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# On the Geometry of Weighted Manifolds

Tese de Doutorado

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Orientador: *Prof. Marcos Petrúcio de A. Cavalcante*

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*To my beloved wife Thatiane.*

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—HEBREUS 4:16

# Abstract

In this thesis we present contributions to the study of weighted manifolds in the intrinsic and extrinsic setting.

Firstly, we prove generalizations of Myers compactness theorem due to Ambrose and Galloway for the Bakry-Émery Ricci tensor. As application we obtain closure theorems for the weighted spacetime.

After that, using maximum principles for $f$-Laplacian, we obtain results type Bernstein and height estimates for hypersurfaces immersed in a semi-Riemannian manifold of type $\varepsilon I \times_\rho \mathbb{P}_f$.

**Keywords:** Weighted manifolds, Bakry-Émery Ricci tensor, $f$-mean curvature, Bernstein type theorems, height estimates.

# Resumo

Nesta tese, nós apresentamos contribuições para o estudo das variedades ponderadas no sentido intrínseco e extrínseco.

Primeiramente, nós provamos generalizações do teorema de compacidade de Myers, devido a Ambrose and Galloway, para o tensor Bakry-Émery Ricci. Como aplicação, nós obtemos teoremas do fechamento para o espaço-tempo ponderado.

Depois disso, usando pricípios do máximo para o $f$-Laplaciano, nós obtemos resultados tipo Bernstein e estimativas de altura para hipersuperfícies imersas em uma variedade semi-Riemanniana ponderada do tipo $\varepsilon I \times_{\rho} \mathbb{P}_f$.

**Palavras-chave:** Variedades ponderadas, tensor Bakry-Émery Ricci, curvatura $f$-média, teoremas tipo Bernstein, estimativas de altura.

# Contents

# Introduction

Many problems lead us to consider Riemannian manifolds endowed with a measure that has a smooth positive density with respect to the Riemannian one. As consequence, the interest in the study of the weighted manifolds have been growing up.

In this thesis, we obtain intrinsic and extrinsic results for weighted manifolds, most of them generalizing well known results in the standard Riemannian and Lorentzian cases. Our results are part of the papers [22], [66], [23], [24] and [25].

In the first chapter, we fix some notations, give some basic definitions, and state results that we will use in the other chapters.

In Chapter 2, based in the papers [22], in collaboration with M. Cavalcante and J. Oliveira, and [66], we prove six generalizations of Myers compactness theorem under different conditions on its generalized Ricci curvature tensor. As applications of the our weighted compactness theorems, we present new closures theorems, that is, theorems which has as conclusion the finiteness of the spatial part of a spacetime manifold.

In Chapter 3, based in the papers [24] and [25], in collaboration with M. Cavalcante and H. de Lima, our aim is to investigate Bernstein properties of hypersurfaces immersed in a weighted semi-Riemannian manifold of the type $\varepsilon I \times_\rho \mathbb{P}$, where $\varepsilon = \pm 1$. In this context, an important point is to explore the Bakry-Émery-Ricci tensor of such a hypersurface in order to ensure the existence of Omori-Yau type sequences. The existence of these sequences will constitute an important tool in the study of the uniqueness of the hypersurfaces.

In Chapter 4, based in the paper [23], in collaboration with M. Cavalcante and H. de Lima, we prove height estimates concerning compact hypersurfaces with nonzero constant weighted mean curvature and whose boundary is contained into a slice of weighted product spaces of nonnegative Bakry-Émery Ricci curvature (cf. Theorems 4.2.1 and 4.3.1). As applications of our estimates, we obtain nonexistence results related to complete noncompact hypersurfaces properly immersed in these weighted product spaces (cf. Theorems 4.2.2 and 4.3.2).

CHAPTER 1

# Preliminaries

Here we establish the notations, some basic definitions and results, concerning weighted manifolds, that will be used along this thesis.

## 1.1 Curvature and Maximum principle

Given a complete $n$-dimensional Riemannian manifold $(M^n, g)$ and a smooth function $f : M^n \to \mathbb{R}$, the *weighted manifold* $M^n_f$ associated to $M^n$ and $f$ is the triple $(M^n, g, d\mu = e^{-f} dM)$, where $dM$ denotes the standard volume element of $M^n$. Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many others, weighted manifolds are proved to be important nontrivial generalizations of Riemannian manifolds and, nowadays, there are several geometric investigations concerning them. For a brief overview of results in this scope, we refer the article [70] of Wey and Willie. We point out that a theory of Ricci curvature for weighted manifolds goes back to Lichnerowicz [48, 49] and it was later developed by Bakry and Émery in the work [11].

In this thesis we use three notions of Ricci curvature for weighted manifolds. Firstly, given a constant $k > 0$, the *k-Bakry-Émery-Ricci* tensor is given by

$$\mathrm{Ric}_f^k = \mathrm{Ric} + \mathrm{Hess}_f - \frac{1}{k} df \otimes df,$$

where Ric stands for the usual Ricci tensor of $(M, g)$ (see [11]).

The second one is the *$\infty$-Bakry-Émery-Ricci* tensor or just Bakry-Émery-Ricci tensor. It is given by

$$\mathrm{Ric}_f := \mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess}_f.$$

We also consider a more general notion to the Ricci curvature as follows. Given a smooth vector field $V$ on $M$ let $V^*$ be its metric-dual vector field and let denote by $\mathscr{L}_V g$ the Lie derivative. Given a constant $k > 0$, the *modified Ricci tensor* with respect to $V$ and $k$ (see [50]) is defined as

$$\mathrm{Ric}_V^k = \mathrm{Ric} + \mathscr{L}_V g - \frac{1}{k} V^* \otimes V^*.$$

Of course, in the above notations, $\mathrm{Ric}_f^{4k} = \mathrm{Ric}^k_{\frac{\nabla f}{2}}$.

In order to exemplify these notions of Ricci curvature we recall that a *Ricci soliton* is just a weighted manifold satisfying the equation bellow

$$\mathrm{Ric}_f = \lambda g,$$

where $\lambda \in \mathbb{R}$. Moreover, a Ricci soliton is called *shrinking, steady,* or *expanding* when $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Ricci solitons are a generalization of Einstein manifolds and its importance is due to Perelman's solution of Poincaré conjecture. They correspond to self-similar solutions to Hamilton's Ricci flow and often arise as limits of dilations of singularities developed along the Ricci flow. Moreover one defines a metric *quasi-Einstein* just by replacing the Bakry-Émery Ricci tensor by $\mathrm{Ric}_f^k$, see [60] and [56]. In the next chapter we deal with a bigger class of manifolds, because we use inequalities instead equalities.

Now, we define the *$f$-divergence* of a vector field $X \in \mathfrak{X}(M^n)$ as

$$\mathrm{div}_f(X) = e^f \mathrm{div}(e^{-f}X),$$

where div is the usual divergence on $M$. From this, the drift Laplacian it is defined by

$$\Delta_f u = \mathrm{div}_f(\nabla u) = \Delta u - \langle \nabla u, \nabla f \rangle,$$

where $u$ is a smooth function on $M$. We will also refer such operator as the *$f$-Laplacian* of $M$.

From the above definitions we have a Bochner type formula for weighted manifolds. Namely:

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess} u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u), \tag{1.1}$$

where $u$ is a smooth function on $M$. See [70] for more details.

An important tool along this work is the Omori-Yau maximum principle for the $f$-Laplacian. According to [61], we have the following definition

**Definition 1.1.1.** *Let $M_f$ be a weighted manifold. We say that the full Omori-Yau maximum principle for $\Delta_f$ holds if for any $C^2$ function $u : M \longrightarrow \mathbb{R}$ satisfying $\sup_M u = u^* < \infty$ there exists a sequence $\{x_n\} \subset M$ along which*

$$(i)\lim_n u(x_n) = \sup_M u, \;\; (ii)\lim_n |\nabla u(x_n)| = 0 \;\text{ and }\; (iii)\limsup_n \Delta_f u(x_n) \leq 0.$$

*If the condition $(ii)$ don't hold, then we say just that the weak Omori-Yau maximum principle for $\Delta_f$ holds on $M$.*

From the classical Omori-Yau maximum principle (see [57] and [71]) we conclude that if

$|\nabla f|$ is bounded and Ric is bounded from below, then the full Omori-Yau maximum principle for $\Delta_f$ holds on $M$.

On the other hand, using volume growth conditions and stochastic process on a weighted manifold, Rimoldi [61], showed that the validity of the weak Omory-Yau maximum principle is guaranteed by a lower bound on the Bakry-Émery Ricci tensor. Namely:

**Theorem 1.1.1.** *Let $M_f$ be a weighted manifold. If $\mathrm{Ric}_f \geq \lambda$ for some $\lambda \in \mathbb{R}$, then the weak Omori-Yau maximum principle holds on $M$.*

## 1.2 Weighted mean curvature

Here, we define a notion of mean curvature in the weighted context. Indeed, let $\Sigma^n$ be a hypersurface immersed in a weighted Riemannian manifold $M_f^{n+1}$ and denote by $\overline{\nabla}$ the gradient with respect to the metric of $M^{n+1}$. According to Gromov [41], the *weighted mean curvature*, or simply *$f$-mean curvature*, $H_f$ of $\Sigma^n$ is given by

$$nH_f = nH + \langle N, \overline{\nabla} f \rangle, \tag{1.2}$$

where $H$ denotes the standard mean curvature of $\Sigma^n$ with respect to its orientation $N$. In this context, it is natural to consider the first variation for the *weighted volume*

$$\mathrm{vol}_f(\Omega) = \int_\Omega e^{-f} d\Omega,$$

where $\Omega$ is a bounded domain in $\Sigma$. According to Bayle [12] the first variation formula is given by

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}_f(\Omega_t) = \int_\Omega H_f \langle N, V \rangle e^{-f} d\Omega,$$

where $V$ is the variational vector field.

In the Euclidean space $\mathbb{R}^{n+1}$, taking $f = \frac{|x|^2}{2}$, the hypersurfaces with $f$-mean curvature $H_f = 0$ are the well known *self-shrinkers*, that is, a hypersurface immersed in $\mathbb{R}^{n+1}$ satisfying

$$H = \langle x, N \rangle,$$

where $x$ is the position vector in $\mathbb{R}^{n+1}$, see for instance [29].

## 1.3 Hypersurfaces in weighted semi-Riemannian manifolds

In what follows, let us consider an $(n+1)$-dimensional product space $\bar{M}^{n+1}$ of the form $I \times \mathbb{P}^n$, where $I \subset \mathbb{R}$ is an open interval, $\mathbb{P}^n$ is an $n$-dimensional connected Riemannian manifold and $\bar{M}^{n+1}$ is endowed with the standard product metric

$$\langle , \rangle = \varepsilon \pi_I^*(dt^2) + \pi_{\mathbb{P}}^*(\langle , \rangle_{\mathbb{P}}),$$

where $\varepsilon = \pm 1$, $\pi_I$ and $\pi_{\mathbb{P}}$ denote the canonical projections from $I \times \mathbb{P}^n$ onto each factor, and $\langle , \rangle_{\mathbb{P}}$ is the Riemannian metric on $\mathbb{P}^n$. For simplicity, we will just write $\bar{M}^{n+1} = \varepsilon I \times \mathbb{P}^n$ and $\langle , \rangle = \varepsilon dt^2 + \langle , \rangle_{\mathbb{P}}$. In this setting, for a fixed $t_0 \in I$, we say that $\mathbb{P}^n_{t_0} = \{t_0\} \times \mathbb{P}^n$ is a *slice* of $\bar{M}^{n+1}$.

