} \in PB_n(M)$: the i-th string of $a_{i,r}$ is $(s_{i,r}(t), t) \in M \times [0, 1]$, where $s_{i,r}$ is a loop in M based at P_i which goes through the wall α_r; it goes upwards if r is odd and downwards if r is even. The j-th string of $a_{i,r}$ is (P_j, t) (the trivial string) for all $j \neq i$. Also, $\Omega = \{\omega_1, \ldots, \omega_{2g}\}$ is a set of generators of $\pi_1(M)$, where g is the genus of M, taken such that

$$\pi_1(M) = \langle \Omega; \omega_1 \cdots \omega_{2g} \omega_1^{-1} \cdots \omega_{2g}^{-1} \rangle.$$

For all $\gamma \in \pi_1(M)$, choose the unique word $\tilde{\gamma}$ over $\Omega \cup \Omega^{-1}$ representing γ. We denote by $\tilde{\gamma}_{(i)}$ the pure braid obtained from $\tilde{\gamma}$ replacing $\omega_r^{\pm 1}$ by $a_{i,r}^{\pm}$. Now, for all $i, j \in \{1, \ldots, n\}$, $i < j$, recall the braid

$$t_{i,j} = \sigma_i \cdots \sigma_{j-2} \sigma_{j-1}^2 \sigma_{j-2}^{-1} \cdots \sigma_i^{-1} \in PB_n(M).$$

Finally, for all $i, j \in \{1, \ldots, n\}$, $i \neq j$, and for all $\gamma \in \pi_1(M)$, we define

$$f_{i,j,\gamma} = \tilde{\gamma}_{(i)} t_{i,j} \tilde{\gamma}_{(i)}^{-1}. \text{ ”}$$

The following result follows from Lemmas 1.3.23, 1.3.24 and 1.3.25.

Theorem 1.3.26. [GM2, Theorem 2.4] *One has $\mathbb{K}_n = (\mathbb{F}_n \rtimes (\mathbb{F}_{n-1} \rtimes (\ldots (\mathbb{F}_3 \rtimes \mathbb{F}_2) \ldots)))$,* *where for all $i = 1, \dots, n-1$, $\mathbb{F}_{(n+1)-i}$ is the free group freely generated by*

$$\mathcal{F}_{i,n} = \{f_{i,j,\gamma};\ \ i < j \leq n,\ \ \gamma \in \pi_1(M)\}.$$

Moreover, for all $m = 2, \dots, n-1$, $\mathbb{K}_m = (\mathbb{F}_m \rtimes (\mathbb{F}_{m-1} \rtimes (\dots (\mathbb{F}_3 \rtimes \mathbb{F}_2) \dots)))$ acts trivially on $H_1(\mathbb{F}_{m+1})$.

Corollary 1.3.27. [GM3, Corollary 2.5] $\mathbb{K}_n$ *is bi-orderable.*

Proof: " We argue by induction on n. If $n = 2$, then $\mathbb{K}_2 = \mathbb{F}_2$ is a free group (of infinite rank), so it is bi-orderable. Suppose that $n > 2$, and that $\mathbb{K}_{n-1}$ is bi-orderable. By Theorem 1.3.26, we have an exact sequence:

$$1 \longrightarrow \mathbb{F}_n \longrightarrow \mathbb{K}_n \longrightarrow \mathbb{K}_{n-1} \longrightarrow 1\ ,$$

where $\mathbb{K}_n = \mathbb{F}_n \rtimes \mathbb{K}_{n-1}$. By definition of bi-order, a conjugation by an element of $\mathbb{F}_n$ is an automorphism of $\mathbb{F}_n$ which preserves the Magnus Order. We also know, by Theorem 1.3.26, that a conjugation by an element of $\mathbb{K}_{n-1}$ is an automorphism of $\mathbb{F}_n$ which is trivial on $H_1(\mathbb{F}_n)$. Since the Magnus Order on any free group $\mathbb{F}$ is preserved by any automorphism in $Aut(\mathbb{F})$ (which induces the identity on the abelianization $H_1(\mathbb{F}) = \frac{\mathbb{F}}{[\mathbb{F},\mathbb{F}]}$), thus it also preserves the Magnus Order on $\mathbb{F}_n$. Therefore, a conjugation by an element of $\mathbb{K}_n$ preserves the Magnus order of $\mathbb{F}_n$ and thus, by Lemma 1.3.7, $\mathbb{K}_n$ is bi-orderable. "

■

Gonzalez–Meneses defines an explicit bi-order on $\mathbb{K}_n$, as follows.

" First, for all $i = 1, \dots, n-1$, we order $\mathcal{F}_{i,n}$ as follows:

$$f_{i,j,\gamma} < f_{i,k,\delta} \Leftrightarrow j < k \quad \text{or} \quad j = k \quad \text{and} \quad \gamma <_{\pi_1} \delta,$$

where $<_{\pi_1}$ is a fixed bi-order of $\pi_1(M)$. Then, we consider the Magnus Order on each $\mathbb{F}_{(n+1)-i}$ corresponding to this order on $\mathcal{F}_{i,n}$. The bi-order on $\mathbb{K}_n$ which comes from Corollary 1.3.27 is the following: for $k, k' \in \mathbb{K}_n$, write $k = k_1 k_2 \cdots k_{n-1}$ and $k' = k'_1 k'_2 \cdots k'_{n-1}$, where $k_i, k'_i \in \mathbb{F}_{(n+1)-i}$. Then $k < k'$, if and only if, $k_j < k'_j$ for the greatest j such that $k_j \neq k'_j$. "

$PB_n(M)$ is bi-orderable

This section was extracted from *Ordering Pure Braid on Compact, Connected Surfaces* by González–Meneses in [GM2].

Theorem 1.3.28. [GM2, Theorem 1.6] *If M is a closed, connected and orientable surface, then $PB_n(M)$ is bi-orderable.*

Proof: " The direct product of bi-orderable groups is bi-orderable, hence by Theorem 1.3.22, $\pi_1(M)^n$ is bi-orderable. So, by Lemma 1.3.7, we only need to show that a conjugation by an element of $PB_n(M)$ is an automorphism of $\mathbb{K}_n$ which preserves the order.

A conjugation by an element of $\mathbb{K}_n$ preserves the order, by definition of bi-order. Hence, it suffices to show the above claim for the conjugation of the generators of $\pi_1(M)^n$ by the pre–images under θ_n. A set of such pre-images is

$$\{a_{i,r};\ \ i = 1, \dots, n,\ \ r = 1, \dots, 2g\}.$$

Now, in [GM3, Lemma 3.15] it is shown that the following relations hold in $H_1(\mathbb{K}_n)$:

$$a_{i,r} f_{j,k,\gamma} a_{i,r}^{-1} \equiv \begin{cases} f_{j,k,\gamma}, & i \neq j, k; \\ f_{j,k,(\omega_r \gamma)}, & i = j; \\ f_{j,k,(\gamma \omega_r^{-1})}, & i = k. \end{cases}$$

We claim that the action of $a_{i,r}$ preserves the Magnus Order on each $\mathbb{F}_m$, $m = 2, \dots, n-1$, and hence, it preserves the order on $\mathbb{K}_n$. Clearly, the action of $a_{i,r}$ on $\mathbb{K}_n$ is the composition of an automorphism $\Psi_{i,r}$ which permutes the generators of each $\mathbb{F}_m$, with an automorphism $\Phi_{i,r}$ which is trivial on $H_1(\mathbb{K}_n)$. Therefore, by Theorems 1.3.20 and 1.3.21, it suffices to prove that the permutation induced by $\Psi_{i,r}$ on $\mathcal{F}_{j,n}$, for $j = 1, \dots, n-1$, preserves the defined order on $\mathcal{F}_{j,n}$. Let then $f_{j,k,\gamma}, f_{j,l,\delta} \in \mathcal{F}_{j,n}$, where $f_{j,k,\gamma} < f_{j,l,\delta}$.

Case 1. If $k < l$, then

$$\Psi_{i,r}(f_{j,k,\gamma}) = f_{j,k,\gamma'} < f_{j,l,\delta'} = \Psi_{i,r}(f_{j,l,\delta}),$$

where γ' and δ' are determined by the above relations.

Case 2. If $k = l$ and $\gamma <_{\pi_1} \delta$, then there are three possibilities. First, if $i \neq j, k$, one has

$$\Psi_{i,r}(f_{j,k,\gamma}) = f_{j,k,\delta} < f_{j,k,\delta} = \Psi_{i,r}(f_{j,k,\delta}).$$

If $i = j$, one has

$$\Psi_{i,r}(f_{j,k,\gamma}) = f_{j,k,(\omega_r\gamma)} < f_{j,k,(\omega_r\delta)} = \Psi_{i,r}(f_{j,k,\delta}),$$

since $\omega_r\gamma <_{\pi_1} \omega_r\delta$ ($<_{\pi_1}$ is a left-order). Finally, if $i = k$,

$$\Psi_{i,r}(f_{j,k,\gamma}) = f_{j,k,(\gamma\omega_r^{-1})} < f_{j,k,\delta\omega_r^{-1}} = \Psi_{i,r}(f_{j,k,\delta}),$$

since $\gamma\omega_r^{-1} <_{\pi_1} \delta\omega_r^{-1}$ ($<_{\pi_1}$ is a right-order).

Therefore, $PB_n(M)$ is a bi-orderable group. ” ∎

Chapter 2

Homotopy string links over surfaces

This chapter was extracted from Ekaterina Yurasovskaya's PhD thesis [Y]. To follow the existing literature it will used the notation set before in [Mil], [L] and [HL].

2.1 A presentation for the string links over surfaces

2.1.1 Definitions and notations

Definition 2.1.1. [Y, Chapter 3, p. 12] Let M be a closed (compact without boundary), connected and orientable surface of genus $g \geq 1$. Choose n points $\mathcal{P} = \{P_1, \ldots, P_n\}$ to lie in the interior of M. Let $I_1, \ldots, I_n$ be n copies of the unit interval $I = [0, 1]$. Let $\coprod_{i=1}^{n} I_i$ denote the disjoint union of these intervals.

A *string link σ on n strands over a surface M* is a smooth or piecewise linear proper imbedding:

$$\coprod_{i=1}^{n} I_i \to M \times I,$$

such that $\sigma_{|(Ii(0))} = (P_i, 0)$ and $\sigma_{|(Ii(1))} = (P_i, 1)$.

" When the surface M is understood, σ shall be called simply a *string link*. Informally, we can say that a string link is a pure braid with the monotonicity requirement relaxed, whose endpoints are still fixed and whose strands may knot upon themselves and on other strands. We orient the strands downwards from $M \times 0$ to $M \times 1$. Every pure braid is in fact a string link in itself. Figure 2.1 shows an example of a string link on 2 strands in the cylinder $\mathbb{D} \times I$, where $\mathbb{D}$ stands for the unit disk. An *ambient isotopy* (or simply isotopy) between string links σ and σ' is an orientation-preserving diffeomorphism of $M \times I$ which maps σ onto σ' while keeping the boundary $M \times \{0,1\}$ point-wise fixed and is isotopic to the identity, relative to $M \times \{0,1\}$. ”

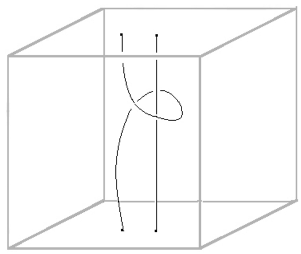


Figure 2.1: A string link σ.

<u>Definition</u> 2.1.2. [HL] *We say that two string links σ and σ' are* link-homotopic *if there is a homotopy of the strings in $M \times I$, fixing $M \times \{0,1\}$; and deforming σ to σ', such that the images of different strings remain disjoint during the deformation.*

“ During the course of deformation, each individual strand is allowed to pass through itself but not through other strands. As an example, we see that the string link of Figure 2.1 is not ambient isotopic with a braid, but is link-homotopic to the braid σ_1^{-2}. The following alternative definition of link-homotopy is more convenient for our purposes (see [Mil], [L], and [HL]): link-homotopy is an equivalence relation on string links, that is generated by a sequence of ambient isotopies of $M \times I$ fixing $M \times \{0,1\}$ and local crossing changes of arcs from the same strand of a string link. See Figure 2.2. ”

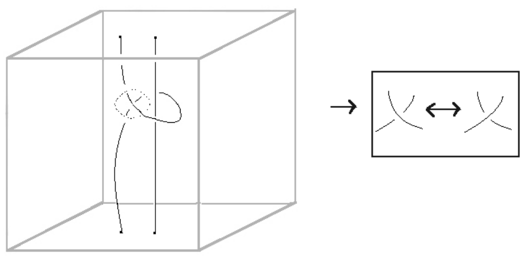


Figure 2.2: Crossing Change.

2.1.2 Homotopy string links form a group

The next result is due to Roger Fenn and Dale Rolfsen and shows that any string link is link-homotopic to a pure braid.

Theorem 2.1.3. *Every n-strand string link over a surface M is link-homotopic to a pure braid.*

Proof: See [Y, Theorem 3.7]. ∎

Because Theorem 2.1.3, we can use the term link-homotopy pure braids instead string links.

Define $H_n(M)$ the set of all pure braids in $PB_n(M)$ which are link-homotopic to the trivial braid. This set is called the set of the *link-homotopically trivial braids*. In symbols:

$$H_n(M) = \{\beta \in PB_n(M); \;\; \beta \sim 1\},$$

where $\sim$ denotes the link-homotopy equivalence relation.

Lemma 2.1.4. [Y, Lemma 3.8] *Link-homotopically trivial surface braids $H_n(M)$ is a normal subgroup of $PB_n(M)$.*

Proof: " The product of two link-homotopically trivial braids produces a link-homotopically trivial braid. If a braid β is link-homotopically trivial, then β^{-1} is also link-homotopically trivial. To see this, move β^{-1} by isotopy to be in the mirror-reflection position with respect to β and use the reflection of the link-homotopy. Note that intermediate stages of link-homotopy may be string links, rather than braids. If β is link-homotopically trivial, then clearly, for any $x \in PB_n(M)$, $x\beta x^{-1}$ is link-homotopically trivial, hence $H_n(M)$ is normal in $PB_n(M)$. " ∎

Let us denote by $\widehat{PB}_n(M)$ the set of link-homotopy classes of string links over the surface M, which it will called simply homotopy string links over surfaces.

Proposition 2.1.5. [Y, Proposition 3.9] *Under concatenation $\widehat{PB}_n(M)$ is a group isomorphic to the quotient of the pure braid group $PB_n(M)$ by the subgroup of link-homotopically trivial braids $H_n(M)$:*

$$\widehat{PB}_n(M) = \frac{PB_n(M)}{H_n(M)}.$$

Proof: " By Theorem 2.1.3, each string link is link-homotopic to a pure braid. Lemma 2.1.4 lets us express $\widehat{PB}_n(M)$ as a quotient of the pure braid group $PB_n(M)$:

$$\widehat{PB}_n(M) \simeq \frac{PB_n(M)}{H_n(M)}.$$

A quotient of a group by its normal subgroup is a group. Thus $\widehat{PB}_n(M)$ inherits from $PB_n(M)$:

operation - concatenation (product) of homotopy string links.

inverse - mirror reflection, up to link-homotopy equivalence. " ∎

Recall the notation

$$t^g = gtg^{-1},$$

for the conjugate of t by g, for elements t and g in a given group G.

Let us recall the generators of $PB_n(M)$ that are the same generators of $\widehat{PB}_n(M)$ given in Figures 1.11 and 1.12.

It was proved in [Y] that the presentation of $\widehat{PB}_n(M)$ has the same generators set as $PB_n(M)$ and the same relations with one more special relation, which is called *link-homotopy relation*, defined by the commutator

$$[t_{i,j}, t_{i,j}^h] = 1,$$

where $t_{i,j} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1}$ and $h \in \mathbb{F}(2g+n-i)$, $i = 1,\dots,n$. Also, we recall that $\mathbb{F}(2g+n-i)$ is the notation for the free group $\pi_1(S \setminus \mathcal{P}_{n-i}, P_i)$ generated by

$$\{\{a_{i,r}\} \cup \{t_{i,j}\};\ i+1 \leq j \leq n,\ 1 \leq r \leq 2g\}\},$$

where S is the surface M with a single point deleted and $\mathcal{P}_{n-i} = \{P_{i+1},\dots,P_n\}$. The representations of $a_{i,r}$ and $t_{i,j}$ are given in the Figures 1.9 and 1.12, respectively.

<u>**Remark**</u> **2.1.6.** *The following description of $\pi_1(S \setminus \mathcal{P}_{n-i}, P_i)$ was given in* [Y, Chapter 4, pp. 24–25, Figures 4.4–4.6].

"We can view $\pi_1(S \setminus \mathcal{P}_{n-i}, P_i)$ as a free subgroup of $\mathbb{PB}_n(S)$, which we shall denote by $\mathbb{F}(2g+n-i)$. The strands based at $\{P_1,\dots,P_{i-1},P_{i+1},\dots,P_n\}$ are trivial and go vertically down without winding. The P_i-based strand winds around the straight strands

based at $\{P_{i+1}, \ldots, P_n\}$ and through the walls of $M \times I$. Generators of the free subgroup $\mathbb{F}(2g+n-i)$ correspond precisely to those of $\pi_1(S \setminus \mathcal{P}_{n-i}, P_i)$."

Let us consider a particular case of the relation $[t_{1,2}, t_{1,2}^{a_{1,1}}] = 1$ in the Figure 2.3, which shows the explicit process of link-homotopy (for details, see [Y, Chapter 4, pp. 30–32]).

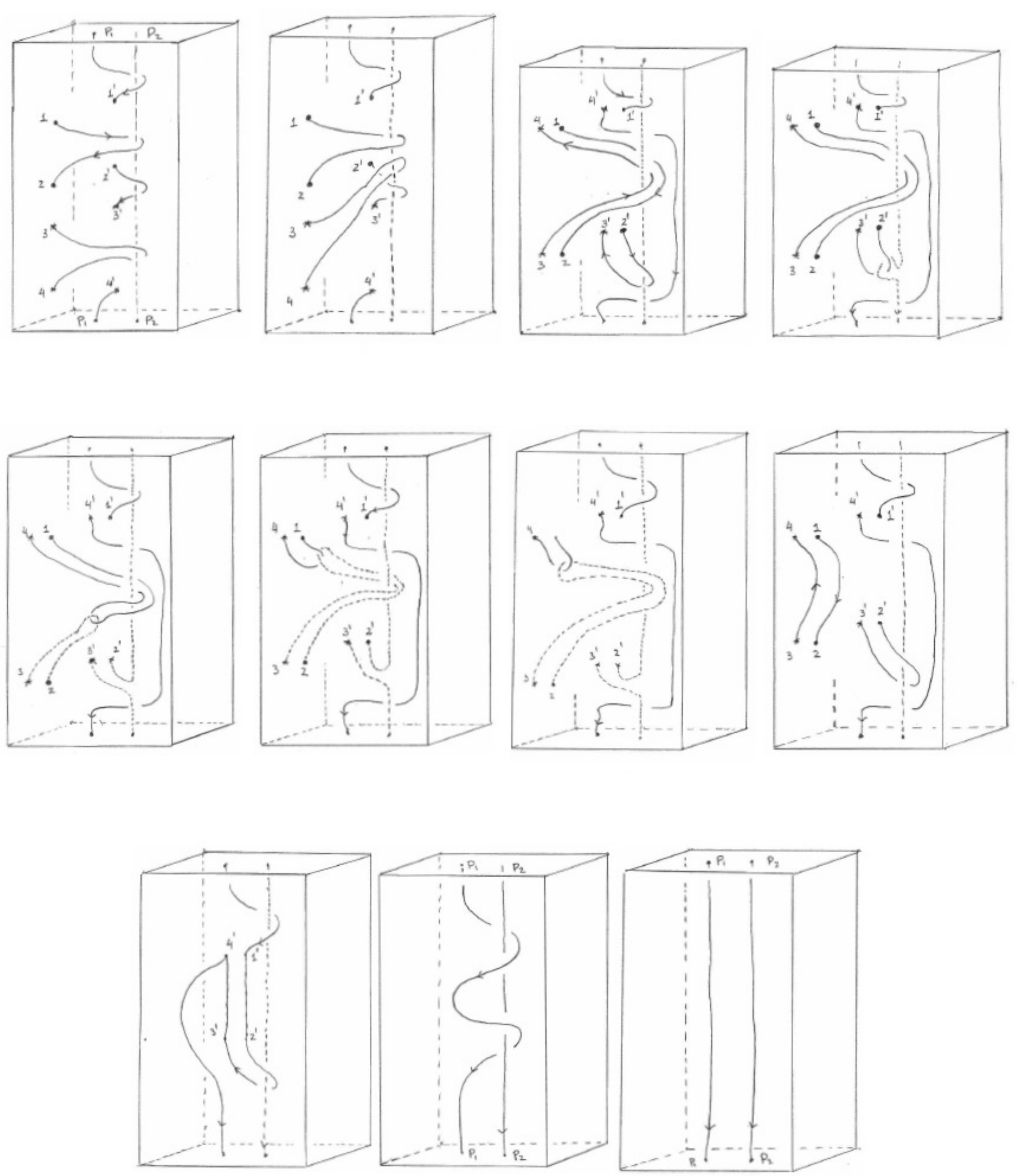


Figure 2.3: Link-homotopy process [Y].

" The points marked as $1, 2, 3, 4$ are located on the front face of the cylinder $\mathbb{T}_2 \times I$, where $\mathbb{T}_2$ denotes the 2-dimensional torus, and are identified with the points $1', 2', 3', 4'$ on the back face of $\mathbb{T}_2 \times I$. Note that in the process of link-homotopy the non-trivial strand of the braid $\tau = [t_{1,2}, t_{1,2}^{a_{1,1}}]$ may crosses itself. "

Theorem 2.1.7. [Y, Theorem 6.3] Let M be a closed, compact, connected and orientable surface of genus $g \geq 1$. The group of homotopy string links $\widehat{PB}_n(M)$ admits the presentation:

Generators: $\{a_{i,r};\ \ 1 \leq i \leq n;\ \ 1 \leq r \leq 2g\} \cup \{t_{j,k};\ \ 1 \leq j < k \leq n\}$.

Relations:

(LH1) $[t_{i,j}, t_{i,j}^h] = 1, \qquad h \in \mathbb{F}(2g + n - i);$

(PR1) $a_{n,1}^{-1} a_{n,2}^{-1} \cdots a_{n,2g}^{-1} a_{n,1} a_{n,2} \cdots a_{n,2g} = \prod_{i=1}^{n-1} T_{i,n-1}^{-1} T_{i,n};$

(PR2) $a_{i,r} A_{j,s} = A_{j,s} a_{i,r}, \qquad 1 \leq i < j \leq n,\ 1 \leq r \leq 2g;\ 1 \leq s \leq 2g-1;\ \ r \neq s;$

(PR3) $(a_{i,1} \cdots a_{i,r}) A_{j,r} (a_{i,r}^{-1} \cdots a_{i,1}^{-1}) A_{j,r}^{-1} = T_{i,j} T_{i,j-1}^{-1}, \qquad 1 \leq i < j \leq n,\ \ 1 \leq r \leq 2g-1;$

(PR4) $T_{i,j} T_{k,l} = T_{k,l} T_{i,j}, \qquad 1 \leq i < j < k < l \leq n$ or $1 \leq i < k < l \leq j \leq n;$

(PR5) $T_{k,l} T_{i,j} T_{k,l}^{-1} = T_{i,k-1} T_{i,k}^{-1} T_{i,j} T_{i,l}^{-1} T_{i,k} T_{i,k-1}^{-1} T_{i,l}, \qquad 1 \leq i < k \leq j < l \leq n;$

(PR6) $a_{i,r} T_{j,k} = T_{j,k} a_{i,r}, \qquad 1 \leq i < j < k \leq n$ or $1 \leq j < k < i \leq n,\ \ 1 \leq r \leq 2g;$

(PR7) $a_{i,r} (a_{j,2g}^{-1} \cdots a_{j,1}^{-1} T_{j,k} a_{j,2g} \cdots a_{j,1}) = (a_{j,2g}^{-1} \cdots a_{j,1}^{-1} T_{j,k} a_{j,2g} \cdots a_{j,1}) a_{i,r},\ 1 \leq j < i \leq k$
$\leq n;$

(PR8) $T_{j,n} = \left(\prod_{i=1}^{j-1} a_{i,2g}^{-1} \cdots a_{i,1}^{-1} T_{i,j-1} T_{i,j}^{-1} a_{i,1} \cdots a_{i,2g} \right) a_{j,1} \cdots a_{j,2g} a_{j,1}^{-1} \cdots a_{j,2g}^{-1};$

where $A_{j,s} = a_{j,1} \cdots a_{j,s-1} a_{j,s+1}^{-1} \cdots a_{j,2g}^{-1}$ and $T_{i,j} = t_{i,j} \cdots t_{i,i+1}$.

Remark 2.1.8. *Note that we can simplify the relation*

$$(LH1)[t_{i,j}^f, t_{i,j}^g] = 1, f, g \in \mathbb{F}(2g + n - i)$$

in [Y, Theorem 6.3, pp. 53–54] *by the relation*

$$(LH1)[t_{i,j}, t_{i,j}^h] = 1, h \in \mathbb{F}(2g + n - i)$$

of Theorem 2.1.7, i.e., these relations are equivalent. Indeed:

$$\begin{aligned} 1 = [t_{i,j}^f, t_{i,j}^g] &\Leftrightarrow f t_{i,j} f^{-1} g t_{i,j} g^{-1} f t_{i,j}^{-1} f^{-1} g t_{i,j}^{-1} g^{-1} = 1 \\ &\Leftrightarrow t_{i,j} f^{-1} g t_{i,j} g^{-1} f t_{i,j}^{-1} f^{-1} g t_{i,j}^{-1} g^{-1} f = 1 \\ &\Leftrightarrow t_{i,j} h t_{i,j} h^{-1} t_{i,j}^{-1} h t_{i,j}^{-1} h^{-1} f = 1, \\ &\Leftrightarrow [t_{i,j}, t_{i,j}^h] = 1. \end{aligned}$$

with $h = f^{-1} g \in \mathbb{F}(2g + n - i)$.

Let M be the surface on the conditions of Theorem 2.1.7. So, we have the following:

Corollary 2.1.9. [Y] $H_n(M)$ *is the smallest normal subgroup of* $PB_n(M)$ *generated by* $(LH1)$. *In symbols:*

$$H_n(M) = \langle \{[t_{i,j}, t_{i,j}^h],\ 1 \leq i < j \leq n,\ h \in \mathbb{F}(2g + n - i)\} \rangle^N,$$

where $\langle \quad \rangle^N$ *denotes the normal closure.*

Remark 2.1.10. *The presentation of the homotopy string links over a surface S obtained by deleting a single point of the surface M is the same presentation given in Theorem 2.1.7 with the exception of the relation* $(PR1)$.

2.2 Ordering reduced free groups

This section was extracted from [Y, Chapter 7].

“ In his search for invariants to classify links up to link-homotopy, John Milnor defined a certain quotient of the free group, which we shall call *reduced free group*, following [HL]. Milnor defined an expansion $\hat{\mu}$ of the reduced free group into a certain polynomial ring with integer coefficients, and showed $\hat{\mu}$ to be injective (see [Mil]). This $\hat{\mu}$ is the Magnus expansion for reduced free groups.

Let $\mathbb{F}$ denote a free group on the set of generators $\{x_1, \ldots, x_k\}$. ”

Definition 2.2.1. [Y, Definition 7.1, p.55] Take a quotient of $\mathbb{F}$ by relations that say each x_i commutes with its conjugates. The resulting group is the classical *reduced free group*

$\mathbb{RF}(k)$ as defined in [HL] and [Mil].

" Let J_1 denote the subgroup of $\mathbb{F}$ generated by commutators of conjugates of x_i:

$$J_1 = \langle [x_i', x_i'']; \;\; 1 \leq i \leq k \rangle.$$

It is possible to show that $\mathbb{RF}(k)$ has the presentation $\mathbb{F}/J_1$. There is an alternative presentation of $\mathbb{RF}(k)$, first given by Jerome Levine in [L], which shall be very useful in the construction of invariants. "

Definition 2.2.2. [Y, Definition 7.2, p. 55] The reduced free group $\mathbb{RF}(k)$ is obtained as a quotient of the free group $\mathbb{F}$ by relations which set to $1_{\mathbb{F}}$ every commutator C in $\{x_i\}$ with repeats.

" To make precise the meaning of a "commutator with repeats" we first recall the definitions of commutators and lower central series. We define the *commutators* in $\{x_i\}$ recursively, as follows:

1. Commutators of weight 1 are $x_1, \ldots, x_k$.

2. Commutators of weight n are words $[C_1, C_2]$, where C_1, C_2 are distinct commutators of weight $< n$, and $n = \text{wt } C_1 + \text{wt } C_2$.

Commutators of weight $\geq n$ generate a normal subgroup $\mathbb{F}_n$. The series

$$\mathbb{F} = \mathbb{F}_1 \supseteq \mathbb{F}_2 \supseteq \mathbb{F}_3 \supseteq \ldots$$

is called the lower central series of $\mathbb{F}$. It is a well-known fact that a free group $\mathbb{F}$ is residually nilpotent, which means that the intersection of all its lower central series subgroups is the identity: $\cap_{i=1}^{\infty} \mathbb{F}_i = \{1_{\mathbb{F}}\}$. "

" Following Levine in [L] we say that x_i occurs r times in a commutator C as follows:

1. If $C = x_j$, then $r = 1$, if $i = j$ and $r = 0$, otherwise.

2. If $C = [C_1, C_2]$, then $r = r_1 + r_2$, where x_i occurs r_1 times in C_1 and r_2 times in C_2.

We say that a commutator C has repeats if some x_i occurs at least twice in C. Finally, let J_2 denote the normal subgroup generated by commutators with repeats. Levine shows that $J_1 = J_2$. We see that Definitions 2.2.1 and 2.2.2 of the reduced free group $\mathbb{RF}(k)$ are indeed equivalent. From now on we denote $J_1 = J_2 = J$. To prove that $J_1 = J_2$ Levine uses P. Hall's construction of basic commutators in a free group, which we recall in the next section. "

2.2.1 Basic commutators

Definition 2.2.3. [F] *Basic commutators in a given set A are defined inductively as follows:*

1. *Each basic commutator C has a weight* wt C*, taking one of the values* $1, 2, 3 \ldots$.

2. *The basic commutators of weight* 1 *are the elements of* $\{x_i\}$*. A basic commutator of weight* > 1 *is a word of the form* $C = [C_1, C_2]$*, where* C_1, C_2 *are previously defined basic commutators and* $\mathrm{wt}\ C = \mathrm{wt}\ C_1 + \mathrm{wt}\ C_2$.

3. *Basic commutators are ordered so as to satisfy the following:*

   (a) *Basic commutators of the same weight are ordered arbitrarily.*

   (b) *If* $\mathrm{wt}\ C > \mathrm{wt}\ C'$*, then* $C > C'$.

4. (a) *If* $\mathrm{wt}\ C > 1$ *and* $C = [C_1, C_2]$*, then* $C_1 < C_2$.

   (b) *If* $\mathrm{wt}\ C > 2$ *and* $C = [C_1,\ [C_2, C_3]]$*, then* $C_1 \geq C_2$.

Note that the set A in Definition 2.2.3 could be a set with finite or infinite rank, for instance (not necessarily a free group of finite rank).

" The next theorem due to P. Hall illustrates the main purpose of basic commutators: to serve as a basis for quotients of free group by its lower central series subgroups. "

Theorem 2.2.4. [Mag] *There exists a set of basic commutators, for any m. Given a set of basic commutators $C_1 < C_2 < \ldots$, then every element of $\mathbb{F}/\mathbb{F}_q$ has a unique representative as a monomial*

$$C_1^{e_1} C_2^{e_2} \cdots C_n^{e_n},$$

where $C_1, C_2, \dots, C_n$ are all the basic commutators of weight $< q$.

2.2.2 Magnus expansion and basic commutators with repeats

From now let us use of the well-know expansion of a free group $\mathbb{F}$ generated by $\{x_1, \dots, x_k\}$ given in Section 1.3.2.

" Let $\Lambda = \mathbb{Z}\langle\langle X_1, \dots, X_n\rangle\rangle$ denote a ring of non-commuting power series in the variables

$$\{X_1, \dots, X_k\},$$

with integral coefficients. The Magnus expansion μ is an injective homomorphism of $\mathbb{F}$ into the group of units U of Λ, defined by $\mu : \mathbb{F} \to \Lambda$ given by

$$x_i \mapsto \mu(x_i) = 1 + X_i \text{ and } x_i^{-1} \mapsto \mu(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots .$$

Every element of U can be written uniquely as $\pm 1 + \rho + h$, where ρ is homogeneous of degree > 0, and every term of h has degree higher than the degree of ρ. We call ρ the *principal part*. The following lemma is proved by induction in [L]. "

Lemma 2.2.5. [L] *If C is a commutator of weight n in $\{x_i\}$ then:*

(i) *The principal part ρ of $\mu(C)$ is of degree n, and*

(ii) *Each variable X_i appears in every term of ρ with a total degree equal to the number of occurrences of x_i in C.*

Example 2.2.6. [Y, Example 7.6] *A simple computation shows that the Magnus expansion μ takes the commutator $[x_2,\ [x_1, x_2]]$ of weight 3 to an element of U:*

$$1 + 2x_2x_1x_2 - x_2^2x_1 - x_1x_2^2 + O(4),$$

where $O(4)$ denotes terms of order 4 and higher. Note that if we assume the ordering $x_1 < x_2 < \cdots < x_k$, then $[x_2,\ [x_1, x_2]]$ is a basic commutator.