In Chapter 3 and 4, we will consider a connected hypersurface $\Sigma^n$ immersed into $\bar{M}^{n+1}$. In the case where $\bar{M}^{n+1}$ is Lorentzian (that is, when $\varepsilon = -1$) we will assume that $\Sigma^n$ is a *spacelike hypersurface*, that is, the metric induced on $\Sigma^n$ via the immersion is a Riemannian metric. Since $\partial_t$ is a globally defined timelike vector field on $-I \times \mathbb{P}^n$, it follows that there exists an unique unitary timelike normal field $N$ globally defined on $\Sigma^n$ and, therefore, the *function angle* $\Theta = \langle N, \partial_t \rangle$ satisfies $|\Theta| \geq 1$. On the other hand, when $\bar{M}^{n+1}$ is Riemannian (that is, when $\varepsilon = 1$), $\Sigma^n$ is assumed to be a *two-sided hypersurface* in $\bar{M}^{n+1}$. This condition means that there is a globally defined unit normal vector field $N$.

Denoting by $\overline{\nabla}$, $\nabla$ and $\widetilde{\nabla}$ the gradients with respect to the metrics of $\varepsilon I \times_\rho \mathbb{P}^n$, $\Sigma^n$ and $\mathbb{P}$, respectively, a simple computation shows that the gradient of $\pi_I$ on $\bar{M}^{n+1}$ is given by

$$\overline{\nabla} \pi_I = \varepsilon \langle \overline{\nabla} \pi_I, \partial_t \rangle \partial_t = \varepsilon \partial_t. \tag{1.3}$$

So, from (1.3) we conclude that the gradient of the (vertical) height function $h = (\pi_I)|_\Sigma$ of $\Sigma^n$ is given by

$$\nabla h = (\overline{\nabla} \pi_I)^\top = \varepsilon \partial_t^\top = \varepsilon \partial_t - \Theta N, \tag{1.4}$$

where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\bar{M}^{n+1})$ along $\Sigma^n$. Thus, we get the following relation

$$|\nabla h|^2 = \varepsilon(1 - \Theta^2). \tag{1.5}$$

In this setting, the *weighted mean curvature* $H_f$ of $\Sigma^n$ is given by

$$nH_f = nH + \varepsilon \langle \overline{\nabla} f, N \rangle. \tag{1.6}$$

CHAPTER 2

# Compactness of weighted manifolds

The results in this chapter are part of the works [22] and [66].

## 2.1 Introduction

The Myers Theorem has been generalized and its deepness is shown in its many applications. In the following we recall some important generalizations. In the first one, due to Ambrose [8], the condition on the lower bound for the Ricci tensor is replaced by a condition on its integral along geodesics. Namely:

**Theorem A** (Ambrose)**.** *Suppose there exists a point $p$ in a complete Riemannian manifold $M$ for which every geodesic $\gamma(t)$ emanating from $p$ satisfies*

$$\int_0^\infty \mathrm{Ric}(\gamma'(s),\gamma'(s))ds = \infty.$$

*Then $M$ is compact.*

The second important generalization we want to mention here is due to Galloway [38] where a perturbed version of Myers theorem were considered.

**Theorem B** (Galloway)**.** *Let $M^n$ be a complete Riemannian manifold, and $\gamma$ a geodesic joining two points of $M$. Assume that*

$$\mathrm{Ric}(\gamma'(s),\gamma'(s)) \geq a + \frac{d\phi}{dt}$$

*holds along $\gamma$, where $a$ is a positive constant and $\phi$ is any smooth function satisfying $|\phi| \leq c$.*

*Then $M$ is compact and its diameter is bounded from above by*

$$\mathrm{diam}(M) \leq \frac{\pi}{a}\big(c + \sqrt{c^2 + a(n-1)}\big).$$

Finally, we recall an interesting result due to C. Sprouse (see [68]). Namely:

**Theorem C** (Sprouse)**.** *Let $(M,g)$ be a complete Riemannian manifold of dimension $n$ satisfying $\mathrm{Ric}(v,v) \geq -a(n-1)$ for all unit vectors $v$ and some $a > 0$. Then for any $R, \delta > 0$ there* *exists* $\varepsilon = \varepsilon(n,a,R,\delta)$ *such that if*

$$\sup_x \frac{1}{\mathrm{Vol}(B(x,R))}\int_{B(x,R)} \max\{(n-1)-\mathrm{Ric}_-(x),0\}\,d\mathrm{vol} < \varepsilon(n,k,R,\delta),$$

*then* $M$ *is compact with* $\mathrm{diam}(M) \leq \pi+\delta$.

These theorems above have applications in Relativity Cosmology (see [38], [37] and [73]) and in the theory of Ricci Solitons (see [35] and [58]).

In this chapter we generalize Theorems **A**, **B** and **C** to the context of weighted manifolds, actually, we generalize an improved versions of Theorem C due to Yun (see [73]) for complete Riemannian manifolds and for globally hyperbolic spacetimes. Our results have applications for closure theorems of spatial hypersurfaces in mathematical relativity.

## 2.2 Weighted Ambrose's Theorem

We denote by $m_f$ the weighted Laplacian of the distance function from a fixed point $p \in M$. From the Bochner equality (1.1) it is easy to see that $m_f$ satisfies the following Riccati inequality (see Appendix A of [70]):

$$\mathrm{Ric}_f^k(\partial r,\partial r) \leq -m_f' - \frac{1}{k+n-1}m_f^2, \tag{2.1}$$

where $k \in (0,\infty)$ and $m_f'$ stands for the derivative of $m_f$ with respect to $r$.

Now we are in position to state and prove our first result.

**Theorem 2.2.1** (Theorem 2.1 [22])**.** *Let* $M_f$ *be a complete weighted manifold. Suppose there exists a point* $p \in M_f$ *such that every geodesic* $\gamma(t)$ *emanating from* $p$ *satisfies*

$$\int_0^\infty \mathrm{Ric}_f^k(\gamma'(s),\gamma'(s))\,ds = \infty,$$

*where* $k \in (0,\infty)$. *Then* $M$ *is compact.*

*Proof.* Suppose by contradiction that $M$ is not compact and let us assume that $\gamma(t)$ is a unit speed ray issuing from $p$. Then, the function $m_f(t)$ is smooth for all $t>0$ along $\gamma(t)$.

Integrating the inequality (2.1) we obtain

$$\int_1^t \mathrm{Ric}_f^k(\gamma'(s),\gamma'(s))ds \leq \int_1^t (-m_f'(s) - \frac{1}{k+n-1}m_f^2(s))ds.$$

So we conclude that

$$\lim_{t\to\infty}(-m_f(t) - \frac{1}{k+n-1}\int_1^t m_f^2(s)ds) = \infty. \tag{2.2}$$

In particular,

$$\lim_{t\to\infty} -m_f(t) = \infty.$$

In the following we show that there exists a finite number $T > 0$ such that $\lim\limits_{t\to T^-} -m_f(t) = \infty$ which contradicts the smoothness of $m_f(t)$.

From (2.2), given $c > k+n-1$ there exists $t_1 > 1$ such that

$$-m_f(t) - \frac{1}{k+n-1}\int_1^t m_f^2(s)ds \geq \frac{c}{k+n-1} > 1, \tag{2.3}$$

for all $t \geq t_1$, where $n = \dim(M)$.

Let denote by $\alpha = (k+n-1)$ and let us consider $\{t_\ell\}$ the sequence defined inductively by

$$t_{\ell+1} = t_\ell + \alpha\left(\frac{\alpha}{c}\right)^{\ell-1}, \quad \text{for } \ell \geq 1.$$

Notice that $\{t_\ell\}$ is an increasing sequence converging to $T = t_1 + \alpha\frac{c/\alpha}{(c/\alpha)-1}$.

Given $\ell \in \mathbb{N}$ we claim that $-m_f(t) \geq \left(\frac{c}{\alpha}\right)^\ell$ for all $t \geq t_\ell$. In fact, from inequality (2.3) we have that it is true for $\ell = 1$. Now, assume that claim holds for all $t \geq t_\ell$ and fix $t \geq t_{\ell+1}$. Then using inequality (2.3) again:

$$\begin{aligned}
-m_f(t) &\geq \frac{c}{k+n-1} + \frac{1}{k+n-1}\int_1^{t_\ell} m_f^2(s)ds + \frac{1}{k+n-1}\int_{t_\ell}^{t_{\ell+1}} m_f^2(s)ds \\
&\geq \frac{1}{k+n-1}\int_{t_\ell}^{t_{\ell+1}} m_f^2(s)ds \\
&\geq \frac{1}{(k+n-1)}\frac{c^{2\ell}}{(k+n-1)^{2\ell}}\frac{\alpha^\ell}{c^{\ell-1}} = \left(\frac{c}{\alpha}\right)^{\ell+1}.
\end{aligned}$$

In particular, $\lim\limits_{t\to T^-} -m_f(t) = \infty$ which is the desired contraction.

□

Using the same techniques we are able to prove Ambrose's theorem for the ∞-Bakry-Émery Ricci tensor. In this case a condition on $f$ is required. See also [74]. Namely:

**Theorem 2.2.2** (Theorem 2.2 [22]). *Let $M_f$ be a complete weighted manifold, where $\frac{df}{dt} \leq 0$ along $\gamma$. Suppose there exists a point $p \in M_f$ such that every geodesic $\gamma(t)$ emanating from $p$ satisfies*

$$\int_0^\infty \mathrm{Ric}_f(\gamma'(s), \gamma'(s))\, ds = \infty.$$

*Then $M$ is compact.*

Applying this theorem to the universal cover of $M$ we obtain the finiteness of the fundamental group.

**Corollary 2.2.1.** *Let $(M,g)$ be a Riemannian manifold in the conditions of either Theorem 2.2.1 or Theorem 2.2.2. Then the first fundamental group of $M$ is finite.*

## 2.3 Weighted Galloway's Theorems

Our first result in this section is Galloway's Theorem for the modified Ricci tensor.