<u>**Proposition**</u> **2.2.7.** [L] *Let $\{C_i\}$ be a set of basic commutators in $\{x_i\}$. Let us suppose that $C_1 < C_2 < \cdots < C_n$ are those of weight $\leq q$. If an element of J has representation $C_1^{e_1} \cdots C_n^{e_n}$ modulo $\mathbb{F}_{q+1}$, then whenever $e_i \neq 0$, C_i must be a commutator with repeats.*

Proof: See [Y, Proposition 7.7]. ∎

<u>**Corollary**</u> **2.2.8.** [L] *Let $\{C_i\}$ be a set of basic commutators in $\{x_i\}$. Let us suppose that $C_1 < C_2 < \cdots < C_n$ are those of weight $\leq k$. Let $f \in \mathbb{F}$ have representation $C_1^{e_1} \cdots C_n^{e_n}$ modulo $\mathbb{F}_{k+1}$, where k is the rank of the free group $\mathbb{F}$. Then $f \in J$, if and only, if each C_i is a commutator with repeats.*

Proof: See [Y, Corollary 7.8]. ∎

2.2.3 Injective expansion of $\mathbb{RF}(k)$

“ We now return to the Magnus expansion μ of a free group $\mathbb{F}$ of rank k into the group of units U of the power series ring Λ in non-commuting variables $\{X_1, \ldots, X_k\}$. Let $\mathcal{R}$ denote the subset of Λ generated by monomials with repeats of a variable. An example of an element of $\mathcal{R}$ is $4X_1X_2X_1 + X_3^3X_1$. $\mathcal{R}$ is an additive subgroup of Λ and $\mathcal{R}$ is closed under right and left multiplication by elements of Λ. It follows that $\mathcal{R}$ is a two-sided ideal of Λ. Let $1 + \mathcal{R}$ denote the set of elements of the form “1 + monomials with repeat of a variable”. ”

<u>**Proposition**</u> **2.2.9.** [Y, Proposition 7.10] *If each term of Magnus expansion $\mu(f)$ has repeats then f is an element of J, i.e.,*

$$\textit{if } \mu(f) \in 1 + \mathcal{R}, \ \ \textit{then } f \in J.$$

Let us consider an expansion of $\mathbb{RF}(k)$ given in [Y, Section 7.3, pp. 57–58], as follows.

“ Let $\hat{\Lambda}$ denote the quotient ring of Λ by the two-sided ideal $\mathcal{R}$, generated by monomials with repeats. In words we can describe Λ as a polynomial ring in non-commutative variables $\{X_1, \ldots, X_k\}$, such that any monomial with repeat of variable X_i, for some $1 \leq i \leq k$, is set to zero.

Define the reduced Magnus expansion $\widehat{\mu} : \mathbb{RF}(k) \to \hat{\Lambda}$ given by

$$\widehat{\mu}(\widehat{x_i}) = 1 + X_i \text{ and } \widehat{\mu}(\widehat{x_i}^{-1}) = 1 - X_i,$$

where $\widehat{x_i}$, for $1 \leq i \leq n$, are the generators of $\mathbb{RF}(k)$. "

Theorem 2.2.10. [Y, Theorem 7.11] *The reduced Magnus expansion $\widehat{\mu} : \mathbb{RF}(k) \to \hat{\Lambda}$ is injective.*

Proof: " $\mathbb{RF}(k)$ is defined as a quotient group of free group $\mathbb{F}$ by J. Consider the following diagram with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & J \arrow[r] & \mathbb{F} \arrow[r, "p"] \arrow[d, "\mu"] & \mathbb{RF}(k) \arrow[r] \arrow[d, "\widehat{\mu}"] & 1 \\ 1 \arrow[r] & \mathcal{R} \arrow[r] & \Lambda \arrow[r, "\pi"] & \hat{\Lambda} \arrow[r] & 1 \end{tikzcd} \tag{2.2.1}$$

Diagram (2.2.1) is commutative. In fact, given a generator x_i of $\mathbb{F}$:

$$\begin{aligned} \widehat{\mu} \circ p(x_i) &= \widehat{\mu}(\widehat{x_i}) = 1 + X_i \\ \pi \circ \mu(x_i) &= \pi(1 + X_i) = 1 + X_i \\ \widehat{\mu} \circ p(x_i^{-1}) &= \widehat{\mu}(\widehat{x_i}^{-1}) = 1 - X_i \\ \pi \circ \mu(x_i^{-1}) &= \pi(1 - X_i + X_i^2 - \cdots) = 1 - X_i. \end{aligned}$$

The rest of the proof follows from Diagram 2.2.1. Let $\hat{f} \in \mathbb{RF}(k)$ and suppose that $\widehat{\mu}(\hat{f}) = 1_{\hat{\Lambda}}$. Let $f \in \mathbb{F}$ be a pre-image of $\hat{f}$. By commutativity of the diagram and by definition of $\hat{\Lambda} = \frac{\Lambda}{\mathcal{R}}$ we see that $\mu(f)$ is an element of $1 + \mathcal{R}$. Proposition 2.2.9 implies that f is an element of $J \subset \mathbb{F}$, hence $\hat{f} = p(f) = 1_{\mathbb{RF}(k)}$. Therefore, $\widehat{\mu}$ is injective. " ■

Definition 2.2.11. [Y, Definition 8.11, p. 68] Let us define an ordering on $\hat{\Lambda}$, which we shall call the reduced Magnus ordering. Let f and g be polynomials in $\hat{\Lambda}$. We first arrange the monomials within f and g by degree. Let us now assume an alphabetical order on the variables $X_1, \ldots, X_k$, for example, X_1 is the first letter, X_2 is the second letter, etc.

Within each degree we arrange the monomials lexicographically. We now compare f and g degree by degree, term by term. We find the first term at which f and g differ and look at its leading coefficients ϵ_f and ϵ_g. We declare that $f > g$, if $\epsilon_f > \epsilon_g$.

Example 2.2.12. [Y, Example 8.12] *Let $f = 1 + X_2$ and $g = 1 + X_1 - X_2$, then:*

$$f = 1 + 0X_1 + X_2 < 1 + X_1 - X_2 = g.$$

Note that reduced Magnus ordering does not define a left-order on $\hat{\Lambda}$. By definition, order must be invariant under left multiplication, however

$$(-1)(1 + X_1) = -1 - X_1 < 0, \quad \text{while} \quad 1 + X_1 > 0.$$

Let $\mathcal{H}$ denote the set of elements of $\hat{\Lambda}$ of the form (1 + higher order terms). It is easy to see that $\mathcal{H}$ is a multiplicative subgroup of $\hat{\Lambda}$. Consider an element g of $\mathcal{H}$ of the form $g = 1 + G$, where G consists of terms of non-zero degree. Then, the inverse element g^{-1} is given by

$$g^{-1} = 1 - G + G^2 - G^3 + \cdots + (-1)^k G^k,$$

because:

$$\begin{aligned} gg^{-1} &= (1 + G)(1 - G + G^2 - G^3 + \cdots + (-1)^k G^k) \\ &= 1 + (-1)^{k+1} G^{k+1} \\ &= 1. \end{aligned}$$

" Note that every monomial of total degree greater than n is bound to have repeats of some variable and hence will be set to zero in $\hat{\Lambda}$. "

Proposition 2.2.13. [Y, Proposition 8.13] *The reduced Magnus ordering induces a bi-ordering on the subgroup $\mathcal{H}$, and hence on the classical reduced free group $\mathbb{RF}(k)$.*

Proposition 2.2.14. [Y, Theorem 8.14] *The reduced Magnus ordering is preserved under any automorphism of $\mathbb{RF}(k)$ that induces the identity on the abelianization of $\mathbb{RF}(k)$ which* *we denote by*

$$\mathbb{RF}(k)_{ab} = \frac{\mathbb{RF}(k)}{[\mathbb{RF}(k), \mathbb{RF}(k)]}.$$

Chapter

3

Ordering homotopy string links over surfaces

Our aim in this chapter is to prove that the homotopy string links group on a closed (compact without boundary), connected and orientable surface M of genus $g \geq 1$, namely, $\widehat{PB}_n(M)$, is bi-orderable. Here, we can use the term *link-homotopy pure braid groups on surfaces* instead of homotopy string links over surfaces since each string-link is homotopic to a pure braid, which was proved by R. A. Fenn and D. Rolfsen (see [Y], Theorem 3.7, Chapter 3).

Let us observe that since by Proposition 2.1.5, $\widehat{PB}_n(M) \simeq \frac{PB_n(M)}{H_n(M)}$, when we refer to a string link in $\widehat{PB}_n(M)$ we will denote it by $\hat{\beta} = [\beta] = [(\beta_1, \ldots, \beta_n)]$ and when we refer to a pure braid in $PB_n(M)$ we will just denote it by $\beta = (\beta_1, \ldots, \beta_n)$.

3.1 Left-ordering $\widehat{PB}_n(S)$ and $\widehat{PB}_n(M)$

Let us observe that in the case of pure braids, [GM2] made a special construction as we presented in Section (1.3.4). It was constructed a special free group $\mathbb{F}_n$ of infinite rank, whose generators come from the conjugation of generators of $PB_n(\mathbb{D})$, where $\mathbb{D}$ denotes the unit disk, by elements in the fundamental group $\pi_1(M)$, allowing to prove that the action by conjugation of $\mathbb{K}_{n-1}$ on the abelianization of $\mathbb{F}_n$ is trivial in the split exact sequence:

$$1 \longrightarrow \mathbb{F}_n \longrightarrow \mathbb{K}_n \longrightarrow \mathbb{K}_{n-1} \longrightarrow 1.$$

This strategy avoids the problem of analyzing the conjugation by elements of the form $a_{i,r} \in PB_{n-1}(M)$ on the elements of the group $\mathbb{F}(2g+n-i)$.

We will use the same strategy for the case of the string links.

3.1.1 $\widehat{PB}_n(S)$ is left-orderable

Let M a closed, connected and orientable surface of $g \geq 1$. Consider S a surface obtained by deleting one single point x_0 of the surface M. From [Y, Corollary 4.7]:

$$\widehat{PB}_n(S) \simeq \widehat{\mathbb{F}}(2g+n-1) \rtimes (\widehat{\mathbb{F}}(2g+n-2) \rtimes (\cdots (\widehat{\mathbb{F}}(2g+1) \rtimes \mathbb{F}(2g)) \cdots)),$$

where

$$\widehat{\mathbb{F}}(2g+n-i) = \frac{\mathbb{F}(2g+n-i)}{H_n(S) \cap \mathbb{F}(2g+n-i)}, \quad \text{(see [Y, Proposition 4.5])},$$

with $\mathbb{F}(2g+n-i)$ the free group on $2g+n-i$ generators $\pi_1(S \setminus \mathcal{P}_{n-i}, P_i))$ and $\mathcal{P}_{n-i} = \{P_{i+1}, \ldots, P_n\}$, for $i = 1, \ldots, n-1$ and $\mathbb{F}(2g)$ is the free group $\pi_1(S, P_n)$ on $2g$ generators.

Recall the split exact sequence proved in [GG, Theorem 6]

$$1 \longrightarrow \mathbb{F}(2g+n-1) \longrightarrow PB_n(S) \xrightarrow{\varrho} PB_{n-1}(S) \longrightarrow 1\ ,$$

and let us consider the section of the homomorphism ϱ, namely, $\sigma : PB_{n-1}(S) \to PB_n(S)$ given by $\sigma(\beta_2, \ldots, \beta_n) = (1, \beta_2, \ldots, \beta_n)$. Geometrically, this section means to add a trivial strand that goes parallel to $x_0 \times I$ in a braid with $(n-1)$ strands. The result will be a braid with n strands (see [Y, pp.24]).

Moreover, from [Y, Corollary 4.8] there is a short split exact sequence of groups

$$1 \longrightarrow \widehat{\mathbb{F}}(2g+n-1) \longrightarrow \widehat{PB}_n(S) \xrightarrow{\hat{\varrho}} \widehat{PB}_{n-1}(S) \longrightarrow 1\ , \tag{3.1.1}$$

where $\hat{\varrho}$ is the homomorphism defined by $\hat{\varrho}([(\beta_1, \ldots, \beta_n)]) = [(\beta_2, \ldots, \beta_n)]$ and induced by the forgetting homomorphism $\varrho : PB_n(S) \to PB_{n-1}(S)$, given by $\varrho(\beta_1, \ldots, \beta_n) = (\beta_2, \ldots, \beta_n)$. Thus, there is a section $\hat{\sigma} : \widehat{PB}_{n-1}(S) \to \widehat{PB}_n(S)$ given by $\hat{\sigma}([(\beta_2, \ldots, \beta_n)]) = [(1, \beta_2, \ldots, \beta_n)]$.

Proposition 3.1.1. *Let S be a surface obtained by deleting a single point x_0 of a closed, connected and orientable surface of genus $g \geq 1$. Thus, $\widehat{PB}_n(S)$ is left-orderable, for all $n \geq 1$.*

Proof: The proof is given by induction on n. For $n = 1$, the sequence (3.1.1) gives us

$$\widehat{PB}_1(S) \simeq \mathbb{F}(2g),$$

where $\mathbb{F}(2g)$ is the free group $\pi_1(S, P_n)$ on $2g$ generators. Thus, $\widehat{PB}_1(S)$ is left-orderable. Let us suppose that $\widehat{PB}_{n-1}(S)$ is left-orderable. Now, the groups $\widehat{\mathbb{F}}(2g + n - 1)$ and $\widehat{PB}_{n-1}(S)$ in the sequence (3.1.1) are left-orderable. By Lemma 1.3.7, $\widehat{PB}_n(S)$ is left-orderable. ■

3.1.2 The structure of $\widehat{\mathbb{K}}_n$

For our purposes, we define here an important structure: the group $\widehat{\mathbb{K}}_n$. This group is the quotient by link-homotopy on the structure of $\mathbb{K}_n$. We recall such structure to construct our structure. Later, we give a characterization for it, which will be very useful to prove that $\widehat{PB}_n(M)$ is bi-orderable, extending the result proved by González–Meneses in [GM2, Theorem 1.6].

Let M be a closed, connected and orientable surface of genus $g \geq 1$, as previously. Given a string link $\hat{\sigma} = [(\sigma_1, \ldots, \sigma_n)]$ over M in $\widehat{PB}_n(M)$, we can consider, for all $i = 1, \ldots, n$ the loop μ_i in M constructed as follows: take the i-th string σ_i (which is a path in $M \times [0, 1]$) and call as μ_i its projection over the first coordinate (i.e., over M). Since $\hat{\sigma} \in \widehat{PB}_n(M)$, μ_i is a loop in M based at P_i, for all $i = 1, \ldots, n$ which represents an element of $\pi_1(M, P_i) \simeq \pi_1(M)$. This defines an epimorphism $\hat{\theta}_n : \widehat{PB}_n(M) \to \pi_1(M)^n$ which sends $\hat{\sigma} = [(\sigma_1, \ldots, \sigma_n)]$ to $\mu = (\mu_1, \ldots, \mu_n)$.

Lemma 3.1.2. *$\hat{\theta}_n$ is a well defined surjective homomorphism.*

Proof:

$$\begin{tikzcd} PB_n(M) \arrow[r, "\theta_n"] \arrow[d, "p_n"'] & \pi_1(M)^n \\ \widehat{PB}_n(M) \arrow[ur, dotted, "\exists\hat{\theta}_n"'] & \end{tikzcd}$$

Recall the surjective homomorphism $\theta_n : PB_n(M) \to \pi_1(M)^n$ defined in [B] by the following: take the i-th string β_i of the pure braid β (which is a path in $M \times [0,1]$) and call as μ_i its projection over the first coordinate (i.e., over M). Since $\beta \in PB_n(M)$, μ_i is a loop in M based at P_i, for all $i = 1, \ldots, n$ which represents an element of $\pi_1(M, P_i) \simeq \pi_1(M)$. Let p_n be the projection homomorphism that maps a pure braid $\beta \in PB_n(M)$ to its link-homotopy class $\hat{\beta} = [\beta] \in \widehat{PB}_n(M)$.