**Theorem 2.3.1** (Theorem 3.1 [22]). *Let $M^n$ be a complete Riemannian manifold and let $V$ be a smooth vector field on $M$. Suppose that for every pair of points in $M^n$ and any normalized minimizing geodesic $\gamma$ joining these points the modified Ricci tensor satisfies*

$$\mathrm{Ric}_V^k(\gamma', \gamma') \geq (n-1)c + \frac{d\phi}{dt}, \tag{2.4}$$

*where $k$ and $c$ are positive constants and $\phi$ is a smooth function such that $|\phi| \leq b$ for some $b \geq 0$. So $M$ is compact and*

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{(n-1)c}} \left[ \frac{b}{\sqrt{(n-1)c}} + \sqrt{\frac{b^2}{(n-1)c} + n - 1 + 4k} \right].$$

*Proof.* Let $p,q$ be distinct points in $M$ and let $\gamma$ be a normalized geodesic that minimizes distance between $p$ and $q$. Assume that the length of $\gamma$ is $\ell$. Consider a parallel orthonormal frame $\{E_1 = \gamma', E_2, ...., E_n\}$ along $\gamma$. Let $h \in C^\infty([0,l])$ such that $h(0) = h(\ell) = 0$ and set $V_i(t) = h(t)E_i(t)$ along $\gamma$. Firstly, the index formula implies

$$I(V_i, V_i) = \int_0^\ell ((h')^2 - h^2 \langle R(\gamma', E_i)\gamma', E_i \rangle) dt, \quad i = 2, \ldots n.$$

So

$$S := \sum_{i=2}^{n} I(V_i, V_i) = \int_0^\ell \left\{(n-1)(h')^2 - h^2 \mathrm{Ric}(\gamma', \gamma')\right\} dt. \tag{2.5}$$

From the condition (2.4) we have

$$S \leq \int_0^\ell (n-1)\left[(h')^2 - h^2 c\right] dt + \int_0^\ell h^2 \left[-\frac{d\phi}{dt} + 2\frac{d}{dt}\langle \gamma', V\rangle - \frac{1}{k}\langle \gamma', V\rangle^2\right] dt.$$

On the other hand, integrating by parts we obtain

$$2\int_0^\ell h^2 \frac{d}{dt}\langle \gamma', V\rangle dt \leq \int_0^\ell 4k(h')^2 + \frac{1}{k}h^2 \langle \gamma', V\rangle^2 dt.$$

So,

$$S \leq (n-1+4k)\int_0^\ell (h')^2 dt - (n-1)c\int_0^\ell h^2 dt - \int_0^\ell h^2 \frac{d\phi}{dt} dt. \tag{2.6}$$

Now we choose $h(t) = \sin\left(\frac{\pi t}{\ell}\right)$.
Then from (2.6), we have

$$S \leq \frac{\pi^2}{2\ell}(n-1+4k) - (n-1)c\frac{\ell}{2} - \int_0^\ell h^2 \frac{d\phi}{dt} dt.$$

Integrating by parts once more we get

$$\int_0^\ell h^2 \frac{d\phi}{dt} dt = -\frac{\pi}{\ell}\int_0^\ell \phi \sin\left(\frac{2\pi t}{\ell}\right) dt \geq -b\pi.$$

Thus we have

$$S \leq \frac{1}{2\ell}\left\{-(n-1)c\ell^2 + 2b\pi\ell + \pi^2(n-1+4k)\right\}.$$

Finally, because $\gamma$ is minimizing we have $S \geq 0$ and therefore

$$\ell \leq \frac{\pi}{\sqrt{(n-1)c}}\left[\frac{b}{\sqrt{(n-1)c}} + \sqrt{\frac{b^2}{(n-1)c} + n - 1 + 4k}\right].$$

It finishes the proof.

□

Following some ideas of [51] we also obtain an extension of Galloway's Theorem for weighted manifolds using the ∞-Bakry-Émery-Ricci tensor.

**Theorem 2.3.2** (Theorem 3.2 [22]). *Let $M_f$ be a weighted complete Riemannian manifold with $|f| \leq a$, where $a$ is a positive constant. Suppose there exist constants $c > 0$ and $b \geq 0$ such that* *for every pair of points in $M_f$ and normalized minimizing geodesic $\gamma$ joining these points we have*

$$\mathrm{Ric}_f(\gamma', \gamma') \geq (n-1)c + \frac{d\phi}{dt}, \tag{2.7}$$

*where $\phi$ is a smooth function satisfying $|\phi| \leq b$. Then $M$ is compact and*

$$\mathrm{diam}(M) \leq \frac{\pi}{c(n-1)}\left[b + \sqrt{b^2 + c(n-1)\lambda}\right], \tag{2.8}$$

*where $\lambda = 2\sqrt{2}a + (n-1)$.*

*Proof.* From (2.7) and (2.5) we have

$$S \leq \int_0^\ell (n-1)\left\{(h')^2 - h^2 c\right\} dt + \int_0^\ell h^2 \mathrm{Hess}_f(\gamma', \gamma') dt - \int_0^\ell h^2 \frac{d\phi}{dt} dt. \tag{2.9}$$

We note that

$$\begin{aligned} h^2 \mathrm{Hess}_f(\gamma', \gamma') &= h^2 \frac{d}{dt}\left\langle \nabla f, \gamma' \right\rangle dt \\ &= \frac{d}{dt}(h^2 \left\langle \nabla f, \gamma' \right\rangle) - 2hh' \left\langle \nabla f, \gamma' \right\rangle \\ &= \frac{d}{dt}(h^2 \left\langle \nabla f, \gamma' \right\rangle) - 2\left[\frac{d}{dt}(fhh') - f\frac{d}{dt}(hh')\right]. \end{aligned}$$

Integrating the last identity above we have

$$\int_0^\ell h^2 \mathrm{Hess}_f(\gamma', \gamma') dt = 2\int_0^\ell f \frac{d}{dt}(hh') dt,$$

since $h(0) = h(\ell) = 0$.

Thus

$$\int_0^\ell h^2 \mathrm{Hess}_f(\gamma', \gamma') dt \leq 2a\sqrt{\ell}\left(\int_0^\ell \left[\frac{d}{dt}(hh')\right]^2 dt\right)^{\frac{1}{2}}.$$

Choosing $h(t) = \sin(\frac{\pi t}{\ell})$ we have

$$\int_0^\ell h^2 \mathrm{Hess}_f(\gamma', \gamma') dt \leq \frac{a\sqrt{2}\pi^2}{\ell}. \tag{2.10}$$

On the other hand a direct computation yelds

$$\int_0^\ell (n-1)(h'^2 - ch^2)dt = (n-1)\left[\frac{\pi^2}{2\ell} - \frac{c\ell}{2}\right]. \tag{2.11}$$

Therefore, from (2.9), (2.10) and (2.11) we have

$$S \le -\frac{1}{2\ell}\left[-2a\pi^2\sqrt{2} + c(n-1)\ell^2 - \pi^2(n-1)\right] + \pi b \tag{2.12}$$

Since $S \ge 0$ we get

$$\ell \le \frac{\pi}{c(n-1)}\left[b + \sqrt{b^2 + c(n-1)\lambda}\right],$$

where $\lambda = 2\sqrt{2}a + (n-1)$ and so $M$ is compact and satisfies (2.8). □

## 2.4 Weighted Riemannian Yun-Sprouse's Theorem

In this section, following the ideas of [73], we provide a weighted version of **Theorem C** as follows.

**Theorem 2.4.1** (Theorem 1.1 [66]). *Let $M_f^n$ be a weighted complete Riemannian manifold. Then for any $\delta > 0$, and $a > 0$, there exists an $\varepsilon = \varepsilon(n,a,\delta)$ satisfying the following:*

*If there is a point $p$ such that along each geodesic $\gamma$ emanating from $p$, the $\mathrm{Ric}_f^k$ curvature satisfies*

$$\int_0^\infty \max\left\{(n-1)a - \mathrm{Ric}_f^k(\gamma',\gamma'), 0\right\} dt < \varepsilon(n,a,\delta) \tag{2.13}$$

*then $M$ is compact with* $\mathrm{diam}(M) \le \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$.

*Proof.* For any small positive $\varepsilon < a^2$ to be determined later, consider the following sets

$$E_1 = \left\{t \in [0,\infty); \mathrm{Ric}_f^k(\gamma'(t),\gamma'(t)) \ge (n-1)(a - \sqrt{\varepsilon})\right\}$$

and

$$E_2 = \left\{t \in [0,\infty); \mathrm{Ric}_f^k(\gamma'(t),\gamma'(t)) < (n-1)(a - \sqrt{\varepsilon})\right\}.$$

From the inequality (2.1) we have on $E_1$

$$\frac{\frac{m_f'}{n+k-1}}{\left(\frac{m_f}{n+k-1}\right)^2 + \frac{(n-1)(a-\sqrt{\varepsilon})}{n+k-1}} \le -1. \tag{2.14}$$

On the other hand, on $E_2$ we have

$$\frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2+\frac{(n-1)(a-\sqrt{\varepsilon})}{n+k-1}} \leq \frac{a-\sqrt{\varepsilon}-\mathrm{Ric}_f^k(\gamma',\gamma')/(n-1)}{a-\sqrt{\varepsilon}}. \tag{2.15}$$

Now, using the assumption (2.13) on the Bakry-Émery Ricci tensor we get

$$\begin{aligned}
\varepsilon &> \int_0^\infty \max\left\{(n-1)a-\mathrm{Ric}_f^k(\gamma'(t),\gamma'(t)),0\right\} \\
&> \int_{E_2}\left\{(n-1)a-\mathrm{Ric}_f^k(\gamma'(t),\gamma'(t))\right\}dt \\
&> \int_{E_2}\left\{(n-1)a-(n-1)(a-\sqrt{\varepsilon})\right\}dt \\
&= \mu(E_2)(n-1)\sqrt{\varepsilon}.
\end{aligned}$$

That is,

$$\mu(E_2) < \frac{\sqrt{\varepsilon}}{n-1}, \tag{2.16}$$

where $\mu$ is the Lebesgue measure on $\mathbb{R}$.

Using inequalities (2.14)−(2.16) we obtain

$$\begin{aligned}
\int_0^r \frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2+\frac{(n-1)(a-\sqrt{\varepsilon})}{n+k-1}}dt &\leq \int_{[0,r]\cap E_1}\frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2+\frac{(n-1)(a-\sqrt{\varepsilon})}{n+k-1}}dt \\
&\quad + \int_{[0,r]\cap E_2}\frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2+\frac{(n-1)(a-\sqrt{\varepsilon})}{n+k-1}}dt \\
&\leq -\mu\{[0,r]\cap E_1\}+\frac{\varepsilon}{(n-1)(a-\sqrt{\varepsilon})} \\
&\leq -r+\mu\{[0,r]\cap E_2\}+\frac{\varepsilon}{(n-1)(a-\sqrt{\varepsilon})} \\
&\leq -r+\frac{\sqrt{\varepsilon}}{n-1}+\frac{\varepsilon}{(n-1)(a-\sqrt{\varepsilon})}.
\end{aligned}$$

Define $\tau(\varepsilon)=\frac{\sqrt{\varepsilon}}{n-1}+\frac{\varepsilon}{(n-1)(a-\sqrt{\varepsilon})}$. The integral of the left hand side can be computed explicitly and therefore we get

$$\arctan\left(\frac{m_f(r)}{(n+k-1)a(\varepsilon)}\right) \leq a(\varepsilon)(-r+\tau(\varepsilon))+\frac{\pi}{2},$$

where $a(\varepsilon) = \sqrt{\frac{(n-1)(a-\sqrt{\varepsilon})}{n+k-1}}$.

So

$$m_f(r) \le -(n+k-1)a(\varepsilon)\cot(a(\varepsilon)(-r+\tau(\varepsilon))),$$

for any $r$ such that $\tau(\varepsilon) < r < \frac{\pi}{a(\varepsilon)} + \tau(\varepsilon)$.

In particular, $m_f(\gamma(r))$ goes to $-\infty$ as $r \to (\frac{\pi}{a(\varepsilon)} + \tau(\varepsilon))^+$. It implies that $\gamma$ can not be minimal beyond $\frac{\pi}{a(\varepsilon)} + \tau(\varepsilon)$. Otherwise $m_f$ would be a smooth function at $r = \frac{\pi}{a(\varepsilon)} + \tau(\varepsilon)$. Taking $\varepsilon$ explicitly so that $\frac{\pi}{a(\varepsilon)} + \tau(\varepsilon) = \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$ and using the completeness of $M$ we have the desired result. $\square$

## 2.5 Weighted Lorentzian Yun-Sprouse's Theorem

Now let us discuss the Lorentzian version of Theorem 2.4.1. Let $M$ be a time-oriented Lorentzian manifold. Given $p \in M$ we set

$$J^+(p) = \{q \in M : \text{ there exist a future pointing causal curve from } p \text{ to } q\},$$

called the *causal future* of $p$. The *causal past* $J^-(p)$ is defined similarly. We say that $M$ is *globally hyperbolic* if the set $J(p,q) := J^+(p) \cap J^-(q)$ is compact for all $p$ and $q$ joined by a causal curve (see [13]). Mathematically, global hyperbolicity often plays a role analogous to geodesic completeness in Riemannian geometry.