Now, let $\hat{\sigma} = [\sigma]$ and $\hat{\beta} = [\beta]$ be two string links such that $\hat{\sigma} = \hat{\beta}$. By definition, β and σ are link-homotopic, i.e., there is a homotopy of the strings in $M \times I$, fixing $M \times \{0,1\}$ and deforming σ to β, such that the images of different strings remain disjoint during the deformation. In symbols, there is, for all $i = 1, \ldots, n$, continuous maps $H_i : I \times I \to M \times I$ such that $H_i(t,0) = \sigma_i(t)$, $H_i(t,1) = \beta_i(t)$, for all $t \in I$ and $H_i(0,s) = P_i$, $H_i(1,s) = P_i$, for all $s \in I$. Let p the projection to the first coordinate given by $p(m,t) = m$, for all $(m,t) \in M \times I$.

Suppose that $\hat{\theta}_n(\hat{\sigma}) = (\mu_1, \ldots, \mu_n)$ and $\hat{\theta}_n(\hat{\beta}) = (\epsilon_1, \ldots, \epsilon_n)$. We need to show that there is an homotopy in the fundamental group such that $\mu_i \simeq \epsilon_i$, for all $i = 1, \ldots, n$. Indeed, define $\mathcal{H}_i : I \times I \to M$ by the composition $\mathcal{H}_i = p \circ H_i$, for all $i = 1, \ldots, n$. We have $\mathcal{H}_i$ is continuous, since it is a composition of continuous maps, for all $i = 1, \ldots, n$. Moreover:

$$\begin{aligned} \mathcal{H}_i(t,0) &= p \circ H_i(t,0) = \mu_i \text{ and} \\ \mathcal{H}_i(t,1) &= p \circ H_i(t,1) = \epsilon_i, \forall t \in I. \\ \mathcal{H}_i(0,s) &= p \circ H_i(0,s) = P_i \text{ and} \\ \mathcal{H}_i(1,s) &= p \circ H_i(1,s) = P_i, \forall s \in I. \end{aligned}$$

Therefore, there is an homotopy in the fundamental group, for each $i = 1, \ldots, n$ as required and then $\hat{\theta}_n$ is well defined.

Now, given $\mu = (\mu_1, \ldots, \mu_n) \in \pi_1(M)^n$, since θ_n is surjective, there is $\beta = (\beta_1, \ldots, \beta_n) \in PB_n(M)$ such that $\theta_n(\beta) = \mu$. Thus,

$$\mu = \theta_n(\beta) = \hat{\theta}_n \circ p_n(\beta) = \hat{\theta}_n([\beta]) = \hat{\theta}_n(\hat{\beta}).$$

By this way, we have $\hat{\theta}_n$ is surjective. Finally, let $\hat{\sigma}, \hat{\beta} \in \widehat{PB}_n(M)$ with σ and β their representatives, respectively. Thus,

$$\begin{aligned} \hat{\theta}_n(\hat{\sigma}\hat{\beta}) &= \hat{\theta}_n(\widehat{\sigma\beta}) = \hat{\theta}_n \circ p_n(\sigma\beta) = \theta_n(\sigma\beta) = \theta_n(\sigma)\theta_n(\beta) \\ &= (\hat{\theta}_n \circ p_n(\sigma))(\hat{\theta}_n \circ p_n(\beta)) = \hat{\theta}_n(\hat{\sigma})\hat{\theta}_n(\hat{\beta}). \end{aligned}$$

Therefore, $\hat{\theta}_n$ is a well defined surjective homomorphism as required. ∎

Remark 3.1.3. *We can represent the homomorphism defined in Lemma 3.1.2 geometrically: indeed, since $\hat{\theta}_n$ is a well defined homomorphism, we have*

$$\theta_n(H_n(M)) \subseteq \{1 = (1, \ldots, 1)\},$$

where $\{1 = (1, \ldots, 1)\}$ denotes the trivial subgroup of $\pi_1(M)^n$. This is equivalent to $H_n(M) \subseteq \ker \theta_n$. In fact, since by Corollary 2.1.9

$$H_n(M) = \langle \{[t_{i,j}, t_{i,j}^h], h \in \mathbb{F}(2g + n - i), 1 \leq i < j \leq n\} \rangle^N,$$

let us observe that the generators of $H_n(M)$, which have the form $\alpha[t_{i,j}, t_{i,j}^h]\alpha^{-1}$, with $\alpha \in PB_n(M)$, are contained in the kernel of θ_n. As a geometric example, let us consider $[t_{i,j}, t_{i,j}^h] = t_{i,j}ht_{i,j}h^{-1}t_{i,j}^{-1}ht_{i,j}^{-1}h^{-1}$. We have the part of the word braid that contains $t_{i,j}$ and its inverses are loops homotopic to the base point in $\pi_1(M)^n$. Its products with the braids which cross the walls on the surface form a loop homotopic to the base point in $\pi_1(M)^n$ as well. The sketch of the word $[t_{i,j}, t_{i,j}^h]$ is given by the particular case $[t_{1,2}, t_{1,2}^{a_{1,1}}]$ in the following picture (for the case when r is even, the other case is analogous). The left side of the figure represents $[t_{1,2}, t_{1,2}^{a_{1,1}}]$ viewed in $PB_2(\mathbb{T})$ and the right side viewed in $\pi_1(\mathbb{T})$, where $\mathbb{T}$ denotes the 2-dimensional torus.

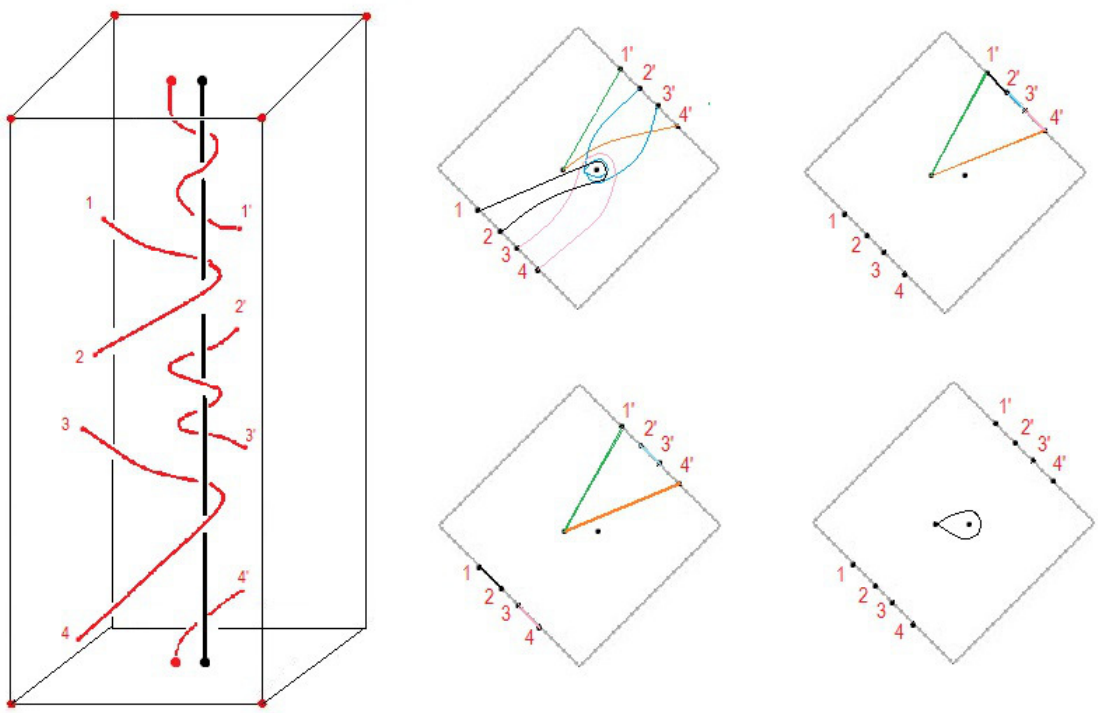


Figure 3.1: $[t_{1,2}, t_{1,2}^{a_{1,1}}]$ in $PB_2(\mathbb{T})$.

<u>Remark</u> 3.1.4. *Let us consider the kernel of the homomorphism defined in Lemma 3.1.2, which we will denote by*

$$\widehat{\mathbb{K}}_n = \ker \hat{\theta}_n = \{\hat{\sigma} = [(\sigma_1, \ldots, \sigma_n)] \in \widehat{PB}_n(M); \hat{\theta}_n(\hat{\sigma}) = (1, \ldots, 1)\},$$

where 1 *denotes the trivial loop in* $\pi_1(M)$*. One has the exact sequence:*

$$1 \longrightarrow \widehat{\mathbb{K}}_n \xrightarrow{\subseteq} \widehat{PB}_n(M) \xrightarrow{\hat{\theta}_n} \pi_1(M)^n \longrightarrow 1 \tag{3.1.2}$$

which is induced by the sequence obtained by [GM2]

$$1 \longrightarrow \mathbb{K}_n \xrightarrow{\subseteq} PB_n(M) \xrightarrow{\theta_n} \pi_1(M)^n \longrightarrow 1\,, \tag{3.1.3}$$

defined for the surjective homomorphism θ_n*, where* $\mathbb{K}_n = \ker \theta_n$ *(for details, see Section 1.3.4).*

Lemma 3.1.5. $\widehat{\mathbb{K}}_n = \frac{\mathbb{K}_n}{H_n(M)}$, *where $H_n(M)$ is the normal subgroup of $PB_n(M)$ of the link-homotopically trivial braids.*

Proof:

(i) First, let us observe that $H_n(M) \subseteq \mathbb{K}_n = \ker(\theta_n)$. A simple braid diagram gives us that $H_n(M)$ is a normal subgroup of $\mathbb{K}_n$. Therefore, we have $\frac{\mathbb{K}_n}{H_n(M)}$ well defined.

(ii) Now, we show that $\ker \hat{\theta}_n = \frac{\mathbb{K}_n}{H_n(M)}$. Indeed, if $\hat{\beta} = [\beta] \in \ker \hat{\theta}_n$, then $\hat{\beta} \in \widehat{PB}_n(M)$ and $\hat{\theta}_n(\hat{\beta}) = 1$. So, we have $1 = \hat{\theta}_n(\hat{\beta}) = \hat{\theta}_n(p_n(\beta)) = \theta_n(\beta)$, i.e., $\beta \in \ker \theta_n$. Therefore, $\ker \hat{\theta}_n \subseteq \frac{\mathbb{K}_n}{H_n(M)}$.

Conversely, if $\hat{\beta} \in \frac{\mathbb{K}_n}{H_n(M)}$, then $\hat{\theta}_n(\hat{\beta}) = \hat{\theta}_n(p_n(\beta)) = \theta_n(\beta) = 1$, i.e., $\hat{\beta} \in \ker \hat{\theta}_n$. ∎

Lemma 3.1.6. *The following diagram commutes:*

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{K}_n \arrow[r, "i_1"] \arrow[d, "q_1"'] & PB_n(M) \arrow[r, "\theta_n"] \arrow[d, "q_2"'] & \pi_1(M)^n \arrow[r] \arrow[d, "id"'] & 1 \\
1 \arrow[r] & \widehat{\mathbb{K}}_n \arrow[r, "i_2"] & \widehat{PB}_n(M) \arrow[r, "\hat{\theta}_n"] & \pi_1(M)^n \arrow[r] & 1
\end{tikzcd}$$

where q_1, q_2 are the respective projections, i_1, i_2 are the respective inclusions and id is the identity.

Proof:

(i) We claim that $q_2 \circ i_1 = i_2 \circ q_1$. Indeed, $\forall \beta \in \mathbb{K}_n$,

$$q_2 \circ i_1(\beta) = q_2 \circ i_1(\beta_1, \ldots, \beta_n) = q_2(\beta_1, \ldots, \beta_n) = q_2(\beta) = [\beta],$$

and $i_2 \circ q_1(\beta) = i_2([\beta]) = [\beta]$. Therefore, $q_2 \circ i_1 = i_2 \circ q_1$.

(ii) We claim that $id \circ \theta_n = \hat{\theta}_n \circ q_2$. Indeed, $\forall \beta \in PB_n(M)$,

$$id \circ \theta_n(\beta) = id \circ \theta_n(\beta_1, \ldots, \beta_n) = id(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_n) = \mu,$$

and $\hat{\theta}_n \circ q_2(\beta) = \hat{\theta}_n([\beta]) = (\mu_1, \ldots, \mu_n) = \mu$. Therefore, $id \circ \theta_n = \hat{\theta}_n \circ q_2$. ∎

3.1.3 A special construction: $\widehat{PB}_n(M)$ is left-orderable

In this section we make a construction that will be useful to help us to guarantee that $\widehat{PB}_n(M)$ is torsion free.

Let $\phi_2 : PB_n(M) \to \pi_1(M, P_2)$ be the homomorphism defined by

$$\phi_2((\beta_1, \beta_2, \ldots, \beta_n)) = \mu_2,$$

where μ_2 is the strand based on P_2 viewed as a loop in $\pi_1(M, P_2)$. The kernel of this homomorphism is $PB_{n-1}(M \setminus \{P_2\})$ (this homomorphism was constructed by Birman and details can be found in [B]). Moreover, by Gonzalez-Meneses' construction in the proof of [GM3, Lemma 2.6], $\mathbb{K}_n$ lies in the kernel of ϕ_2 as a subgroup, namely,

$$\mathbb{K}_n \subseteq PB_{n-1}(M \setminus \{P_2\}).$$

We will show that $\widehat{\mathbb{K}}_n \subseteq \widehat{PB}_{n-1}(M \setminus \{P_2\})$, as follows.

Proposition 3.1.7. *Let M be a closed, connected and orientable surface of genus $g \geq 1$. Define $\widehat{\phi}_2 : \widehat{PB}_n(M) \to \pi_1(M, P_2)$ given by $\widehat{\phi}_2([(\beta_1, \beta_2, \ldots, \beta_n)]) = \mu_2$, where μ_2 is the strand β_2 viewed as a loop in $\pi_1(M, P_2)$. The following conditions hold:*

(i) *$\widehat{\phi}_2$ is the homomorphism induced by ϕ_2.*

(ii) $\ker \widehat{\phi}_2 = \widehat{PB}_{n-1}(M \setminus \{P_2\})$.

(iii) *$\widehat{\mathbb{K}}_n$ is a subgroup of $\widehat{PB}_{n-1}(M \setminus \{P_2\})$.*

Proof:

(i) We need to show that $\phi_2(H_n(M)) \subseteq \{1\}$, where $\{1\}$ denotes the identity of the fundamental group $\pi_1(M, P_2)$. Let $\beta \in H_n(M)$. Since β is link-homotopically trivial, we have that all strands are deformed as straight lines in a link-homotopic process. Thus, μ_2 is a trivial loop in $\pi_1(M, P_2)$. By this way, $\widehat{\phi}_2$ is a well defined homomorphism induced by ϕ_2 and we have the following diagram, where p_n is the natural projection.

$$\begin{tikzcd} PB_n(M) \arrow[r, "\phi_2"] \arrow[d, "p_n"'] & \pi_1(M, P_2) \\ \widehat{PB}_n(M) \arrow[ur, dotted, "\exists\widehat{\phi}_2"'] & \end{tikzcd}$$

(ii) Let us show that $\ker\widehat{\phi}_2 = \widehat{\ker\phi_2}$. First, is clear that $H_n(M)$ is a normal subgroup of $\ker\phi_2$. Now, if $\hat{\beta} \in \ker\widehat{\phi}_2$, then $\widehat{\phi}_2(\hat{\beta}) = 1$. On the other hand, there is a $\beta \in PB_n(M)$ such that $p_n(\beta) = \hat{\beta}$. Hence we have $\widehat{\phi}_2 \circ p_n(\beta) = 1$, wich implies $\phi_2(\beta) = 1$, i.e., $\beta \in \ker\phi_2$. Therefore, $\widehat{\beta} = \beta H_n(M)$, with $\beta \in \ker\phi_2$, i.e., $\hat{\beta} \in \widehat{\ker\phi_2}$. Conversely, let $\hat{\beta} \in \widehat{\ker\phi_2}$. Thus, $\hat{\beta} = \beta H_n(M)$, with $\beta \in \ker\phi_2$. So, $\phi_2(\beta) = 1$. Hence we have: $\widehat{\phi}_2(\hat{\beta}) = \widehat{\phi}_2 \circ p_n(\beta) = \phi_2(\beta) = 1$.