Let $\gamma : [a,b] \longrightarrow M$ be a future-directed timelike unit-speed geodesic. Given $\{E_1, E_2, \dots, E_n\}$ an orthonormal frame field along $\gamma$, and for each $i \in \{1, \dots, n\}$ we let $J_i$ be the unique Jacobi field along $c$ such that $J_i(a) = 0$ and $J_i'(0) = E_i$. Denote by $A$ the matrix $A = [J_1 J_2 \dots J_n]$, where each column is just the vector for $J_i$ in the basis defined by $\{E_i\}$. In this situation, we have that $A(t)$ is invertible if and only if $\gamma(t)$ is not conjugate to $\gamma(a)$.

Now we define $B_f = A'A^{-1} - \frac{1}{n-1}(f \circ \gamma)'E$ wherever $A$ is invertible, where $E(t)$ is the identity map on $(\gamma'(t))^{\perp}$. The *$f$-expansion function* is a smooth function defined by $\theta_f = \mathrm{tr} B_f$ (see [21, Definition 2.6]) Note that, if $|\theta_f| \to \infty$ as $t \to t_0$, where $t_0 \in [a,b]$, then $\gamma(t_0)$ is conjugate to $\gamma(a)$.

Recently, Case in [21] obtained the following relation between the Bakry-Emery Ricci Tensor and the *$f$-expansion function* $\theta_f$:

**Lemma 2.5.1.** *Under the above notations,*

$$\theta_f' \leq -\mathrm{Ric}_f^k(\gamma',\gamma') - \frac{\theta_f^2}{k+n-1}. \tag{2.17}$$

The inequality (2.17) is called $(k,f)$-*Raychaudhuri inequality*. This inequality is a generalization of the well known Raychaudhuri inequality (see for instance [36]).

The *distance* between two timelike related points is the supremum of lengths of causal curves joining the points. It follows that the distance between any two timelike related points in a globally hyperbolic spacetime is the length of such a maximal timelike geodesic. The *timelike diameter*, $\mathrm{diam}(M)$, of a Lorentzian manifold is defined to be the supremum of distances $d(p,q)$ between points of $M$.

**Theorem 2.5.1** (Theorem [66]). *Let $M_f^n$ be a weighted globally hyperbolic spacetime. Then for any $\delta > 0$, $a > 0$, there exists an $\varepsilon = \varepsilon(n,a,\delta)$ satisfying the following:*

*If there is a point $p$ such that along each future directed timelike geodesic $\gamma$ emanating from $p$, with $l(\gamma) = \sup\{t \geq 0, d(p,\gamma(t)) = t\}$ the $\mathrm{Ric}_f^k$ curvature satisfies*

$$\int_0^{l(\gamma)} \max\left\{(n-1)a - \mathrm{Ric}_f^k(\gamma',\gamma'), 0\right\} dt < \varepsilon(n,a,\delta),$$

*then the timelike diameter satisfies* $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$.

*Proof.* Let $\varepsilon(n,a,\delta)$ be the explicit constant in the previous theorem. Assume by contradiction that there are two points $p$ and $q$ with $d(p,q) > \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$. On the other hand, since $M$ is globally hyperbolic, there exists a maximal timelike geodesic $\gamma$ joining $p$ and $q$ such that $\ell(\gamma) = d(p,q)$. Following the steps of the proof of Theorem 2.4.1 using the $(k,f)$-Raychaudhuri inequality (2.17) we get

$$\lim_{t\to t_0^+} \theta_f(t) = -\infty,$$

where $t_0 = \frac{\pi}{a(\varepsilon)} + \tau(\varepsilon)$. So, we conclude that $\gamma$ cannot be maximal beyond $\frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$ which is a contradiction.

□

An immediate consequence of Theorem 2.5.1 is the Lorentzian version of the original weighted Myers Theorem obtained by Qian in [59]. Namely:

**Corollary 2.5.1.** *Let $M^n_f$ be a weighted globally hyperbolic spacetime. Let $a$ be a positive constant and assume that $\mathrm{Ric}^k_f(v,v) \geq (n-1)a$, for all unit timelike vector field $v \in TM$. Then, the timelike diameter satisfies $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}}$.*

## 2.6 Closure Theorem via Galloway's Theorem

Let $\bar{M}^{n+1}$ be an $(n+1)$-dimensional *spacetime manifold*, that is, a smooth manifold endowed with a pseudo-Riemannian metric $ds^2$ of signature one. Assume that $\bar{M}^{n+1}$ is complete and time orientable.

An interesting problem in mathematical relativity is to determine whether space-like slices of spacetime manifolds are compact. In this section we present some theorems in this direction, although our results are purely geometric. For the sake of simplicity we will omit the dimension super index and we always use a *bar* for geometric objects related to $\bar{M}$.

Let us denote by $\mathbf{u}$ a unit time-like vector field on $\bar{M}$. Let us assume that $\mathbf{u}$ is *irrotational*, that is the bracket of two vectors orthogonal to $\mathbf{u}$ is still orthogonal to $\mathbf{u}$. In this case, from Frobenius theorem, at any point of $\bar{M}$ there exists a complete connected $n$-dimensional hypersurface $M$ orthogonal to $\mathbf{u}$. Physically, we may interpret the manifold $M$ as the spatial universe at some moment in time. Moreover, in each point of $M^n$ there exist a local coordinate neighborhood $U^n$ and a local coordinate $t$ with values in the interval $(-\varepsilon,\varepsilon)$ such that $W = (-\varepsilon,\varepsilon) \times U$ is an open neighborhood of a point in $M$ and $U_t = \{t\} \times U$ is orthogonal to $\mathbf{u}$.

In $W$, the spacetime metric is given by

$$ds^2 = -\varphi^2 dt^2 + \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(x,t)dx^\alpha dx^\beta,$$

where $x^\alpha$ are local coordinates introduced in $U^n$ and $\mathbf{u} = \frac{1}{\varphi}\frac{\partial}{\partial t}$.

Let $\bar{X}$ be a vector field tangent to $\bar{M}$. $\bar{X}$ is called *invariant under the flow* if

$$[\bar{X},\mathbf{u}] = \overline{\nabla}_{\mathbf{u}}\bar{X} - \overline{\nabla}_{\bar{X}}\mathbf{u} = 0.$$

Denote $\bar{X}^T$ by the projection of the vector field $\bar{X}$ on $M$. So,

$$\bar{X}^T = X + \langle \bar{X},\mathbf{u}\rangle \mathbf{u}.$$

Let $X$ be a vector field tangent to $M$. Extend $X$ along the flow by making it invariant under the flow generated by $\mathbf{u}$. In this setting, we define the *velocity* and *acceleration* of $X$ by the vector fields given respectively by

$$v(X) = \overline{\nabla}_{\mathbf{u}}\overline{X}^T \quad \text{and} \quad a(X) = \overline{\nabla}_{\mathbf{u}}\overline{\nabla}_{\mathbf{u}}\overline{X}^T.$$

The shape operator of $M$ as a hypersurface of $\bar{M}$ is defined by $b(X) = -\overline{\nabla}_X \mathbf{u}$. Let denote by $B$ the second fundamental form of $b$ and by $H$ its *mean curvature* function. We point out that $H = -\overline{\mathrm{div}}\mathbf{u}$, that is, the averaged *Hubble expansion parameter* at points of $M$ in relativistic cosmology (see [62] §3.3.1 or [36], page 161.).

The following lemma is an application of Gauss equation for space-like submanifolds.

**Lemma 2.6.1.** *Let $\gamma(s)$ be a geodesic in a weighted manifold $M_f$ with unit tangent $X$. Then*

$$\begin{aligned} \mathrm{Ric}_f(X,X) &= \overline{\mathrm{Ric}}_f(X,X) - \langle a(X),X\rangle + \langle v(X),X\rangle H_f + \langle v(X),X\rangle^2 + \langle \overline{\nabla}_{\mathbf{u}}\mathbf{u},X\rangle^2 \\ &+ \frac{d}{ds}\langle X,\overline{\nabla}_{\mathbf{u}}\mathbf{u}\rangle + \sum_{j=2}^{n}\langle v(X),e_j\rangle^2 + \left(\frac{1}{\varphi}\frac{\partial\varphi}{\partial s}\right)^2, \end{aligned}$$

*where $\{e_1 = X, e_2, \ldots, e_n\}$ is an orthonormal basis of $T_pM$ and $H_f = H + \langle \mathbf{u}, \overline{\nabla} f\rangle$ denotes the $f$-mean curvature of $M_f$.*

*Proof.* Let $\{\bar{e}_1(t) = \bar{X}(t), \bar{e}_2(t), \ldots, \bar{e}_n(t)\}$ be a set invariant under the flow generated by $\mathbf{u}$, where $\alpha = 1, \ldots, n$. Since $[\bar{e}_\alpha, \mathbf{u}] = 0$ we get that

$$\overline{\nabla}_{\bar{e}_\alpha^T}\mathbf{u} = \overline{\nabla}_{\mathbf{u}}\bar{e}_\alpha^T - \langle \overline{\nabla}_{\mathbf{u}}\mathbf{u}, \bar{e}_\alpha^T\rangle \mathbf{u}.$$

Therefore, a straightforward calculation shows that

$$\langle v(e_\alpha), e_\alpha\rangle = -B(e_\alpha, e_\alpha) \quad \text{and} \quad \widetilde{B}(X, e_\alpha) = \langle v(X), e_\alpha\rangle^2.$$

From the above identities and the Gauss equation we get

$$\begin{aligned} \mathrm{Ric}(X,X) &= \sum_{j=2}^{n}\left\{\overline{K}(X,e_j) - B(X,X)B(e_j,e_j) + \widetilde{B}(X,e_j)\right\} \\ &= \sum_{j=2}^{n}\overline{K}(X,e_j) - B(X,X)(H - B(X,X)) + \sum_{j=2}^{n}\widetilde{B}(X,e_j) \\ &= \overline{\mathrm{Ric}}(X,X) - \overline{K}(X,\mathbf{u}) - B(X,X)(H - B(X,X)) + \sum_{j=2}^{n}\widetilde{B}(X,e_j) \\ &= \overline{\mathrm{Ric}}(X,X) - \overline{K}(X,\mathbf{u}) + \langle v(X),X\rangle H + \langle v(X),X\rangle^2 + \sum_{j=2}^{n}\langle v(X),e_j\rangle^2. \end{aligned}$$

On the other hand, note that

$$\begin{aligned}
\overline{K}(X,\mathbf{u}) &= -\langle \overline{R}(\overline{X}^T,\mathbf{u})\mathbf{u},\overline{X}^T\rangle \\
&= \langle \overline{\nabla}_{\mathbf{u}}\overline{\nabla}_{\overline{X}^T}u,X\rangle - \langle \overline{\nabla}_{\overline{X}^T}\overline{\nabla}_{\mathbf{u}}\mathbf{u},\overline{X}^T\rangle + \langle \overline{\nabla}_{[\overline{X}^T,\mathbf{u}]}\mathbf{u},\overline{X}^T\rangle \\
&= \langle a(X),X\rangle - \langle \overline{\nabla}_{\mathbf{u}}\mathbf{u},X\rangle^2 - \frac{d}{ds}\langle X,\overline{\nabla}_{\mathbf{u}}\mathbf{u}\rangle - \left(\frac{1}{\varphi}\frac{\partial\varphi}{\partial s}\right)^2.
\end{aligned}$$

Note that we may write

$$\nabla f = \overline{\nabla} f + \left\langle \overline{\nabla} f,\mathbf{u}\right\rangle \mathbf{u},$$

and, therefore, we have

$$\overline{\mathrm{Hess}}_f(X,X) = \mathrm{Hess}_f(X,X) - \mathbf{u}(f)\,\langle v(X),X\rangle\,.$$

Thus, using the above equations we obtain the desired result. □

Now inspired by the ideas of Galloway [38] we can find a diameter estimate $M$, under some restrictions.

**Theorem 2.6.1** (Theorem 4.5 [22]). *Let $\bar{M}_f^{n+1}$ be a weighted space time with $|f| \le a$, where $a$ is a positive constant. Let $\mathbf{u}$ be an irrotational unit time-like vector field on $\bar{M}_f^{n+1}$ and let $M^n$ be a complete spatial hypersurface orthogonal to $\mathbf{u}$. Suppose that the following conditions hold on $M^n$.*

*(i) At each point $p \in M^n$, the flow generated by $\mathbf{u}$ is expanding in all directions, i.e,*

$$\langle v(X),X\rangle \ge 0,$$

*where $X$ is a vector field tangent to $M$.*

*(ii) At each point $p \in M^n$, the rate of expansion is decreasing in all directions, i.e,*

$$\langle a(X),X\rangle \le 0,$$

*where $X$ is a vector field tangent to $M$.*

*(iii) $\overline{\mathrm{Ric}}_f$ of $\bar{M}^{n+1}$ satisfies*

$$\inf(\overline{\mathrm{Ric}}_f(\zeta,\zeta) - \frac{n-1}{2n}H^2) = c > 0,$$

*where* $\zeta \in T_pM$, $|\zeta| = 1$, $\langle \zeta, u\rangle = 0$ *and* $H = -\overline{\mathrm{div}}\,\mathrm{u}$.

*(iv) The flow lines are of bounded geodesic curvature on* $M^n$, *i.e,*

$$\sup|\overline{\nabla}_{\mathbf{u}}\mathbf{u}| = \mu < \infty.$$

*(v) On* $M^n$, *we have that*

$$\left\langle \overline{\nabla} f, \mathbf{u} \right\rangle = \mathbf{u}(f) \geq 0.$$

*Then* $M^n$ *is compact and*

$$\mathrm{diam}(M^n) \leq \frac{\pi}{c}\left(\mu + \sqrt{\mu^2 + c\lambda}\right), \tag{2.18}$$

*where* $\lambda = 2(\sqrt{2}a + 1)$.

*Proof.* Let us assume that $\{X = e_1, e_2 \ldots, e_n\}$ is an orthonormal basis of $M$. Consider the notation $v_\alpha = \langle v(e_\alpha), e_\alpha\rangle$. From item $(i)$, $v_\alpha \geq 0$. Note that $H = \sum_{i=1}^{n} v_i$. Then by Schwartz inequality

$$\begin{aligned}
-(\langle v(X), X\rangle^2 + \langle v(X), X\rangle H) &\leq \frac{1}{2}\left[\left(\sum_{i=1}^{n} v_i\right)^2 - \sum_{i=1}^{n} v_i^2\right] \\
&\leq \frac{n-1}{2n}\left(\sum_{i=1}^{n} v_i\right)^2 \\
&= \frac{n-1}{2n}H^2.
\end{aligned} \tag{2.19}$$

From (2.19), Lemma 2.6.1 and item $(ii)$ we get

$$\mathrm{Ric}(X,X) \geq \overline{\mathrm{Ric}}(X,X) - \frac{n-1}{2n}H^2 + \frac{d\phi}{ds}, \tag{2.20}$$

where $\phi = \left\langle X, \overline{\nabla}_{\mathbf{u}}\mathbf{u} \right\rangle$.

Then

$$\mathrm{Ric}_f(X,X) \geq \left(\overline{\mathrm{Ric}}_f(X,X) - \frac{n-1}{2n}H^2\right) + \frac{d\phi}{ds} + \langle B(X,X), \mathbf{u}\rangle\, \mathbf{u}(f). \tag{2.21}$$

Item $(i)$ gives $\langle B(X,X),\mathbf{u}\rangle \leq 0$. From (2.21), $(iii)$ and $(v)$ we get

$$\mathrm{Ric}_f(\zeta,\zeta) \geq c + \frac{d\phi}{dt}.$$

Therefore by Theorem 2.3.2 we have that $M$ is compact and satisfies (2.18).

□

Analogously to the previous theorem, we give a closure theorem for the $k$-Bakry-Émery-Ricci tensor as follows.

**Theorem 2.6.2** (Theorem 4.6 [22]). *Let $\bar{M}_f^{n+1}$ be a weighted space time. Let $\mathbf{u}$ be an irrotational unit time-like vector field on $\bar{M}_f^{n+1}$ and $M^n$ be a complete space-like hypersurface orthogonal to $\mathbf{u}$. Suppose the following conditions hold on $M^n$.*

*(i) At each point $p \in M^n$, the flow generated by $\mathbf{u}$ is expanding in all directions, i.e,*

$$\langle v(X),X\rangle \geq 0,$$

*where $X$ is avector field tangent to $M$.*

*(ii) At each point $p \in M^n$, the rate of expansion is decreasing in all directions, i.e,*

$$\langle a(X),X\rangle \leq 0,$$

*where $X$ is a vector field tangent to $M$.*

*(iii) $\overline{\mathrm{Ric}}_f^k$ of $\bar{M}^{n+1}$ satisfies*

$$\inf(\overline{\mathrm{Ric}}_f^k(\zeta,\zeta) - \frac{n-1}{2n}H^2) = c > 0,$$

*where $k \in (0,\infty)$, $\zeta \in T_pV$, $|\zeta| = 1$, $\langle \zeta,u\rangle = 0$ and $h = -div\mathbf{u}$.*

*(iv) The flow lines are of bounded geodesic curvature on $M^n$, i.e,*

$$\sup|\overline{\nabla}_{\mathbf{u}}\mathbf{u}| = \mu < \infty.$$

*(v) On $M^n$, we have that*

$$\left\langle \overline{\nabla} f,\mathbf{u}\right\rangle = \mathbf{u}(f) \geq 0.$$

*Then $M^n$ is compact and*

$$\operatorname{diam}(M^n) \leq \frac{\pi}{c}\left(\mu + \sqrt{\mu^2 + c\lambda}\right),$$

*where* $\lambda = 2(\sqrt{2}a + 1)$.

## 2.7 Closure Theorem via Ambrose's Theorem

Let $M^n$ be a spatial hypersurface in a weighted spacetime $M_f^{n+1}$. The unit tangent vectors to the future directed geodesics orthogonal to $M^n$ define a smooth unit timelike vector field $\mathbf{u}$ at least in a neighborhood of $M^n$

Let $X$ be a vector tangent to $M$. Extended $X$ along the normal geodesic by making it invariant under the flow generated by $\mathbf{u}$. Following the ideas of Frankel and Galloway [37] we obtain the following closure theorem.

**Theorem 2.7.1** (Theorem 4.2 [22])**.** *Let $M^n$ be a spatial hypersurface in a weighted spacetime $(\bar{M}^{n+1}, f)$. Let $\mathbf{u}$ be a unit normal vector field time-like to $M$. If there is a point $q \in M^n$ such that along each geodesic $\gamma$ of $M^n$ emanating from $q$, we have $\frac{df}{dt} \leq 0$ and*

$$\int_0^\infty \left\{\overline{\operatorname{Ric}}_f(X,X) + H_f\langle v(X), X\rangle - \langle a(X), X\rangle\right\} dt = \infty, \tag{2.22}$$

*where $t$ is arc length along $\gamma$ and $X$ is the unit tangent to $\gamma$, then $M^n$ is compact.*

*Proof.* From Lemma (2.6.1) and identity (2.22) we get

$$\int_0^\infty \operatorname{Ric}_f(X,X)dt = \infty.$$

From Theorem 2.2.2 we conclude that $M^n$ is compact. □

Analogously, we get the following closure theorem.

**Theorem 2.7.2** (Theorem 4.3 [22])**.** *Let $M^n$ be a spatial hypersurface in a weighted space time $\bar{M}_f^n$. Let $\mathbf{u}$ be a unit normal vector field time-like to $M$. If there is a point $q \in M^n$ such that along each geodesic $\gamma$ of $M^n$ emanating from $q$, we have*

$$\int_0^\infty \left\{\overline{\operatorname{Ric}}_f^k(X,X) + H_f\langle v(X), X\rangle - \langle a(X), X\rangle\right\} dt = \infty, \tag{2.23}$$

*where $t$ is arc length along $\gamma$ and $X$ is the unit tangent to $\gamma$, then $M^n$ is compact.*

**Remark 2.7.1.** *Conditions* (2.22) *and* (2.23) *may be interpreted as the mass-energy density on* $M^n$ *(see [37]). So, roughly speaking Theorems 2.7.1 and 2.7.2 say that spatial hypersurfaces with mass-energy density sufficiently large are compact.*

Now, using the same steps of the above theorem and Theorem 2.4.1 we get the following closure theorem.

**Theorem 2.7.3** (Theorem 1.3 [66]). *Let* $M^n$ *be a space-like hypersurface in* $\bar{M}_f^{n+1}$ *and assume that* $M$ *is complete in the induced metric. Then for any* $\delta > 0$, $a > 0$, *there exists an* $\varepsilon = \varepsilon(n,a,\delta)$ *satisfying the following:*

*If there is a point* $p \in M$ *such that along each geodesic* $\gamma$ *in* $M$ *emanating from* $p$, *the condition*

$$\int_0^\infty \max\left\{(n-1)a - \overline{\mathrm{Ric}}_f^k(X,X) - \langle v(X),X\rangle H_f + \langle a(X),X\rangle, 0\right\} dt < \varepsilon(n,a,\delta),$$

*is satisfied, where* $X = \gamma'$, *then* $M$ *is compact and* $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$.

CHAPTER 3

# Bernstein type properties of complete hypersurfaces in weighted semi-Riemannian manifolds

The results of this chapter are part of [24] and [25].

## 3.1 Introduction

The last years have seen a steadily growing interest in the study of the Bernstein-type properties concerning complete hypersurfaces immersed in a warped product of the type $I \times_\rho \mathbb{P}^n$, where $\mathbb{P}^n$ is a connected $n$-dimensional oriented Riemannian manifold, $I \subset \mathbb{R}$ is an open interval and $\rho$ is a positive smooth function defined on $I$. In this context, an important question to pay attention is the uniqueness of complete hypersurfaces in $I \times_\rho \mathbb{P}^n$, under reasonable restrictions on their mean curvatures. Along this branch, we may cite, for instance, the papers [5] to [10], [17], [20], [28] and [55]. A Lorentzian version of the problems discussed above, that was pursued by several authors more recently, was to treat the problem of uniqueness for complete constant mean curvature spacelike hypersurfaces of generalized Robertson-Walker (GRW) spacetimes, that is, Lorentzian warped products with 1-dimensional negative definite base and Riemannian fiber of type $-I \times_\rho \mathbb{P}^n$. In this setting, we may cite, for instance, the works [1, 4, 6, 7, 15, 16, 18, 32, 63, 64].