(iii) Now, let $\hat{\beta} \in \widehat{\mathbb{K}}_n$. Thus, $\hat{\beta} = \beta H_n(M)$, for some $\beta \in \mathbb{K}_n$. Since we have that $\mathbb{K}_n \subseteq PB_{n-1}(M \setminus \{P_2\})$, thus $\beta \in PB_{n-1}(M \setminus \{P_2\})$. Therefore, $\hat{\beta} \in \widehat{PB}_{n-1}(M \setminus \{P_2\})$. By this way, we have $\widehat{\mathbb{K}}_n \subseteq \widehat{PB}_{n-1}(M \setminus \{P_2\})$ as a subgroup. ∎

Proposition 3.1.8. *Let M be a closed, connected and orientable surface of genus $g \geq 1$. Thus, $\widehat{\mathbb{K}}_n$ is left-orderable, for all $n \geq 1$.*

Proof: From Proposition 3.1.7, $\widehat{\mathbb{K}}_n$ is contained (as subgroup) in $\widehat{PB}_{n-1}(S)$, where S denotes the surface obtained by deleting a single point of the surface M. By Proposition 3.1.1, $\widehat{PB}_{n-1}(S)$ is left-orderable, and therefore, so $\widehat{\mathbb{K}}_n$. ∎

Proposition 3.1.9. *Let M be a closed, connected and orientable surface of genus $g \geq 1$. Thus, $\widehat{PB}_n(M)$ is left-orderable, for all $n \geq 1$.*

Proof: In the exact sequence (3.1.2), the groups $\widehat{\mathbb{K}}_n$ and $\pi_1(M)^n$ are left-orderable, by Proposition 3.1.8 and by [Ba], respectively. Therefore, by Lemma 1.3.7, we have $\widehat{PB}_n(M)$ left-orderable, as required. ∎

3.2 Bi-ordering $\widehat{PB}_n(M)$

Provided that we have $\widehat{PB}_n(M)$ left-orderable, for all n, we have by ordering theory that $\widehat{PB}_n(M)$ is torsion free, for all n. Since bi-orderability implies torsion free[1] (but it is

[1]For the details about theory of orderability and torsion free groups, see [DDRW] and [MR].

not a sufficient condition), it induces to think that $\widehat{PB}_n(M)$ can be, in fact, bi-orderable, as we will show next.

In Section (3.2.1), we will introduce the problems that we find to ordering $\widehat{PB}_n(M)$, when we try to use some tools utilized in ordering in pure braid groups.

Finally, in Section (3.2.3), we will prove that $\widehat{PB}_n(M)$ is bi-orderable.

3.2.1 An obstruction for bi-orderability on homotopy string links over surfaces

Here we discuss about one specific problem that appears when we try to equip the homotopy string links over a surface S with the Magnus ordering, where S is obtained by deleting a single point x_0 of a closed, connected and orientable surface M of genus $g \geq 1$. More specifically, Proposition 3.2.1 shows that the problem arises because of the generators related with the genus of the surface considered.

Proposition 3.2.1. *Let S be a surface obtained by deleting a single point of a closed, connected and orientable surface M of genus $g \geq 1$ and let us consider the isomorphism:*

$$\widehat{PB}_n(S) \simeq \widehat{\mathbb{F}}(2g+n-1) \rtimes \widehat{PB}_{n-1}(S), \tag{3.2.1}$$

given in [Y, Chapter 4, Corollary 4.8, p.29]. *The action of $\widehat{PB}_{n-1}(S)$ on $\widehat{\mathbb{F}}(2g+n-1)$ in the abelianization of $\widehat{\mathbb{F}}(2g+n-1)$, induced by conjugation, is not trivial.*

Proof: It follows from [Y, Corollary 4.8] that:

$$\widehat{PB}_n(S) \simeq \widehat{\mathbb{F}}(2g+n-1) \rtimes \widehat{PB}_{n-1}(S).$$

Let us consider the action $\psi : \widehat{PB}_{n-1}(S) \times \widehat{\mathbb{F}}(2g+n-1) \to \widehat{\mathbb{F}}(2g+n-1)$ given by

$$\psi(\hat{\beta}, \hat{f}) = \hat{\sigma}(\hat{\beta})\hat{f}\hat{\sigma}(\hat{\beta})^{-1},$$

where $\hat{\sigma}$ is the section of the homomorphism $\hat{\varrho}$ given in the exact sequence (3.1.1). Recall the generators set of $\widehat{PB}_{n-1}(S)$:

$$\{a_{i,r};\ \ 2 \leq i \leq n,\ \ 1 \leq r \leq 2g\} \cup \{t_{i,j};\ \ 2 \leq j < k \leq n\}.$$

Take the words on the generators above and apply the section $\hat{\sigma}$ means just add the trivial strand as a straight line at P_1 and then we have a word on $\widehat{PB}_n(S)$. Also, recall the generators of $\widehat{\mathbb{F}}(2g+n-1)$:

$$\{a_{1,r};\ \ 1 \leq r \leq 2g\} \cup \{t_{1,j};\ \ 1 < j \leq n\}.$$

Now, take the word $a_{i,r}a_{1,s}a_{i,r}^{-1}$, $i = 1, ..., n$. This word is written as $\hat{\sigma}(\hat{\beta})\hat{f}\hat{\sigma}(\hat{\beta})^{-1}$ on some generators $\hat{\beta}$ and $\hat{f}$, and we can check, using presentation of $\widehat{PB}_n(S)$ (see Remark (2.1.10)), that this action on the abelianization of $\widehat{\mathbb{F}}(2g+n-1)$ is not the identity, for $r \neq s$. ∎

<u>Remark</u> 3.2.2. *Let us observe that by Proposition 3.2.1 we can not apply Proposition 2.2.14 for the automorphism given by conjugation.*

3.2.2 The structure of $\widehat{\mathbb{F}}_n$

Consider the "forgetting" homomorphism: $\varrho : PB_n(M) \longrightarrow PB_{n-1}(M)$ given by $\varrho(\beta) = \varrho(\beta_1, \ldots, \beta_n) = (\beta_2, \ldots, \beta_n)$. Let $\mathbb{F}_n = \ker \varrho \cap \mathbb{K}_n$ be the group defined by González–Meneses in [GM3] (for details see Section 1.3.4). Also, he obtained the well defined exact sequence:

$$1 \longrightarrow \mathbb{F}_n \xrightarrow{\subseteq} \mathbb{K}_n \xrightarrow{\varrho} \mathbb{K}_{n-1} \longrightarrow 1\ ,$$

since $\varrho(\mathbb{K}_n) = \mathbb{K}_{n-1}$. Moreover, he proved that this exact sequence splits, i.e., there is a section, namely $\sigma : \mathbb{K}_{n-1} \to \mathbb{K}_n$ such that $\varrho \circ \sigma = id_{\mathbb{K}_n}$, where $id_{\mathbb{K}_n}$ is the identity homomorphism in $\mathbb{K}_n$.

We recall that from Theorem 1.3.26, we have $\mathbb{K}_n = (\mathbb{F}_n \rtimes (\mathbb{F}_{n-1} \rtimes (\cdots (\mathbb{F}_3 \rtimes \mathbb{F}_2) \cdots)))$, where for all $i = 1, \dots, n-1$, $\mathbb{F}_{(n+1)-i}$ is the free group freely generated by

$$\{\tilde{\gamma}_{(i)} t_{i,j} \tilde{\gamma}_{(i)}^{-1}; i < j \leq n, \gamma \in \pi_1(M)\}.$$

Moreover, for all $m = 2, \dots, n-1$, $\mathbb{K}_m = (\mathbb{F}_m \rtimes (\mathbb{F}_{m-1} \rtimes (\cdots (\mathbb{F}_3 \rtimes \mathbb{F}_2) \cdots)))$ acts trivially on $H_1(\mathbb{F}_{m+1})$. We also remember that this trivial action on the abelianization $H_1(\mathbb{F}_{m+1})$, guarantee that the Magnus order is preserved.

<u>Theorem</u> 3.2.3. *One has* $\widehat{\mathbb{K}}_n = (\widehat{\mathbb{F}}_n \rtimes (\widehat{\mathbb{F}}_{n-1} \rtimes (\cdots (\widehat{\mathbb{F}}_3 \rtimes \widehat{\mathbb{F}}_2) \cdots)))$, *where for all* $i = 1, \dots, n-1$, $\widehat{\mathbb{F}}_{(n+1)-i}$ *is the reduced free group generated by*

$$\{\widehat{f_{i,j,\gamma}}; \ \ i < j \leq n, \ \ \gamma \in \pi_1(M)\},$$

where $f_{i,j,\gamma} = \tilde{\gamma}_{(i)} t_{i,j} \tilde{\gamma_{(i)}}^{-1} \in \mathbb{F}_{(n+1)-i}$, *for all* $i = 1, \dots, n-1$. *Moreover,*

$$\widehat{\mathbb{K}}_m = (\widehat{\mathbb{F}}_m \rtimes (\widehat{\mathbb{F}}_{m-1} \rtimes (\cdots (\widehat{\mathbb{F}}_3 \rtimes \widehat{\mathbb{F}}_2) \cdots)))$$

acts trivially on $H_1(\widehat{\mathbb{F}}_{m+1})$, *for all* $m = 2, \dots, n-1$.

Proof: Let us consider the following commutative diagram, obtained from diagram (1.3.1) (in which the rows and columns are exacts), by adding the split exact sequence $(\star)$.

$$\begin{tikzcd}
 & 1 & 1 & 1 & \\
1 \arrow[r] & \pi_1(M, P_1) \arrow[r] \arrow[u] & \pi_1(M)^n \arrow[r] \arrow[u] & \pi_1(M)^{n-1} \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & \pi_1(M \setminus \mathcal{P}_{n-1}, P_1) \arrow[r] \arrow[u] \arrow[d, phantom, "\cup"] & PB_n(M) \arrow[r, "\varrho"] \arrow[u, "\theta_n"] \arrow[d, phantom, "\cup"] & PB_{n-1}(M) \arrow[r] \arrow[u, "\theta_{n-1}"] \arrow[d, phantom, "\cup"] & 1 \\
1 \arrow[r] & \mathbb{F}(2g+n-2) \arrow[r] & PB_{n-1}(S) \arrow[r, "\varrho"] & PB_{n-2}(S) \arrow[r] & 1 & (\star) \\
1 \arrow[r] & \mathbb{F}_n \arrow[r] \arrow[u] & \mathbb{K}_n \arrow[r, "\varrho"] \arrow[u] & \mathbb{K}_{n-1} \arrow[r] \arrow[u] & 1 \\
 & 1 \arrow[u] & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd} \tag{3.2.2}$$

So, we can consider, up to link-homotopy, the following commutative diagram, with exact rows and columns:

$$\begin{tikzcd}
 & & 1 & & \\
 & & \pi_1(M)^n \arrow[u] & & \\
 & & \widehat{PB}_n(M) \arrow[u, "\theta_n"] & & \\
 & & \cup & & \\
1 \arrow[r] & \widehat{\mathbb{F}}(2g+n-2) \arrow[r] & \widehat{PB}_{n-1}(S) \arrow[r, "\hat{\varrho}"] & \widehat{PB}_{n-2}(S) \arrow[r] & 1 \quad (\star\star) \\
1 \arrow[r] & \widehat{\mathbb{F}}_n \arrow[r] \arrow[u] & \widehat{\mathbb{K}}_n \arrow[r, "\hat{\varrho}"] \arrow[u] & \widehat{\mathbb{K}}_{n-1} \arrow[r] \arrow[u] & 1 \\
 & 1 \arrow[u] & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd} \tag{3.2.3}$$

where the row $(\star\star)$ is the split exact sequence proved in [Y, Corollary 4.8], which sends by restriction to the following split exact sequence:

$$1 \longrightarrow \widehat{\mathbb{F}}_n \longrightarrow \widehat{\mathbb{K}}_n \xrightarrow{\hat{\varrho}} \widehat{\mathbb{K}}_{n-1} \longrightarrow 1\ , \tag{3.2.4}$$

where $\widehat{\mathbb{F}}_n = \ker(\hat{\varrho}) \cap \widehat{\mathbb{K}}_n$ is a subgroup of the reduced free group $\widehat{\mathbb{F}}(2g+n-2)$, and therefore $\widehat{\mathbb{F}}_n$ is bi-orderable. Then, there is a section $\hat{\sigma} : \widehat{\mathbb{K}}_{n-1} \to \widehat{\mathbb{K}}_n$ of $\hat{\varrho}$, which is the restriction of the section $\hat{\sigma}$ obtained in [Y, Corollary 4.8]. Thus,

$$\widehat{\mathbb{K}}_n \simeq \widehat{\mathbb{F}}_n \rtimes \widehat{\mathbb{K}}_{n-1}. \tag{3.2.5}$$

Therefore, inductively, we have

$$\widehat{\mathbb{K}}_n = (\widehat{\mathbb{F}}_n \rtimes (\widehat{\mathbb{F}}_{n-1} \rtimes (\cdots (\widehat{\mathbb{F}}_3 \rtimes \widehat{\mathbb{F}}_2) \cdots))).$$

Now, we will prove that the action of $\widehat{\mathbb{K}}_{n-1}$ on the abelianization of $\widehat{\mathbb{F}}_n$ is trivial and then we will have that Magnus order is preserved in $\widehat{\mathbb{K}}_n$.

From Lemma 1.3.25, recall that $\mathbb{K}_{n-1}$ acts on $\mathbb{F}_n$ in the following way: given $\beta \in \mathbb{K}_{n-1}$, the action induced by β sends $f \in \mathbb{F}_n$ to $\sigma(\beta) f \sigma(\beta)^{-1}$. This action induces an action of $\mathbb{K}_{n-1}$, which is trivial on the abelianization $H_1(\mathbb{F}_n) = \frac{\mathbb{F}_n}{[\mathbb{F}_n, \mathbb{F}_n]}$ of $\mathbb{F}_n$, where $[\mathbb{F}_n, \mathbb{F}_n]$ denotes the commutator subgroup of $\mathbb{F}_n$. In symbols:

$$\begin{aligned} \psi : \mathbb{K}_{n-1} \times \mathbb{F}_n &\to \mathbb{F}_n, \\ (\beta, f) &\longmapsto \sigma(\beta) f \sigma(\beta)^{-1} \end{aligned}$$

and its trivial abelianization:

$$\begin{aligned} \psi_{ab} : \mathbb{K}_{n-1} \times \frac{\mathbb{F}_n}{[\mathbb{F}_n, \mathbb{F}_n]} &\to \frac{\mathbb{F}_n}{[\mathbb{F}_n, \mathbb{F}_n]}, \\ (\beta, [f]_{ab}) &\longmapsto [\sigma(\beta) f \sigma(\beta)^{-1}]_{ab} \end{aligned}$$

where $[\quad]_{ab}$ denotes the quotient on the commutator group.

Now, considering the homomorphism $\hat{\varrho}$ and its section $\hat{\sigma}$, one defines the action:

$$\begin{aligned} \hat{\psi} : \widehat{\mathbb{K}}_{n-1} \times \widehat{\mathbb{F}}_n &\to \widehat{\mathbb{F}}_n. \\ (\hat{\beta}, \hat{f}) &\longmapsto \hat{\sigma}(\hat{\beta}) \hat{f} \hat{\sigma}(\hat{\beta})^{-1} \end{aligned}$$

This action induces an action of $\widehat{\mathbb{K}}_{n-1}$ on the abelianization $H_1(\widehat{\mathbb{F}}_n)$, namely:

$$\begin{aligned} \hat{\psi}_{ab} : \widehat{\mathbb{K}}_{n-1} \times \frac{\widehat{\mathbb{F}}_n}{[\widehat{\mathbb{F}}_n, \widehat{\mathbb{F}}_n]} &\to \frac{\widehat{\mathbb{F}}_n}{[\widehat{\mathbb{F}}_n, \widehat{\mathbb{F}}_n]}, \\ (\hat{\beta}, [\hat{f}]_{ab}) &\longmapsto [\hat{\sigma}(\hat{\beta}) \hat{f} \hat{\sigma}(\hat{\beta})^{-1}]_{\widehat{ab}} \end{aligned}$$

which is trivial, since

$$
\begin{array}{lcl}
\psi_{ab} \text{ is trivial} & \Leftrightarrow & [\psi(\beta,f)]_{ab} = [f]_{ab} \\
& \Leftrightarrow & \psi(\beta,f)f^{-1} \in [\mathbb{F}_n, \mathbb{F}_n] \\
& \Leftrightarrow & \psi(\beta,f)f^{-1} = x_{i_1}^{n_1} \cdots x_{i_p}^{n_p}, \qquad x_{i_k} \in \langle aba^{-1}b^{-1};\ a,b \in \mathbb{F}_n\rangle,\ \ k = 1,\ldots,p \\
& \Leftrightarrow & \sigma(\beta)f\sigma(\beta)^{-1}f^{-1} = x_{i_1}^{n_1} \cdots x_{i_p}^{n_p},\ \ x_{i_k} \in \langle aba^{-1}b^{-1};\ a,b \in \mathbb{F}_n\rangle,\ \ k = 1,\ldots,p \\
& \Rightarrow & \overbrace{\sigma(\beta)f\sigma(\beta)^{-1}}\,\widehat{f^{-1}} = \overbrace{\sigma(\beta)f\sigma(\beta)^{-1}f^{-1}} = \overbrace{x_{i_1}^{n_1} \cdots x_{i_p}^{n_p}} = \widehat{x_{i_1}}^{n_1} \cdots \widehat{x_{i_p}}^{n_p}, \\
\text{with} & & \widehat{x_{i_k}} \in \langle \hat{a}\hat{b}\hat{a}^{-1}\hat{b}^{-1};\ \hat{a},\hat{b} \in \widehat{\mathbb{F}}_n\rangle,\ \ k = 1,\ldots,p \\
& \Leftrightarrow & \overbrace{\sigma(\beta)f\sigma(\beta)^{-1}}\,\widehat{f^{-1}} \in [\widehat{\mathbb{F}}_n, \widehat{\mathbb{F}}_n] \\
& \Leftrightarrow & [\overbrace{\sigma(\beta)f\sigma(\beta)^{-1}}]_{\widehat{ab}} = [\widehat{f}]_{\widehat{ab}} \\
& \Leftrightarrow & [\hat{\sigma}(\hat{\beta})\hat{f}\hat{\sigma}(\hat{\beta})^{-1}]_{\widehat{ab}} = [\widehat{f}]_{\widehat{ab}} \\
& \Leftrightarrow & \widehat{\psi_{ab}} \text{ is trivial.}
\end{array}
$$

■

Corollary 3.2.4. *$\widehat{\mathbb{K}}_n$ is bi-orderable.*

Proof: The proof is given by induction on n. For $n = 2$, $\widehat{\mathbb{K}}_2 = \widehat{\mathbb{F}}_2$ is a reduced free group and then, it is bi-orderable. Let us suppose $n > 2$ and by induction hypothesis that $\widehat{\mathbb{K}}_{n-1}$ is bi-orderable. By Theorem 3.2.3, we have:

$$1 \longrightarrow \widehat{\mathbb{F}}_n \longrightarrow \widehat{\mathbb{K}}_n \longrightarrow \widehat{\mathbb{K}}_{n-1} \longrightarrow 1,$$

where $\widehat{\mathbb{K}}_n = \widehat{\mathbb{F}}_n \rtimes \widehat{\mathbb{K}}_{n-1}$. The definition of a bi-order says that conjugation by an element of $\widehat{\mathbb{F}}_n$ is an automorphism of $\widehat{\mathbb{F}}_n$ which preserves the Magnus order. By Theorem 3.2.3, the conjugation by an element of $\widehat{\mathbb{K}}_{n-1}$ is an automorphism of $\widehat{\mathbb{F}}_n$, which is trivial on $H_1(\widehat{\mathbb{F}}_n)$. So, we have by Proposition 2.2.14 that it also preserves the Magnus order on $\widehat{\mathbb{F}}_n$.

Hence, conjugation by an element of $\widehat{\mathbb{K}}_n$ preserves the Magnus order of $\widehat{\mathbb{F}}_n$ and therefore, by Lemma 1.3.7, $\widehat{\mathbb{K}}_n$ is bi-orderable. ■

Now we have that $\widehat{\mathbb{K}}_n$ is the semidirect product of reduced free groups and since we already know by Section 2.2 that these subgroups are bi-orderable, we can find an explicit order to it. First, for all $i = 1, \ldots, n-1$ we order the generators:

$$\widehat{\mathcal{F}}_{i,n} = \{\widehat{f_{i,j,\gamma}};\ \ i < j \leq n,\ \ \gamma \in \pi_1(M)\},$$

where $f_{i,j,\gamma} \in \mathbb{F}_{(n+1)-i}$, as follows:

$$\widehat{f_{i,j,\gamma}} <_{\widehat{\mathbb{K}}_n} \widehat{f_{i,k,\delta}} \Leftrightarrow j < k \text{ or } j = k \text{ and } \gamma <_{\pi_1} \delta,$$

where $<_{\pi_1}$ is the bi-ordering for $\pi_1(M)$. Then, we consider the reduced Magnus order on each generator of $\widehat{\mathbb{F}}_{(n+1)-i}$. By this way, let us define the bi-ordering for $\widehat{\mathbb{K}}_n$: for $\hat{k}, \hat{k}' \in \hat{\mathbb{K}}_n$, we can write $\hat{k} = \hat{k}_1 \cdots \hat{k}_{n-1}$ and $\hat{k}' = \hat{k}'_1 \cdots \widehat{k'_{n-1}}$, where $\hat{k}_i, \hat{k}'_i \in \widehat{\mathbb{F}}_{(n+1)-i}$. Then:

$$\hat{k} <_{\widehat{\mathbb{K}}_n} \hat{k}' \Leftrightarrow \hat{k}_j <_{\widehat{\mathbb{K}}_n} \hat{k}'_j,\ \text{ for the greatest } j \text{ such that } \hat{k}_j \neq \hat{k}'_j.$$

3.2.3 $\widehat{PB}_n(M)$ is bi-orderable

<u>**Theorem**</u> **3.2.5.** *Let M be a closed, connected and orientable surface of genus $g \geq 1$. Then $\widehat{PB}_n(M)$ is bi-orderable.*

Proof: Recall the short exact sequence obtained in Remark 3.1.4.

$$1 \longrightarrow \widehat{\mathbb{K}}_n \xrightarrow{\subseteq} \widehat{PB}_n(M) \xrightarrow{\hat{\theta}_n} \pi_1(M)^n \longrightarrow 1\ .$$

We will use this sequence and Lemma 1.3.7(ii) to prove our result. We need to show that the conjugation of $\widehat{\mathbb{K}}_n$ by $\widehat{PB}_n(M)$ is order preserving, i.e.,

$$\hat{f} <_{\widehat{\mathbb{K}}_n} \hat{f}' \Leftrightarrow \hat{\beta}\hat{f}(\hat{\beta})^{-1} <_{\widehat{\mathbb{K}}_n} \hat{\beta}\hat{f}'(\hat{\beta})^{-1}, \tag{3.2.6}$$

for all $\hat{\beta} \in \widehat{PB}_n(M)$ and for all $\hat{f}, \hat{f}' \in \widehat{\mathbb{K}}_n$.

1. From Theorem 1.3.22, $\pi_1(M)^n$ is bi-orderable.

2. We had proved in Corollary 3.2.4 that $\widehat{\mathbb{K}}_n$ is bi-orderable.

By this way, we are able to prove the condition (3.2.6). Indeed, since $\widehat{\mathbb{K}}_n$ is bi-orderable, by definition, conjugation by an element of $\widehat{\mathbb{K}}_n$ preserves the order. Hence, it suffices to show (3.2.6) for the conjugation by the pre-images under $\hat{\theta}_n$ of the generators of $\pi_1(M)^n$. A set of such pre-images is $\{a_{i,r};\quad i=1,\ldots,n,\quad 1\leq r\leq 2g\}$. Since isotopy implies link-homotopy, we have by the proof of Theorem 1.3.28 that the following relations hold in $H_1(\widehat{\mathbb{K}}_n)$:

$$a_{i,r}\widehat{f_{j,k,\gamma}}a_{i,r}^{-1}\equiv\begin{cases}\widehat{f_{j,k,\gamma}}, & i\neq j,k,\\ \widehat{f_{j,k,(\omega_r\gamma)}}, & i=j,\\ \widehat{f_{j,k,(\gamma\omega_r^{-1})}}, & i=k.\end{cases}\tag{3.2.7}$$

Note that the action of $a_{i,r}$ preserves Magnus order on each $\widehat{\mathbb{F}}_m$, $m=2,\ldots,n-1$ and hence, it preserves the order on $\widehat{\mathbb{K}}_n$. Indeed, by the relations found, we have that the action of $a_{i,r}$ on $\widehat{\mathbb{K}}_n$ is the composition of an automorphism $\hat{\Psi}_{i,r}$ which permutes the generators of each $\widehat{\mathbb{F}}_n$ with an automorphism $\hat{\Phi}_{i,r}$ which is trivial on $H_1(\widehat{\mathbb{K}}_n)$.

Therefore, by Proposition 2.2.14 and Theorem 1.3.21, it is suffices to prove that the permutations induced by $\hat{\Psi}_{i,r}$ on $\widehat{\mathcal{F}}_{j,n}$, for $j=1,\ldots,n-1$, preserves the defined order on $\widehat{\mathcal{F}}_{j,n}$. Let us consider $\widehat{f_{j,k,\gamma}}$, $\widehat{f_{j,l,\delta}}$, where $\widehat{f_{j,k,\gamma}}<_{\widehat{\mathbb{K}}_n}\widehat{f_{j,l,\delta}}$. We analyze the following cases:

Case 1. If $k<l$, then

$$a_{i,r}\widehat{f_{j,k,\gamma}}a_{i,r}^{-1}=\widehat{\Psi}_{i,r}(\widehat{f_{j,k,\gamma}})=\widehat{f_{j,k,\gamma'}}<_{\widehat{\mathbb{K}}_n}\widehat{f_{j,l,\delta'}}=\widehat{\Psi}_{i,r}(\widehat{f_{j,l,\delta}})=a_{i,r}\widehat{f_{j,l,\delta}}a_{i,r}^{-1},$$

where γ' and δ' are determined by the relations in (3.2.7).

Case 2. If $k=l$ and $\gamma<_{\pi_1}\delta$, then we have more 3 cases to analyze:

First, if $i\neq j,k$, we have

$$a_{i,r}\widehat{f_{j,k,\gamma}}a_{i,r}^{-1}=\widehat{\Psi}_{i,r}(\widehat{f_{j,k,\gamma}})=\widehat{f_{j,k,\gamma}}<_{\widehat{\mathbb{K}}_n}\widehat{f_{j,k,\delta}}=\widehat{\Psi}_{i,r}(\widehat{f_{j,k,\delta}})=a_{i,r}\widehat{f_{j,k,\delta}}a_{i,r}^{-1}.$$

If $i=j$, thus

$$a_{i,r}\widehat{f_{j,k,\gamma}}a_{i,r}^{-1}=\widehat{\Psi}_{i,r}(\widehat{f_{j,k,\gamma}})=\widehat{f_{j,k,(\omega_r\gamma)}}<_{\widehat{\mathbb{K}}_n}\widehat{f_{j,k,(\omega_r\delta)}}=\widehat{\Psi}_{i,r}(\widehat{f_{j,k,\delta}})=a_{i,r}\widehat{f_{j,k,\delta}}a_{i,r}^{-1},$$

since $\omega_r\gamma <_{\pi_1} \omega_r\delta$, with $<_{\pi_1}$ left-order.

If $i = k$,

$$a_{i,r}\widehat{f_{j,k,\gamma}}a_{i,r}^{-1} = \widehat{\Psi}_{i,r}(\widehat{f_{j,k,\gamma}}) = \widehat{f_{j,k,(\gamma\omega_r^{-1})}} <_{\widehat{\mathbb{K}}_n} \widehat{f_{j,k,(\delta\omega_r^{-1})}} = \widehat{\Psi}_{i,r}(\widehat{f_{j,k,\delta}}) = a_{i,r}\widehat{f_{j,k,\delta}}a_{i,r}^{-1},$$

since $\gamma\omega_r^{-1} <_{\pi_1} \delta\omega_r^{-1}$, with $<_{\pi_1}$ right-order. Therefore we finish our proof with the result required. ∎

Corollary 3.2.6. *Let S be a surface obtained by deleting a single point of a closed, connected and orientable surface M of genus $g \geq 1$. Then, $\widehat{PB}_n(S)$ is bi-orderable, for all n.*

Proof: The proof follows from of the fact that $\widehat{PB}_{n-1}(S) \subset \widehat{PB}_n(M)$ as a subgroup (see sequence (3.2.3)). ∎

3.3 An exact sequence for link-homotopy braid groups

In this section we obtain an extension of a result proved by Charles H. Goldberg in [Go, Theorem 1]. To prove this result, we will use as tool the surjective homomorphism $\hat{\theta}_n$ defined in Section 3.1.2.

3.3.1 Definitions and main theorem

Let us consider a closed, connected and orientable surface M, of genus $g \geq 1$, i.e., M is not the sphere. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a set of n distinct fixed points chosen arbitrarily in the interior of M. Now, let us define the following map:

$$\hat{f}_n : \widehat{PB}_n(\mathbb{D}) \to \widehat{PB}_n(M)$$

given by $\hat{f}_n(\hat{\beta}) = \hat{\beta}$, for each $\hat{\beta}$ in $\widehat{PB}_n(\mathbb{D})$, where $\mathbb{D}$ denotes the unit disk. Also, we have $\hat{\theta}_n : \widehat{PB}_n(M) \to \pi_1(M)^n$ given by $\hat{\theta}_n(\hat{\alpha}) = \hat{\theta}_n([(\alpha_1, \ldots, \alpha_n)]) = (\mu_1, \ldots, \mu_n)$, where each μ_i is the strand α_i of $\hat{\alpha}$ viewed as a loop in the fundamental group of M, $i = 1, \ldots, n$, for all $\hat{\alpha} = [(\alpha_1, \ldots, \alpha_n)]$ in $\widehat{PB}_n(M)$. By Lemma 3.1.2, $\hat{\theta}_n$ is a well defined surjective homomorphism.

Theorem 3.3.1. *If M is a closed, connected and orientable surface, of genus $g \geq 1$, then in the following sequence of groups (not necessarily abelian):*

$$1 \longrightarrow \widehat{PB}_n(\mathbb{D}) \xrightarrow{\hat{f}_n} \widehat{PB}_n(M) \xrightarrow{\hat{\theta}_n} \pi_1(M)^n \longrightarrow 1 \ ,$$

the kernel of each homomorphism is equal to the normal closure of the image of the previous homomorphism in the sequence, i.e., $\ker(\hat{\theta}_n) = \langle \mathrm{Im}(\hat{f}_n) \rangle^N$.

3.3.2 The well definition of $\hat{f}_n$ and proof of Theorem 3.3.1

The proof will be given by the following results:

Lemma 3.3.2. *The map $\hat{f}_n$ is a well defined injective homomorphism.*

Proof: Recall the inclusion homomorphism $f_n : PB_n(\mathbb{D}) \to PB_n(M)$, defined by Birman in [B]. Now, let β be an element link-homotopically trivial in $PB_n(\mathbb{D})$. Clearly, $f_n(\beta) = \beta$ is a link-homotopically trivial element of $PB_n(M)$, i.e., $f_n(H_n(\mathbb{D})) \subseteq H_n(M)$. Thus, $\hat{f}_n$ is the well defined homomorphism induced by the injection f_n. We need to prove that $\hat{f}_n$ is injective. Indeed, let $\hat{\beta} \in \widehat{PB}_n(\mathbb{D})$ such that $\hat{f}_n(\hat{\beta}) = 1$, where 1 denotes the identity in $\widehat{PB}_n(M)$. So, we have $\hat{\beta} \in \widehat{PB}_n(\mathbb{D})$ and $\hat{\beta} \in H_n(M)$. By Goldsmith in [G], we have $\hat{\beta} \in H_n(\mathbb{D})$. Therefore, $\ker(\hat{f}_n) = H_n(\mathbb{D})$, i.e., $\hat{f}_n$ is injective. ■

Let us consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & PB_n(\mathbb{D}) \arrow[r, "f_n"] \arrow[d, "p_1"] & PB_n(M) \arrow[r, "\theta_n"] \arrow[d, "p_2"] & \pi_1(M)^n \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & \widehat{PB}_n(\mathbb{D}) \arrow[r, "\hat{f}_n"] & \widehat{PB}_n(M) \arrow[r, "\hat{\theta}_n"] & \pi_1(M)^n \arrow[r] & 1
\end{tikzcd} \tag{3.3.1}$$

where p_1, p_2 are the respective projections and id is the identity in $\pi_1(M)$.