The goal of the Chapter 3 is to study the unicity of hypersurfaces $\Sigma$ immersed in a semi-Riemannian manifold of type $\varepsilon I \times_\rho \mathbb{P}^n$.

## 3.2 Uniqueness results in weighted warped products

It follows from a splitting theorem due to Fang, Li and Zhang (see [34], Theorem 1.1) that if a product manifold $I \times \mathbb{P}$ with bounded weighted function $f$ is such that $\overline{\mathrm{Ric}}_f \geq 0$, then $f$ must be constant along $\mathbb{R}$. So, motivated by this result, along this section, we will consider weighted products $\mathbb{P}^n \times \mathbb{R}$ whose weighted function $f$ does not depend on the parameter $t \in I$, that is $\langle \overline{\nabla} f, \partial_t \rangle = 0$ and, for sake of simplicity, we will denote them by $\mathbb{P}^n_f \times \mathbb{R}$.

In order to prove our Bernstein type theorems in weighted warped products of the type

$I \times_\rho \mathbb{P}_f$, we will need some auxiliary lemmas. The first one is an extension of Proposition 4.1 of [6].

**Lemma 3.2.1.** *Let $\Sigma^n$ be a hypersurface immersed in a weighted warped product $I \times_\rho \mathbb{P}^n_f$, with height function $h$. Then,*

*(i)* $\Delta_f h = (\log \rho)'(h)(n - |\nabla h|^2) + n\Theta H_f$;

*(ii)* $\Delta_f \sigma(h) = n(\rho'(h) + \rho(h)\Theta H_f)$;

*where* $\sigma(t) = \int_{t_0}^t \rho(s)dt$.

*Proof.* Let us prove item (i). Taking into account that $f$ is constant along $\mathbb{R}$, from (1.3) we get that

$$\langle \nabla f, \nabla h \rangle = -\Theta \langle \overline{\nabla} f, N \rangle. \tag{3.1}$$

On the other hand, from Proposition 4.1 of [6] (see also Proposition 3.2 of [20]) we have

$$\Delta h = (\log \rho'(h)(n - |\nabla h|^2) + nH\Theta. \tag{3.2}$$

Hence, from (1.6), (3.1) and (3.2) we obtain

$$\begin{aligned} \Delta_f h &= (\log \rho)'(h)(n - |\nabla h|^2) + nH\Theta - \langle \nabla f, \nabla h \rangle \\ &= (\log \rho)'(h)(n + |\nabla h|^2) + nH\Theta + \Theta \langle \overline{\nabla} f, N \rangle \\ &= (\log \rho)'(h)(n + |\nabla h|^2) + n\Theta H_f. \end{aligned}$$

Moreover, since Proposition 4.1 of [6] also gives that

$$\Delta \sigma(h) = n(\rho'(h) + \rho(h)\Theta H),$$

in a similar way we also prove item (ii). □

Let us denote by $\mathscr{L}^1_f(\Sigma)$ the space of the integrable functions on $\Sigma^n$, in relation to weighted volume element $d\mu = e^{-f} d\Sigma$. Since $\mathrm{div}_f(X) = e^f \mathrm{div}(e^{-f} X)$, it is not difficult to see that from Proposition 2.1 of [19] we get the following extension of a result due to Yau in [72].

**Lemma 3.2.2.** *Let $u$ be a smooth function on a complete weighted Riemannian manifold $\Sigma^n$ with weighted function $f$, such that $\Delta_f u$ does not change sign on $\Sigma^n$. If $|\nabla u| \in \mathscr{L}^1_f(\Sigma)$, then $\Delta_f u$ vanishes identically on $\Sigma^n$.*

We recall that a *slab* of a warped product $I \times_\rho \mathbb{P}^n$ is a region of the type

$$[t_1, t_2] \times \mathbb{P}^n = \{(t, q) \in I \times_\rho \mathbb{P}^n : t_1 \leq t \leq t_2\}.$$

Now, we are in position to state and prove our first result.

**Theorem 3.2.1** (Theorem 1 [24]). *Let $\Sigma^n$ be a complete two-sided hypersurface which lies in a slab of a weighted warped product $I \times_\rho \mathbb{P}^n_f$. Suppose that $\Theta \leq 0$ and that $H_f$ satisfies*

$$0 < H_f \leq \inf_\Sigma (\log \rho)'(h). \tag{3.3}$$

*If $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*Proof.* From Lemma 3.2.1, since we are assuming that $-1 \leq \Theta \leq 0$, we get

$$\begin{aligned} \Delta_f \sigma(h) &= n\rho(h)\left((\log\rho)'(h) + \Theta H_f\right) \\ &\geq n\rho(h)\left((\log\rho)'(h) - H_f\right) \\ &\geq n\rho(h)\left(\inf_\Sigma (\log\rho)'(h) - H_f\right). \end{aligned} \tag{3.4}$$

Thus, taking into account our hypothesis (3.3), from (3.4) we have that $\Delta_f \sigma(h) \geq 0$.

On the other hand, since $\Sigma^n$ lies in a slab of $I \times_\rho \mathbb{P}^n$, we have that exists a positive constant $C$ such that

$$|\nabla \sigma(h)| = \rho(h)|\nabla h| \leq C|\nabla h|.$$

Consequently, our hypothesis $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$ implies that $|\nabla \sigma(h)| \in \mathscr{L}^1_f(\Sigma)$.

So, we can apply Lemma 3.2.2 to assure that $\Delta_f \sigma(h) = 0$ on $\Sigma^n$. Hence, returning to (3.4) we get

$$(\log \rho)'(h) = -\Theta H_f. \tag{3.5}$$

Consequently, using once more hypothesis (3.3), from (3.5) we have that

$$H_f \leq \inf_\Sigma (\log\rho)'(h) \leq (\log\rho)'(h) = -\Theta H_f. \tag{3.6}$$

Therefore, from (3.6) we conclude that $\Theta = -1$ and, hence, $\Sigma^n$ must be a slice $\{t\} \times \mathbb{P}$. □

**Remark 3.2.1.** *Concerning Theorem 3.2.1, we note that if $\Sigma^n$ is locally a graph over $\mathbb{P}^n$, then its angle function $\Theta$ is either $\Theta < 0$ or $\Theta > 0$ along $\Sigma$. Hence, the assumption that $\Theta$ does not change sign is generally weaker than that of $\Sigma^n$ being a local graph. Moreover, as it was* *already observed by Espinar and Rosenberg [33] when they made allusion to immersions into the Euclidean space, the condition that $\Theta$ does not change sign can also be regarded as the image of the Gauss map of the hypersurface lying in a closed hemisphere of the Euclidean sphere.*

*On the other hand, from Proposition* 1 *of [55] and relation* (1.6)*, if we consider on the slice* $\{t\} \times \mathbb{P}$ *of* $I \times_\rho \mathbb{P}_f$ *the orientation given by* $N = -\partial_t$*, then its* $f$*-mean curvature is given by*

$$H_f(t) = H(t) = (\log \rho)'(t).$$

*Consequently, the differential inequality* (3.3) *means that, at each point* $(t,x)$ *of the hypersurface* $\Sigma^n$*, the weighted mean curvature of* $\Sigma^n$ *can be any value less than or equal to the value of the weighted mean curvature of the slice* $\{t\} \times \mathbb{P}^n$*, with respect to the orientation given by* $-\partial_t$*. Hence, we only suppose here a natural comparison inequality between weighted mean curvature quantities, without to require that the weighted mean curvature of* $\Sigma^n$ *be constant. In this sense,* (3.3) *is a mild hypothesis.*

According to the classical terminology in linear potential theory, a weighted manifold $\Sigma$ with weighted function $f$ is said to be *$f$-parabolic* if every bounded solution of $\Delta_f u \geq 0$ must be identically constant. So, from Theorem 3.2.1 we obtain

**Corollary 3.2.1.** *Let* $\Sigma^n$ *be a complete two-sided hypersurface which lies in a slab of a weighted warped product* $I \times_\rho \mathbb{P}^n_f$*. Suppose that* $\Theta \leq 0$ *and that* $H_f$ *satisfies*

$$0 < H_f \leq \inf_\Sigma (\log \rho)'(h).$$

*If* $\Sigma^n$ *is* $f$*-parabolic, then* $\Sigma^n$ *is a slice* $\{t\} \times \mathbb{P}$*.*

**Remark 3.2.2.** *As it was observed by Impera and Rimoldi in Remark* 3.8 *of [46], the* $f$*-parabolicity of* $\Sigma^n$ *holds if it has finite* $f$*-volume. On the other hand, in the case that* $\Sigma^n$ *is complete noncompact with* $\mathrm{Ric}_f$ *nonnegative, from Theorem* 1.1 *of [69] a sufficient condition for* $\Sigma^n$ *to have finite* $f$*-volume is that the space of* $L^2_f$ *harmonic one-forms be nontrivial.*

From the proof of Theorem 3.2.1 we also get the following

**Corollary 3.2.2.** *Let* $\Sigma^n$ *be a complete two-sided hypersurface which lies in a slab of a weighted warped product* $I \times_\rho \mathbb{P}^n_f$*. Suppose that* $\Theta \leq 0$ *and that* $H_f$ *is constant and satisfies*

$$0 \leq H_f \leq \inf_\Sigma (\log \rho)'(h).$$

*If either $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$ or $\Sigma^n$ is $f$-parabolic, then $\Sigma^n$ is either a $f$-minimal hypersurface or a slice $\{t\} \times \mathbb{P}$.*

Extending ideas of [6], [9], [10] and [55], in our next result we will assume that the ambient space is a weighted warped product $I \times_\rho \mathbb{P}^n_f$ which obeys the following convergence condition

$$K_{\mathbb{P}} \geq \sup_I((\rho')^2 - \rho\rho''), \tag{3.7}$$

where $K_{\mathbb{P}}$ stands for the sectional curvature of the fiber $\mathbb{P}^n$.

**Theorem 3.2.2** (Theorem 2 [24]). *Let $I \times_\rho \mathbb{P}^n_f$ be a weighted warped product which satisfies the convergence condition (3.7). Let $\Sigma^n$ be a complete two-sided hypersurface which lies in a slab of $I \times_\rho \mathbb{P}^n_f$. Suppose that $\overline{\nabla} f$ and the Weingarten operator $A$ of $\Sigma^n$ are bounded. If $\Theta$ does not change sign on $\Sigma^n$ and*

$$|\nabla h| \leq \inf_\Sigma \left\{ (\log \rho)'(h) - |H_f| \right\}, \tag{3.8}$$

*then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*Proof.* From Lemma 3.2.1 we get

$$\begin{aligned} \Delta_f h &= (\log\rho)'(h)(n - |\nabla h|^2) + nH_f\Theta \\ &\geq (\log\rho)'(h)(n - |\nabla h|^2) - n|H_f| \\ &\geq n((\log\rho)'(h) - |H_f|) - (\log\rho)'(h)|\nabla h|^2. \end{aligned} \tag{3.9}$$

On the other hand, since from the Cauchy-Schwarz inequality we get $nH^2 \leq |A|^2$, we have that $H$ is also bounded. Thus, since we are also assuming that $\Sigma^n$ lies in a slice of $I \times_\rho \mathbb{P}^n_f$, we can apply Proposition 3.1 of [9] to assure that the Ricci curvature of $\Sigma^n$ is bounded from below.