We claim that this diagram is commutative. Indeed, $\forall \beta \in PB_n(\mathbb{D})$,

$$\hat{f}_n \circ p_1(\beta) = \hat{f}_n([\beta]) = [\beta] \text{ and } p_2 \circ f_n(\beta) = p_2(\beta) = [\beta],$$

that is, $\hat{f}_n \circ p_1 = p_2 \circ f_n$. Moreover, $\forall \alpha \in PB_n(M)$,

$$\hat{\theta}_n \circ p_2(\alpha) = \hat{\theta}_n([\alpha]) = (\mu_1, \ldots, \mu_n) \text{ and } id \circ \theta_n(\alpha) = id(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_n),$$

i.e., $\hat{\theta}_n \circ p_2 = id \circ \theta_n$.

<u>Lemma</u> 3.3.3. $\mathrm{Im}(\hat{f}_n) \subseteq \ker(\hat{\theta}_n)$.

Proof: Let $\hat{\alpha} \in \mathrm{Im}(\hat{f}_n)$. So, there is an element $\hat{\beta} \in \widehat{PB}_n(\mathbb{D})$ such that $\hat{\alpha} = \hat{f}_n(\hat{\beta})$. Thus, $\hat{\theta}_n \circ \hat{f}_n(\hat{\beta}) = \hat{\theta}_n(\hat{\alpha})$. Since $\hat{\beta}$ is in $\widehat{PB}_n(\mathbb{D})$ and p_1 is surjective, there is β in $PB_n(\mathbb{D})$ such that $p_1(\beta) = \hat{\beta}$. So, $\hat{\theta}_n \circ \hat{f}_n \circ p_1(\beta) = \hat{\theta}_n(\hat{\alpha})$, which implies, $\hat{\theta}_n \circ p_2 \circ f_n(\beta) = \hat{\theta}_n(\hat{\alpha})$, i.e., $\theta_n \circ f_n(\beta) = \hat{\theta}_n(\hat{\alpha})$. By [Go], we have $\mathrm{Im}(f_n) \subseteq \ker(\theta_n)$, i.e., $\hat{\theta}_n(\hat{\alpha}) = 1$ and then, $\hat{\alpha} \in \ker(\hat{\theta}_n)$. ∎

<u>Lemma</u> 3.3.4. $\ker(\hat{\theta}_n) \subseteq \langle \mathrm{Im}(\hat{f}_n) \rangle^N$.

Proof: Let $\hat{\gamma}$ be an element in $\ker(\hat{\theta}_n)$. Thus, $\hat{\gamma} \in \widehat{PB}_n(M)$ and $\hat{\theta}_n(\hat{\gamma}) = 1$. By Theorem 2.1.3, each string link is link homotopic to a pure braid. Let $\gamma \in PB_n(M)$ be such pure braid. So, $\theta_n(\gamma) = 1$, where 1 denotes the identity in $\pi_1(M)^n$. By [Go, Theorem 1], we have $\gamma = \prod_k \alpha_k \beta_k \alpha_k^{-1}$, with $\alpha_k \in PB_n(M), \beta_k \in \mathrm{Im}(f_n)$, i.e., $\beta_k = f_n(\gamma_k)$, for some $\gamma_k \in PB_n(\mathbb{D})$ with $p_1(\gamma_k) = \hat{\gamma}_k$, and then, $\hat{f}_n \circ p_1(\gamma_k) = \hat{f}_n(\hat{\gamma}_k)$. Since the diagram (3.3.1) commutes,

$$\hat{f}_n(\hat{\gamma}_k) = \hat{f}_n \circ p_1(\gamma_k) = p_2 \circ f_n(\gamma_k) = p_2(\beta_k) = \hat{\beta}_k,$$

i.e., $\hat{\beta}_k \in \mathrm{Im}(\hat{f}_n)$. Therefore,

$$\hat{\gamma} = p_2(\gamma) = p_2\left(\prod_k \alpha_k \beta_k \alpha_k^{-1}\right) = \prod_k p_2(\alpha_k) p_2(\beta_k) p_2(\alpha_k)^{-1} = \prod_k \hat{\alpha}_k \hat{\beta}_k \hat{\alpha}_k{}^{-1},$$

with $\hat{\alpha}_k \in \widehat{PB}_n(M), \hat{\beta}_k \in \mathrm{Im}(\hat{f}_n)$. ∎

Proof of Theorem 3.3.1: Follows from Lemmas 3.3.2, 3.3.3 and 3.3.4. ∎

Chapter 4

Homotopy generalized string links over surfaces

In this chapter we introduce the generalization of the homotopy string links over surfaces that we call generalized homotopy string links over surfaces. We show that the set of all generalized string links over surfaces form a well defined group and we find a presentation for this group.

4.1 Generalized string links over surfaces

Let M be a closed, connected and orientable surface of genus $g \geq 1$. Choose n different points $\mathcal{P} = \{P_1, \ldots, P_n\}$ to lie in the interior of M. Let $I_1, \ldots, I_n$ be n copies of the unit interval $I = [0, 1]$ and $\coprod_{i=1}^{n} I_i$ denote the disjoint union of these intervals.

Definition 4.1.1. *A* generalized string link σ on n strands over a surface M *is a smooth or piecewise linear proper imbedding:*

$$\sigma : \coprod_{i=1}^{n} I_i \to M \times I,$$

that satisfies the two following conditions:

(i) $\sigma|_{(I_i(0))} = (P_i, 0)$,

(ii) $\sigma|_{(I_i(1))} \in \{(P_1, 1), \ldots, (P_n, 1)\}$,

where $I_i(t) = t \in I_i$, *for all* t *and for all* $i = 1, \ldots, n$.

Here, we orient the strands downwards from $M \times \{0\}$ to $M \times \{1\}$. Also, an *ambient isotopy* between generalized string links σ and σ' is an orientation-preserving diffeomorphism of $M \times I$ which maps σ onto σ' while keeping the boundary $M \times \{0, 1\}$ point-wise fixed and is isotopic to the identity, relative to $M \times \{0, 1\}$.

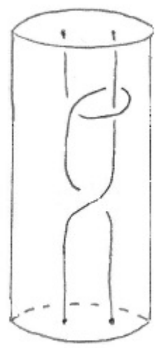


Figure 4.1: Generalized string link σ.

Now, we can talk about the concept of link-homotopy found in [HL], [L] and [Mil] for generalized string links, since we have the non trivial permutation, generalized string links differ of string links. Indeed, we give two different definitions for it that we use throughout this work.

<u>Definition</u> 4.1.2. *We say that two generalized string links* σ *and* σ' *are* link-homotopic *if there is a homotopy of the strings in* $M \times I$, *fixing* $M \times \{0, 1\}$ *and deforming* σ *to* σ', *such that the images of different strings remain disjoint during the deformation.*

During the course of deformation, each individual strand is allowed to pass through itself but not through other strands.

<u>Definition</u> 4.1.3. *Link-homotopy is an equivalence relation on generalized string links that is generated by a sequence of ambient isotopies of* $M \times I$ *fixing* $M \times \{0, 1\}$, *and local* crossing changes *of arcs from the* same strand *of a generalized string link.*

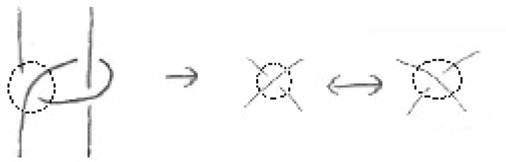


Figure 4.2: Crossing change in the same strand.

4.2 Homotopy generalized string links form a group

Lemma 4.2.1. *Let σ and σ' be generalized string links over a surface M. Denote by $\sim$ the link-homotopy equivalence relation. If $\sigma \sim \sigma'$ then $\sigma\tau \sim \sigma'\tau$ and $\tau\sigma \sim \tau\sigma'$, for all generalized string link τ.*

Proof: In both cases, consider the concatenation of the mentioned generalized string links in its levels in the diagram of braids respectively. By this way, we can deform σ to σ' under homotopy while τ is fixed, for all generalized string link τ. ■

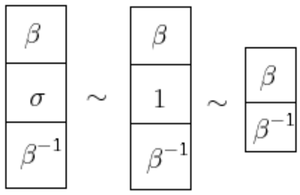
Theorem 4.2.2. *Every generalized string link on n-strands over a surface M is link-homotopic to a braid.*

Proof: Let us denote by $\sim$ the link-homotopy equivalence relation. We want to prove that if σ is a generalized string link then $\sigma \sim \alpha$, for some $\alpha \in B_n(M)$. Consider σ a generalized n-string link and β some braid on n-strands such that the concatenation $\sigma\beta$ is a string link, namely σ'. So, we have $\sigma\beta = \sigma'$. Since σ' is a string link, we have it is link-homotopic to a pure braid on n-strands, namely γ. Thus, we have $\sigma' \sim \gamma$. By the transitivity of the equivalence relation, we have that $\sigma\beta \sim \gamma$. Let β^{-1} be the inverse of the braid β. By Lemma 4.2.1, we have $\sigma\beta\beta^{-1} \sim \gamma\beta^{-1}$, i.e., $\sigma \sim \gamma\beta^{-1}$, where $\gamma\beta^{-1}$ is a braid on n-strands. Put $\gamma\beta^{-1} = \alpha$. Therefore, every generalized string link on n-strands is link-homotopic to a braid on n strands. ■

Remark 4.2.3. *From now we will call a generalized string link on n-strands just by generalized string link, since the chosen n points are fixed. For the next result, recall that a string link σ is link-homotopically trivial if σ is link-homotopic to the trivial braid. Also, let $H_n(M)$ denote the set of link-homotopically trivial surface n-strand braids. We already* *know that $H_n(M)$ is a normal subgroup of $PB_n(M)$ and that the group of link-homotopy classes of string links over a surface M, namely $\widehat{PB}_n(M)$, is isomorphic to $\frac{PB_n(M)}{H_n(M)}$.*

Proposition 4.2.4. *The set of link-homotopically trivial surface braids on n-strands $H_n(M)$ is a normal subgroup of $B_n(M)$.*

Proof: We will show that $\beta H_n(M)\beta^{-1} = H_n(M)$, for all β in $B_n(M)$. Indeed, given $\beta \in B_n(M)$, $\sigma \in H_n(M)$, if we consider the braid diagram for the concatenation $\beta\sigma\beta^{-1}$ and remember that $\sigma \sim 1$, let β and β^{-1} fixed and deform σ under homotopy in the trivial braid 1:



Thus, $\beta\sigma\beta^{-1} \sim 1$ and $1 \in H_n(M)$. Therefore, $\beta H_n(M)\beta^{-1} \subseteq H_n(M)$.

Conversely, let σ be a link-homotopically trivial braid. We show that σ can be written as an element in $\beta H_n(M)\beta^{-1}$, for β in $B_n(M)$. Indeed:

$$\sigma \sim 1\sigma 1 \sim \beta(\beta^{-1}\sigma\beta)\beta^{-1}, \beta \in B_n(M).$$

But we have already proved that the element inside of parentheses is a link-homotopically braid. So, put $\gamma = \beta^{-1}\sigma\beta$. Then, $\sigma = \beta\gamma\beta^{-1}$, and we have $H_n(M) \subseteq \beta H_n(M)\beta^{-1}$, for all $\beta \in B_n(M)$. Therefore, $H_n(M)$ is a normal subgroup of $B_n(M)$ as required. ∎

Remark 4.2.5. (i) *We denote the set of link-homotopy classes of generalized string links over a surface M by $\widehat{B_n}(M)$, which we shall call simply homotopy generalized string links.*

(ii) *$\widehat{B_n}(M)$ equipped with concatenation, is a group. Moreover, $\widehat{PB}_n(M)$ is a normal subgroup of $\widehat{B_n}(M)$.*

(iii) *We say that a braid is deformed to be a string-link through a finite link-homotopic moves, i.e., through a finite number of isotopies and crossing changes.*

Theorem 4.2.6. *Under concatenation, $\widehat{B_n}(M)$ is a group isomorphic to the quotient of the braid group $B_n(M)$ by the subgroup of link-homotopically trivial braids $H_n(M)$:*

$$\widehat{B_n}(M) \simeq \frac{B_n(M)}{H_n(M)}.$$

Proof: Let us consider the following map:

$$\varphi : B_n(M) \to \widehat{B_n}(M),$$

defined by $\varphi(\beta) = \hat{\beta}$, which takes the isotopy class of each braid to its homotopy class. Let us denote by $\approx$ the isotopy equivalence relation and by $\sim$ the homotopy equivalence relation. Remember that $B_n(M)$ is already a quotient group and β denotes the representative of the equivalence class of all braids that is isotopic to β, so we must show first that φ is a well defined homomorphism. Indeed: Let β, γ be two representative braids of an equivalence class. So, we have $\beta \approx \gamma$. Let $\varphi(\beta) = \hat{\beta}$ and $\varphi(\gamma) = \hat{\gamma}$, where $\hat{\beta}$ and $\hat{\gamma}$ are string links provided from β and γ under a finite link-homotopic moves respectively. Thus, $\varphi(\beta) = \hat{\beta} \sim \beta$ and $\varphi(\gamma) = \hat{\gamma} \sim \gamma$. So, we have:

$$\varphi(\gamma) = \hat{\gamma} \sim \gamma \approx \beta \sim \hat{\beta} = \varphi(\beta),$$

and since isotopy implies homotopy, we have $\varphi(\beta) = \varphi(\gamma)$. Now, note that by Theorem 4.2.2, φ is surjective. So, by the Homomorphism Theorem, we have:

$$\frac{B_n(M)}{\ker(\varphi)} \simeq \widehat{B_n}(M).$$

We claim that $\ker(\varphi) = H_n(M)$. Indeed: $\ker(\varphi) = \{\beta \in B_n(M);\ \varphi(\beta) = 1\}$. If $\beta \in \ker(\varphi)$, then we have that $\beta \sim \hat{\beta} \sim 1$, where $\hat{\beta}$ is a generalized string link provided from β under a finite link-homotopic moves. So, we have that $\ker(\varphi) \subseteq H_n(M)$. Conversely, if $\beta \in H_n(M)$, then β is link-homotopic to 1. Choose a generalized string link $\hat{\beta}$ that is link-homotopic to β under a finite link-homotopic moves. Clearly, $\beta \in \ker(\varphi)$. Therefore, we have:

$$\widehat{B_n}(M) \simeq \frac{B_n(M)}{H_n(M)},$$

as required. Thus, $\widehat{B_n}(M)$ inherits from $B_n(M)$:

operation: concatenation of generalized string links;

inverse: mirror reflexion up to link-homotopy. ∎

Now, given a generalized string link σ, let us denote by $\pi(\sigma)$ the permutation associated to σ. Let Σ_n be the symmetric group on n elements. Consider the following map:

$$\psi : \widehat{B_n}(M) \to \Sigma_n$$

defined by $\psi(\sigma) = \pi(\sigma)$, for all $\sigma \in \widehat{B_n}(M)$. We claim that ψ is a well defined homomorphism. Indeed, if σ and σ' are two generalized string links in the same equivalence class, then both generalized string links have the same permutation. Thus, $\psi(\sigma) = \psi(\sigma')$ and the map is well defined as claimed. Clearly, ψ is a homomorphism. Note that the homomorphism ψ is surjective. Thus, by the homomorphism theorem, we have $\frac{\widehat{B_n}(M)}{\ker(\psi)}$ isomorphic to Σ_n. By the definition, $\ker(\psi) = \widehat{PB}_n(M)$. So we have the following result:

Proposition 4.2.7. *$\widehat{PB}_n(M)$ is a normal subgroup of $\widehat{B_n}(M)$. Moreover, under the homomorphism ψ defined previously, we have the well defined short exact sequence:*

$$1 \longrightarrow \widehat{PB}_n(M) \xrightarrow{i} \widehat{B_n}(M) \xrightarrow{\psi} \Sigma_n \longrightarrow 1 \,,$$

where i is the inclusion homomorphism.

∎

4.3 Homotopy generalized string links over an orientable surface

Since we had defined the homotopy generalized string links over an orientable surface M of genus $g \geq 1$, it is interesting to ask about its presentation. In order to find a presentation for this group, we study it from a geometric point of view and collect some important results that will be necessary for our aim.

4.3.1 Geometric view of orientable surface braids

The following geometric view that we show here is the same of [GM] since we give here a presentation for $\widehat{B_n}(M)$ that generalizes the presentation for the orientable case of $B_n(M)$ in the mentioned paper.

For the remainder of this section let M be a closed, orientable surface of genus $g \geq 1$, i.e., for M different than the sphere $\mathbb{S}^2$. Let us represent M by its fundamental polygon L, with $4g$ sides, with pairs labeled $\alpha_1, \ldots, \alpha_{2g}$. Choose n distinct points $\mathcal{P} = \{P_1, \ldots, P_n\}$ as base points across a diameter of L.

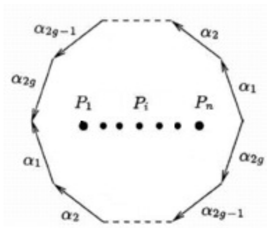


Figure 4.3: Fundamental polygon L over M [GM].

Now let $I = [0, 1]$ be the unit interval. We represent $M \times I$ by the cylinder $L \times I$ with opposite sides identified, following the original identifications of L. Let us assume that $M \times \{0\}$ is the upper level of the cylinder and $M \times \{1\}$ is the lower level of the cylinder. A surface braid β appears in the cylinder with strands downwards. Note that a string of a braid may "go through the wall" of the cylinder $L \times I$ and re-appear from the identified opposite "wall" as shown in the left side of the figure below. We have another way to represent a braid in the cylinder: look on $L \times I$ from the top, as shown in the right side of the figure below.

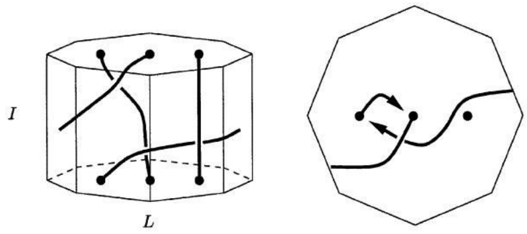


Figure 4.4: Two forms to see a surface braid [GM].

Our aim now is to state our presentation of orientable surface link-homotopy braid groups, defining the generators and showing that the proposed relations are satisfied. So let us start defining elements of $\widehat{B_n}(M)$. After choosing the base points $\mathcal{P} = \{P_1, \ldots, P_n\}$ along a horizontal diameter of L, define:

- For $1 \leq i \leq n$, $1 \leq r \leq 2g$, we denote by $a_{i,r}$ the braids in $\widehat{B_n}(M)$ that "go through the wall" of the cylinder $L \times I$. Note in the figure that $a_{i,r}$ goes upwards in L if r is odd and goes downwards if r is even.

- For $i+1 \leq j \leq n$, we denote by $t_{i,j}$ the braid in $\widehat{B_n}(M)$ that is a loop starting at P_i, going around $P_{i+1}, \ldots, P_j$ and turning back to P_i passing in front of P_j only.

The following figure shows us such braids:

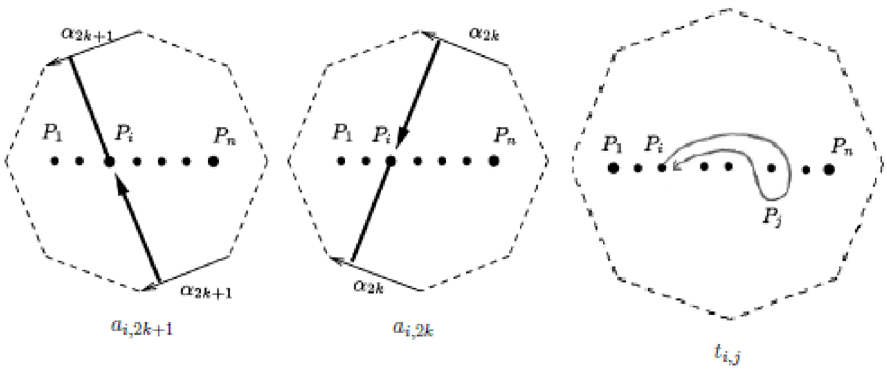


Figure 4.5: Elements of $\widehat{B_n}(M)$, for $i = 1$ [GM].

4.4 A presentation for $\widehat{B_n}(M)$

The goal of this section is to prove the following result:

Theorem 4.4.1. *Let M be a closed, orientable surface of genus $g \geq 1$. The Group of Link-Homotopy Classes of Generalized String Links Over M, namely $\widehat{B_n}(M)$, admits the following presentation:*

Generators: $\{a_{1,1}, \ldots, a_{1,2g}\} \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$;

Relations:

(LH) $[t_{1,j}, t_{1,j}^h] = 1, \qquad h \in \mathbb{F}(2g+n-1);$

(R1) $\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geq 2;$

(R2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad 1 \leq i \leq n-2;$

(R3) $a_{1,1}\cdots a_{1,2g}a_{1,1}^{-1}\cdots a_{1,2g}^{-1} = \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1,$

(R4) $a_{1,r}A_{2,s} = A_{2,s}a_{1,r}, \qquad 1 \leq r, s \leq 2g;\ r \neq s;$

(R5) $(a_{1,1}\cdots a_{1,r})A_{2,r} = \sigma_1^2 A_{2,r}(a_{1,1}\cdots a_{1,r}), \qquad 1 \leq r \leq 2g;$

(R6) $a_{1,r}\sigma_i = \sigma_i a_{1,r}, \qquad 1 \leq r \leq 2g;\ i \geq 2.$

Where:

$$\begin{aligned} t_{1,j} &= \sigma_1\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_1^{-1}, \ \textit{for}\ j = 2,\ldots,n, \\ A_{2,s} &= \sigma_1^{-1}(a_{1,1}\cdots a_{1,s-1}a_{1,s+1}^{-1}\cdots a_{1,2g}^{-1})\sigma_1^{-1}, \ \textit{for}\ s = 1,\ldots,2g. \end{aligned}$$

Remark 4.4.2. *To prove Theorem 4.4.1 we will consider the methods and constructions used by González–Meneses in [GM] (to compute the presentation of surface braids $B_n(M)$ over a closed surface M) discussed in Proposition 1.2.1 and Lemma 1.2.2. Such constructions are very well known and common methods used in several proofs in this theory for finding presentations of groups.*

Here, we use the notations and arguments of [GM] to establish connections with the presentations of the braid groups $B_n(M)$ and the generalized string links groups $\widehat{B_n}(M)$.

4.4.1 The idea of the proof

Now, recalling the short exact sequence from Proposition 4.2.7:

$$1 \longrightarrow \widehat{PB}_n(M) \xrightarrow{i} \widehat{B_n}(M) \xrightarrow{\psi} \Sigma_n \longrightarrow 1\ ,$$

and considering the following presentations:

Presentation of $\widehat{PB}_n(M)$ (see Theorem 2.1.7):

Generators: $\{a_{i,r};\ 1 \leq i \leq n,\ 1 \leq r \leq 2g\} \cup \{t_{j,k};\ 1 \leq j < k \leq n\}$.

Relations: (LH1) $[t_{i,j}, t_{i,j}^h] = 1$, for all $h \in \mathbb{F}(2g+n-i)$;

(PR1) $a_{n,1}^{-1}a_{n,2}^{-1}\cdots a_{n,2g}^{-1}a_{n,1}a_{n,2}...a_{n,2g} = \prod_{i=1}^{n-1} T_{i,n-1}^{-1}T_{i,n}$,

(PR2) $a_{i,r}A_{j,s} = A_{j,s}a_{i,r}$, $1 \leq i < j \leq n;\ 1 \leq r, s \leq 2g;\ r \neq s$;

(PR3) $(a_{i,1}\cdots a_{i,r})A_{j,r}(a_{i,r}^{-1}\cdots a_{i,1}^{-1})A_{j,r}^{-1} = T_{i,j}T_{i,j-1}^{-1}$, $1 \leq i < j \leq n;\ 1 \leq r \leq 2g$;

(PR4) $T_{i,j}T_{k,l} = T_{k,l}T_{i,j}$, $1 \leq i < j < k < l \leq n$ or $1 \leq i < k < l \leq j \leq n$;

(PR5) $T_{k,l}T_{i,j}T_{k,l}^{-1} = T_{i,k-1}T_{i,k}^{-1}T_{i,j}T_{i,l}^{-1}T_{i,k}T_{i,k-1}^{-1}T_{i,l}$, $1 \leq i < k \leq j < l \leq n$;

(PR6) $a_{i,r}T_{j,k} = T_{j,k}a_{i,r}$, $1 \leq i < j < k \leq n$ or $1 \leq j < k < i \leq n;\ 1 \leq r \leq 2g$;

(PR7) $a_{i,r}(a_{j,2g}^{-1}\cdots a_{j,1}^{-1}T_{j,k}a_{j,2g}\cdots a_{j,1}) = (a_{j,2g}^{-1}\cdots a_{j,1}^{-1}T_{j,k}a_{j,2g}\cdots a_{j,1})a_{i,r}$, $1 \leq j < i \leq k \leq n$;

(PR8) $T_{j,n} = \left(\prod_{i=1}^{j-1} a_{i,2g}^{-1}\cdots a_{i,1}^{-1}T_{i,j-1}T_{i,j}^{-1}a_{i,1}\cdots a_{i,2g}\right) a_{j,1}\cdots a_{j,2g}a_{j,1}^{-1}\cdots a_{j,2g}^{-1}$.

Where:

$\mathbb{F}(2g+n-i)$ is generated by $\{a_{i,r}; 1 \leq i \leq n, 1 \leq r \leq 2g\} \cup \{t_{i,j}; i < j \leq n\}$;
$A_{j,s} = a_{j,1}\cdots a_{j,s-1}a_{j,s+1}^{-1}\cdots a_{j,2g}^{-1}$, for $1 \leq s \leq 2g$;
$T_{i,j} = t_{i,j}\cdots t_{i,i+1}$, for $1 \leq i < j \leq n$.

Presentation of Σ_n:

Generators: $\delta_1, \ldots, \delta_{n-1}$.

Relations:

- $\delta_i\delta_j = \delta_j\delta_i$, $|i-j| \geq 2$;
- $\delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1}$, $1 \leq i \leq n-2$;
- $\delta_i^2 = 1$, $1 \leq i \leq n-1$;

where δ_i is the permutation $(i, i+1)$, for all i.

We are now able to find a presentation for $\widehat{B_n}(M)$.

The idea for the proof is to define an abstract group, namely $\mathcal{B}_n$, with the presentation of the Theorem 4.4.1. After, we define a homomorphism $\varphi : \mathcal{B}_n \to \widehat{B_n}(M)$ in a natural way and we will show that φ is an isomorphism. First, we will show that φ is well defined showing that all relations of $\mathcal{B}_n$ holds in $\widehat{B_n}(M)$. After that, we will use Proposition 1.2.1 to the exact sequence:

$$1 \longrightarrow \widehat{PB_n}(M) \xrightarrow{i} \widehat{B_n}(M) \xrightarrow{\psi} \Sigma_n \longrightarrow 1 \tag{4.4.1}$$

to show that φ is an isomorphism.

4.4.2 The proof of Theorem 4.4.1

Let us call $\mathcal{B}_n$ the abstract group that admits the presentation of the Theorem 4.4.1. To show the validity of the presentation we will need to add some generators and relations, keeping equivalent to the other that we have:

new generators:

- $a_{i,r}, \quad 2 \leq i \leq n;$
- $t_{j,k}, \quad 1 \leq j < k \leq n.$

new relations:

(R7) $a_{j+1,r} = \sigma_j a_{j,r} \sigma_j,$ $\quad 1 \leq j \leq n-1; 1 \leq r \leq 2g;$ r even;

(R8) $a_{j+1,r} = \sigma_j^{-1} a_{j,r} \sigma_j^{-1},$ $\quad 1 \leq j \leq n-1; 1 \leq r \leq 2g;$ r odd;

(R9) $t_{j,k} = \sigma_j \sigma_{j+1} \cdots \sigma_{k-2} \sigma_{k-1}^2 \sigma_{k-2}^{-1} \cdots \sigma_{j+1}^{-1} \sigma_j^{-1},$ $\quad 1 \leq j < k \leq n.$

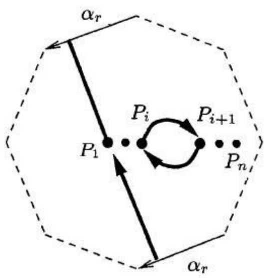
It is easy to see that $(R7)$, $(R8)$ and $(R9)$ hold in $\widehat{B_n}(M)$, using a braid diagram and even observing that they are product of the generators set of Theorem 4.4.1. Moreover, adding these new relations, they still define the same group.

Note that both presentations define the same group, since the "new" generators and relations can be written in terms of the old generators by the relations $(R7), (R8)$ and $(R9)$. Now let us define the mapping in a natural way:

$$\begin{aligned}\varphi : &\mathcal{B}_n \to \widehat{B_n}(M)\\ &\sigma_i \longmapsto \sigma_i, \ \ 1 \leq i \leq n-1\\ &a_{1,r} \longmapsto a_{1,r}, \ \ 1 \leq r \leq 2g\end{aligned}$$

Note that we will keep the notation σ_i and $a_{1,r}$ for the braids that will be the images of the generators σ_i and $a_{1,r}$ of $\mathcal{B}_n$ under the homomorphism φ. Such braids are defined as follows: σ_i are the elementary n-braids and $a_{1,r}$ are the braids that "go through the wall", starting and arriving in P_1, with the remaining strands being trivial.

We claim that φ is well defined. Indeed, by [GM] the relations $(R1)$–$(R9)$ hold in $B_n(M)$. Since isotopy equivalence relation implies link-homotopy equivalence relation (see Definition 4.1.3), we have they still hold in $\widehat{B_n}(M)$. By this way, $(R1)$–$(R9)$ hold in $\widehat{B_n}(M)$. Let us give the braid diagram that shows $(R6)$ $a_{1,r}\sigma_i = \sigma_i a_{1,r}$, for $1 \leq r \leq 2g$, is valid:



For the other cases, see [GM, Section 2].

Now, consider the relation (under φ) $[t_{1,j}, t_{1,j}^h] = 1$, for $h \in \mathbb{F}(2g+n-1)$. Let us recall that the generators of $\mathbb{F}(2g+n-1)$ are $\{a_{1,r}, 1 \leq r \leq 2g\} \cup \{t_{1,j}, 2 \leq j \leq n\}$. Such relation holds in $\widehat{B_n}(M)$, since it is a particular case from the relation $(LH1)$ in $\widehat{PB_n}(M)$ that is contained in $\widehat{B_n}(M)$. So, we have that φ is well defined.

To show that φ is surjective, consider the short exact sequence (4.4.1).

Applying Proposition 1.2.1 to find the generators of $\widehat{B_n}(M)$ we have two types: first, the generators of $\widehat{PB}_n(M)$ that become generators of $\widehat{B_n}(M)$ under the inclusion:

$$\{a_{i,r};\ 1 \leq i \leq n,\ 1 \leq r \leq 2g\} \cup \{t_{j,k};\ 1 \leq j < k \leq n\}.$$

Second, the other type of generators are the elements σ_i that are pre-image of each generator $\delta_i \in \Sigma_n$, for $i = 1, \ldots, n-1$, under the surjective homomorphism ψ in the exact sequence: $\{\sigma_i, 1 \leq i \leq n-1\}$.

So, we have $\{a_{i,r};\ 1 \leq i \leq n,\ 1 \leq r \leq 2g\} \cup \{t_{j,k};\ 1 \leq j < k \leq n\} \cup \{\sigma_i, 1 \leq i \leq n-1\}$ as generators of $\widehat{B_n}(M)$. But note that, by the relation $(R9)$ we have $t_{j,k}$ written as a product of σ_i's and by the relaions $(R7)$–$(R8)$, $a_{i,r}$ can be written as a product of $a_{1,r}$. So we reduce the set of generators that we found to:

$$\{a_{1,r};\ \ 1 \leq r \leq 2g\} \cup \{\sigma_i, 1 \leq i \leq n-1\}.$$

Therefore, we have φ surjective, as required.

To show that φ is injective, we need to show that all relations in $\widehat{B_n}(M)$ still hold in $\mathcal{B}_n$. Indeed, we observe that relations $(R1)$–$(R9)$ in $\widehat{B_n}(M)$ come from the same relations in $B_n(M)$, since isotopy implies link-homotopy. The process made by González–Meneses in [GM] to find all relations of Types 1, 2 and 3 is given in this thesis in Section 1.2. So, the process is the same for $\widehat{B_n}(M)$. The remainder relation that does not appear in cases made by González–Meneses, i.e., the link-homotopy relation (LH) $[t_{1,j}, t_{1,j}^h] = 1$, for $h \in \mathbb{F}(2g+n-1)$, is a relation of Type 1 that comes from $(LH1)$ in the presentation of $\widehat{PB}_n(M)$, when $i = 1$. The Figure 2.3 gives a particular case of this relation. Thus, φ is injective.

By this way, we have that $\mathcal{B}_n$ and $\widehat{B_n}(M)$ have the same generators and relations and, therefore, φ is an isomorphism and $\widehat{B_n}(M)$ has the presentation of the Theorem 4.4.1, as required. ∎

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