Hence, since $|\overline{\nabla} f|$ is bounded on $\Sigma^n$, from maximum principle of Omori-Yau, there exists a sequence $\{p_k\}$ in $\Sigma^n$ such that

$$\lim_k h(p_k) = \sup_\Sigma h, \ \lim_k |\nabla h(p_k)| = 0 \text{ and } \limsup_k \Delta_f h(p_k) \leq 0.$$

Thus, from inequality (3.9) we get

$$0 \geq \limsup_k \Delta_f h(p_k) \geq \lim_k ((\log\rho)'(h) - |H_f|)(p_k) \geq 0.$$

Therefore, we have that $\lim_k((\log\rho)'(h) - |H_f|)(p_k) = 0$ and, taking into account our hypothesis (3.8), we conclude that $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$. $\square$

## 3.3 Rigidity results in weighted product spaces

In this section, we will treat the special case when the ambient space is a weighted product $I \times \mathbb{P}^n_f$. We start with the following uniqueness result.

**Theorem 3.3.1** (Theorem 3 [24]). *Let $\Sigma^n$ be a complete two-sided hypersurface which lies in a slab of the weighted product $I \times \mathbb{P}^n_f$. Suppose that $\Theta$ and $H_f$ do not change sign on $\Sigma^n$. If $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*Proof.* From Lemma 3.2.1 we have that

$$\Delta_f h = nH_f\Theta. \tag{3.10}$$

Since we are supposing that $\Theta$ and $H_f$ do not change sign on $\Sigma^n$, we get that $\Delta_f h$ also does not change sign on $\Sigma^n$. Thus, since $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$, we can apply Lemma 3.2.2 to get that $\Delta_f h = 0$ on $\Sigma^n$.

Consequently, we obtain

$$\Delta_f h^2 = 2h\Delta_f h + 2|\nabla h|^2 = 2|\nabla h|^2 \geq 0. \tag{3.11}$$

But, since $h$ is bounded and using once more that $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$, Lemma 3.2.2 guarantees also that $\Delta_f h^2 = 0$ on $\Sigma^n$. Therefore, returning to (3.11) we conclude that $\Sigma^n$ must be a slice $\{t\} \times \mathbb{P}$. $\square$

From the proof of Theorem 3.3.1 we get

**Corollary 3.3.1.** *Let $\Sigma^n$ be a complete two-sided hypersurface which lies in a slab of the weighted product $I \times \mathbb{P}^n_f$. Suppose that $\Theta$ and $H_f$ do not change sign on $\Sigma^n$. If $\Sigma^n$ is $f$-parabolic, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

Consider the weighted product space $\mathbb{R} \times \mathbb{G}^n$. Recall that $\mathbb{G}^n$ corresponds to the Euclidean space $\mathbb{R}^n$ endowed with the Gaussian measure $d\mu = e^{-\frac{|x|^2}{4}}dx^2$. Hieu and Nam extended the classical Bernstein's theorem [14] showing that the only graphs with weighted mean curvature identically zero given by a function $u(x_2,\dots,x_{n+1}) = x_1$ over $\mathbb{G}^n$ are the hyperplanes $x_1 =$

*constant* (cf. [43], Theorem 4). In this setting, with a straightforward computation we can verify that

$$N = \frac{1}{\sqrt{1+|Du|}}(Du - \partial_t) \tag{3.12}$$

gives an orientation on $\Sigma^n(u)$ such that $-1 \leq \Theta < 0$, where $Du$ stands for the gradient of the function $u$ on $\mathbb{G}^n$. Thus, since $|N^*| = |\nabla h|$, from (3.12) we deduce that

$$|\nabla h|^2 = \frac{|Du|^2}{1+|Du|^2}. \tag{3.13}$$

So, taking into account relation (3.13), from the proof of Theorem 3.3.1 jointly with Theorem 4 of [43] we obtain

**Corollary 3.3.2.** *Let $\Sigma^n(u)$ be a complete graph of a function $u(x_2,\ldots,x_{n+1}) = x_1$ over the Gaussian space $\mathbb{G}^n$. Suppose that the weighted mean curvature of $\Sigma^n(u)$ does change sign. If $|Du| \in \mathscr{L}^1(\mathbb{G})$, then $\Sigma^n(u)$ is a hyperplane $x_1 = constant$.*

To prove our next result, we will need of the following auxiliary lemma.

**Lemma 3.3.1.** *Let $\Sigma^n$ be a hypersurface with constant $f$-mean curvature in a weighted product manifold $\mathbb{P}^n_f \times \mathbb{R}$. Then*

$$\Delta_f \Theta = -(|A|^2 + \widetilde{\mathrm{Ric}}_f(N^*,N^*))\Theta, \tag{3.14}$$

*where $A$ denotes the Weingarten operator of $\Sigma$, $\widetilde{\mathrm{Ric}}_f$ stands for the Bakry-Émery-Ricci tensor of the fiber $\mathbb{P}$ and $N^* = N - \Theta\partial_t$ is the orthonormal projection of $N$ onto $\mathbb{P}$.*

*Proof.* It is well known that

$$\nabla\Theta = -A\partial_t^T - (\overline{\nabla}_N \partial_t)^T \tag{3.15}$$

and

$$\Delta\Theta = -n\partial_t^T(H) - \Theta(\widetilde{\mathrm{Ric}}(N^*,N^*) + |A|^2), \tag{3.16}$$

see for instance [20]. Taking into account that $\langle \partial_t, \overline{\nabla} f\rangle = 0$ we get that

$$\begin{aligned} n\partial_t^T(H) &= \partial_t^T(nH_f - \langle \overline{\nabla} f, N\rangle) \\ &= -\partial_t^T\langle \overline{\nabla} f, N\rangle \\ &= -\langle \overline{\mathrm{Hess}}_f(\partial_t), N\rangle + \Theta\overline{\mathrm{Hess}}_f(N,N) + \langle A\partial_t^T, \overline{\nabla} f\rangle. \end{aligned} \tag{3.17}$$

On the other hand, from (3.15) we get that

$$\begin{aligned}\langle \nabla\Theta, \overline{\nabla} f\rangle &= -\langle A\partial_t^T + (\overline{\nabla}_N \partial_t)^T, \overline{\nabla} f\rangle \\ &= -\langle A\partial_t^T, \overline{\nabla} f\rangle - \langle \overline{\nabla}_N \partial_t - \langle \overline{\nabla}_N \partial_t, N\rangle N, \overline{\nabla} f\rangle \\ &= -\langle A\partial_t^T, \overline{\nabla} f\rangle + \langle \partial_t, \overline{\nabla}_N \overline{\nabla} f\rangle \\ &= -\langle A\partial_t^T, \overline{\nabla} f\rangle + \langle \partial_t, \overline{\mathrm{Hess}}_f(N)\rangle.\end{aligned} \tag{3.18}$$

From (3.17) and (3.18) we have

$$n\partial_t^T(H) = \Theta \overline{\mathrm{Hess}}_f(N,N) - \langle \nabla\Theta, \overline{\nabla} f\rangle. \tag{3.19}$$

Now, taking into account once more that $\langle \partial_t, \overline{\nabla} f\rangle = 0$, it is not difficult to verify that

$$\overline{\mathrm{Hess}}_f(N,N) = \widetilde{\mathrm{Hess}}_f(N^*,N^*). \tag{3.20}$$

Putting (3.16) and (3.20) into (3.19) we have the desired result. □

**Theorem 3.3.2** (Theorem 4 [24]). *Let $\Sigma^n$ be a complete two-sided hypersurface immersed in a weighted product $I \times \mathbb{P}^n_f$. Suppose that $\widetilde{\mathrm{Ric}}_f \geq 0$, $A$ is bounded, $\Theta$ has strict sign $H_f$ does change sign on $\Sigma^n$. If $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$, then $\Sigma^n$ is totally geodesic. Moreover, if $\widetilde{\mathrm{Ric}}_f > 0$, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*Proof.* Since we are supposing that $\Theta$ has strict sign, $H_f$ does not change sign on $\Sigma^n$ and that $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$, from and applying Lemma 3.2.2 we get that $\Sigma$ is $f$-minimal, that is, $H_f = 0$ on $\Sigma^n$.

Note that

$$|\nabla\Theta| \leq |A||\nabla h| \in \mathscr{L}^1_f(\Sigma). \tag{3.21}$$

So, since we are also supposing that $\widetilde{\mathrm{Ric}}_f \geq 0$, from (3.14) and (3.21) we have that $\Delta_f \Theta = 0$ on $\Sigma^n$.

Hence, returning to equation (3.14) we conclude that $\Sigma^n$ is totally geodesic. Furthermore, if $\widetilde{\mathrm{Ric}}_f$ is strictly positive, then $N^* = 0$ on $\Sigma^n$ and, therefore, $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$. □

When the fiber of the weighted product space is compact, we have

**Theorem 3.3.3** (Theorem 5 [24]). *Let $\Sigma^n$ be a complete two-sided hypersurface immersed in a weighted product $I \times \mathbb{P}^n_f$, whose fiber $\mathbb{P}^n$ is compact with positive sectional curvature, and such that the weighted function $f$ is convex. Suppose that $A$ is bounded and $H_f$ is constant. If $\Theta$ is such that either $0 \leq \arccos\Theta \leq \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4} \leq \arccos\Theta \leq \pi$, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}^n$.*

*Proof.* Firstly, we claim that the Bakry-Émery Ricci tensor of $\Sigma^n$, $\mathrm{Ric}_f$, is bounded from below. Indeed, since we are assuming that $\mathbb{P}^n$ is compact with positive sectional curvature, from Gauss equation it follows that

$$\mathrm{Ric}(X,X) \geq (n-1)\kappa\left(1-|\nabla h|^2\right)|X|^2 + nH\langle AX,X\rangle - \langle AX,AX\rangle, \tag{3.22}$$

for all $X \in \mathfrak{X}(\Sigma)$ and some positive constant $\kappa = \kappa(X)$.

On the other hand, taking into account that $f$ is convex and constant along $\mathbb{R}$, we have

$$\begin{aligned} \mathrm{Hess} f(X,X) &= \overline{\mathrm{Hess}} f(X,X) + \langle \overline{\nabla} f, N\rangle\langle AX,X\rangle \\ &= \widetilde{\mathrm{Hess}} f(X^*,X^*) + \langle \overline{\nabla} f, N\rangle\langle AX,X\rangle \\ &\geq \langle \overline{\nabla} f, N\rangle\langle AX,X\rangle, \end{aligned} \tag{3.23}$$

for all $X \in \mathfrak{X}(\Sigma)$.

Thus, from (3.22) and (3.23) we get

$$\mathrm{Ric}_f(X,X) \geq (n-1)\kappa\left(1-|\nabla h|^2\right)|X|^2 + nH_f\langle AX,X\rangle - \langle AX,AX\rangle. \tag{3.24}$$

Consequently, from (3.24) we obtain

$$\mathrm{Ric}_f(X,X) \geq ((n-1)\kappa\left(1-|\nabla h|^2\right) - (n|H_f||A| + |A|^2))|X|^2, \tag{3.25}$$

for all $X \in \mathfrak{X}(\Sigma)$.

We also note that our restriction on $\Theta$ amounts to $|\nabla h|^2 \leq \frac{1}{2}$. Hence, since $A$ is also bounded and $H_f$ is constant, from (3.25) we conclude that $\mathrm{Ric}_f$ is bounded from below.

On the other hand, using once more that $H_f$ is constant, from Lemma 3.2.1 we have

$$\nabla\Delta_f h = nH_f\nabla\Theta. \tag{3.26}$$

Recall that

$$\nabla\Theta = -A(\nabla h). \tag{3.27}$$

Consequently, from (3.26) and (3.27) we get

$$\nabla\Delta_f h = -nH_f A(\nabla h). \tag{3.28}$$

From (1.4) we have that

$$\nabla_X \nabla h = \nabla_X(\partial_t^\top) = AX\Theta. \tag{3.29}$$

Thus, from (3.29) we obtain

$$|\mathrm{Hess}\, h|^2 = |A|^2\Theta^2. \tag{3.30}$$

Consequently, from (1.5) and (3.30) we get

$$|\mathrm{Hess}\, h|^2 = |A|^2 - |\nabla h|^2|A|^2. \tag{3.31}$$

Now, from Bochner's formula (1.1) we also have that

$$\frac{1}{2}\Delta_f|\nabla h|^2 = |\mathrm{Hess}\, h|^2 + \mathrm{Ric}_f(\nabla h, \nabla h) + \langle \nabla \Delta_f h, \nabla h\rangle. \tag{3.32}$$

But, from (3.24) it follows that

$$\begin{aligned} \mathrm{Ric}_f(\nabla h, \nabla h) &\geq (n-1)\kappa(1-|\nabla h|^2)|\nabla h|^2 + nH_f\langle A\nabla h, \nabla h\rangle - \langle A\nabla h, A\nabla h\rangle \\ &\geq (n-1)\kappa(1-|\nabla h|^2)|\nabla h|^2 + nH_f\langle A\nabla h, \nabla h\rangle - |A|^2|\nabla h|^2. \end{aligned} \tag{3.33}$$

Hence, considering (3.28)), (3.31) and (3.33) into (3.32) we get

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq (n-1)\kappa(1-|\nabla h|^2)|\nabla h|^2 + |A|^2(1-2|\nabla h|^2). \tag{3.34}$$

Consequently, from (1.5) and (3.34) jointly with our hypothesis on $\Theta$ we obtain

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq (n-1)\kappa(1-|\nabla h|^2)|\nabla h|^2. \tag{3.35}$$

Since $\mathrm{Ric}_f$ is bounded from below on $\Sigma^n$, from Theorem 1.1.1 we have that there exists a sequence of points $(p_k)_{k\geq 1}$ in $\Sigma^n$ such that

$$\lim_k |\nabla h|^2(p_k) = \sup_\Sigma |\nabla h|^2 \text{ and } \limsup_k \Delta_f|\nabla h|^2(p_k) \leq 0.$$

Thus, from (3.35) we have

$$0 \geq \limsup_k \Delta_f|\nabla h|^2(p_k) \geq (n-1)\kappa(1-\sup_\Sigma|\nabla h|^2)\sup_\Sigma|\nabla h|^2 \geq 0. \tag{3.36}$$

Consequently, taking into account once more our hypothesis on $\Theta$, from (3.36) we conclude that $\sup_\Sigma|\nabla h|^2 = 0$. Therefore, $h$ is constant on $\Sigma^n$ and, hence, $\Sigma^n$ is a slice $\{t\}\times\mathbb{P}^n$. □

**Corollary 3.3.3.** *Let $\Sigma^n$ be a $f$-parabolic complete two-sided hypersurface immersed in a weighted product $I \times \mathbb{P}^n_f$, whose fiber $\mathbb{P}^n$ is complete with nonnegative sectional curvature, and such that the weighted function $f$ is convex. Suppose that $A$ is bounded and $H_f$ is constant. It holds the following:*

*(a) If either $0 \leq \arccos\Theta < \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4} < \arccos\Theta \leq \pi$, then $\Sigma^n$ is totally geodesic.*

*(b) If $K_{\mathbb{P}}$ is positive and either $0 \leq \arccos\Theta \leq \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4} \leq \arccos\Theta \leq \pi$, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}^n$.*

*Proof.* In a very similar way of that was made in order to prove inequality (3.34), taking a local orthonormal frame $\{E_1, \ldots, E_n\}$ on $\Sigma$, we obtain that

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq (n-1)\min_i K_{\mathbb{P}}((\nabla h)^*, E_i^*)(1-|\nabla h|^2)|\nabla h|^2 + |A|^2(1-2|\nabla h|^2). \tag{3.37}$$

If we assume the hypothesis of item (a), since we are also supposing that $K_{\mathbb{P}}$ is nonnegative, from (3.37) we get that $\Delta_f|\nabla h|^2 \geq 0$. Hence, since we are also assuming that $\Sigma^n$ is $f$-parabolic, we have that $|\nabla h|$ is constant on $\Sigma$. Therefore, returning to (3.37) we conclude that $|A| = 0$ on $\Sigma$, that is, $\Sigma$ is totally geodesic.

Now, we assume the hypothesis of item (b). Since $\kappa := \min_i K_{\mathbb{P}}((\nabla h)^*, E_i^*) > 0$, from (3.37) we also get that

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq (n-1)\kappa(1-|\nabla h|^2)|\nabla h|^2 \geq 0. \tag{3.38}$$

Hence, using once more that $\Sigma^n$ is $f$-parabolic and noting that $|\nabla h| < 1$, from (3.38) we conclude that $|\nabla h| = 0$ on $\Sigma^n$, that is, $\Sigma^n$ must be a slice $\{t\} \times \mathbb{P}^n$. □

Proceeding, we obtain an extension of Theorem 3.1 of [31].

**Theorem 3.3.4** (Theorem 6 [24]). *Let $\Sigma^n$ be a complete two-sided hypersurface immersed in a weighted product $I \times \mathbb{P}^n_f$, whose fiber $\mathbb{P}^n$ has sectional curvature $K_{\mathbb{P}}$ satisfying $K_{\mathbb{P}} \geq -\kappa$ and such that $\widetilde{\operatorname{Hess}} f \geq -\gamma$, for some positive constants $\kappa$ and $\gamma$. Suppose that $A$ is bounded, $\Theta$ is bounded away from zero and $H_f$ is constant. If the height function $h$ of $\Sigma^n$ satisfies*

$$|\nabla h|^2 \leq \frac{\alpha}{(n-1)\kappa + \gamma}|A|^2, \tag{3.39}$$

*for some constant $0 < \alpha < 1$, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*Proof.* Since we are assuming that $\Theta$ is bounded away from zero, we can suppose that $\Theta > 0$ and, consequently, $\inf\Theta > 0$.

Moreover, since we are also assuming that the sectional curvature $K_{\mathbb{P}}$ of the base $\mathbb{P}^n$ is such that $K_{\mathbb{P}} \geq -\kappa$ for some $\kappa > 0$, from Gauss equation and with a straightforward computation we get

$$\widetilde{\mathrm{Ric}}(N^*, N^*) \geq -(n-1)\kappa|N^*|^2 = -(n-1)\kappa|\nabla h|^2. \tag{3.40}$$

On the other hand, our restriction on $\widetilde{\mathrm{Hess}} f$ jointly with equation 1.5 give

$$\widetilde{\mathrm{Hess}} f(N^*, N^*) \geq -\gamma|N^*|^2 = -\gamma|\nabla h|^2. \tag{3.41}$$

Thus, from (3.40) and (3.41) we get

$$\widetilde{\mathrm{Ric}}_f(N^*, N^*) \geq -((n-1)\kappa + \gamma)|\nabla h|^2. \tag{3.42}$$

Hence, from (3.42) and (3.14) we obtain

$$\Delta_f \Theta \leq -(|A|^2 - ((n-1)\kappa + \gamma)|\nabla h|^2)\Theta. \tag{3.43}$$

Consequently, from (3.39) and (3.43) we have

$$\Delta_f \Theta \leq -(1-\alpha)|A|^2\Theta. \tag{3.44}$$

Now, we claim that $\mathrm{Ric}_f$ is bounded from below on $\Sigma^n$. Indeed, following similar ideas of that in the proof of Theorem 3.3.3 we get

$$\begin{aligned} \mathrm{Hess} f(X,X) &\geq \langle \overline{\nabla} f, N\rangle\langle AX, X\rangle - \gamma|X^*|^2 \\ &= \langle \overline{\nabla} f, N\rangle\langle AX, X\rangle - \gamma(|X|^2 - \langle X, \partial_t\rangle^2) \\ &\geq \langle \overline{\nabla} f, N\rangle\langle AX, X\rangle - \gamma|X|^2. \end{aligned} \tag{3.45}$$

From (3.22) and (3.45) we obtain

$$\mathrm{Ric}_f(X,X) \geq ((n-1)\kappa\left(1 - |\nabla h|^2\right) - \gamma)|X|^2 + nH_f\langle AX, X\rangle - \langle AX, AX\rangle. \tag{3.46}$$

Thus, since $A$ and $|\nabla h|$ are bounded and $H_f$ is constant, from (3.46) we conclude that $\mathrm{Ric}_f$ is bounded from below.

Now, we are in position to apply Theorem 1.1.1 and guarantee the existence of a sequence of points $p_k \in \Sigma^n$ satisfying

$$\liminf_{k\to\infty} \Delta_f \Theta(p_k) \geq 0 \text{ and } \lim_{k\to\infty} \Theta(p_k) = \inf_{p\in\Sigma} \Theta(p).$$

Consequently, since we are assuming that $A$ is bounded on $\Sigma^n$, from (3.44), up to a subsequence, we get

$$0 \leq \liminf_{k\to\infty} \Delta_f \Theta(p_k) \leq -(1-\alpha) \lim_{k\to\infty} |A|^2(p_k) \inf_{p\in\Sigma} \Theta(p) \leq 0. \tag{3.47}$$

Thus, from (3.47) we obtain that $\lim_{k\to\infty} |A|(p_k) = 0$ and, using (3.39), we get

$$\lim_{k\to\infty} |\nabla h|(p_k) = 0. \tag{3.48}$$

Therefore, from (1.5) and (3.48) we conclude that $\inf_{p\in\Sigma} \Theta(p) = 1$ and, hence, $\Theta \equiv 1$, that is, $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$. □

To close our section, we will apply a Liouville type result due to Huang et al. [45] in order to prove the following:

**Theorem 3.3.5** (Theorem 7 [24]). *Let $\Sigma^n$ be a complete two-sided hypersurface which lies in a slab of a weighted product $I \times \mathbb{P}^n_f$, whose fiber $\mathbb{P}^n$ has sectional curvature $K_{\mathbb{P}}$ bounded from below and such that $\overline{\nabla} f$ is bounded. Suppose that $A$ and $H_f$ are bounded on $\Sigma^n$. If $\Theta$ is not adhere to* 1 *or* $-1$, *then* $\inf_\Sigma H_f = 0$. *Moreover, if $H_f$ is constant and* $\mathrm{Ric}_f$ *is nonnegative, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*Proof.* Following similar steps of the proof of Theorem 3.3.3, our restriction on the sectional curvature of the fiber $\mathbb{P}^n$ jointly with our hypothesis on $A$, $H_f$ and $\overline{\nabla} f$ guarantee that the Ricci curvature of $\Sigma^n$ is bounded from below.

Now, suppose for instance that $H_f \geq 0$ on $\Sigma^n$. Thus, since $\Sigma^n$ lies between two slices of $\mathbb{R} \times \mathbb{P}^n$, from Lemma 3.2.1 and maximum principle of Omori-Yau, we obtain a sequence of points $p_k \in \Sigma^n$ such that

$$0 \geq \limsup_{k\to\infty} \Delta_f h(p_k) = n \limsup_{k\to\infty} \left(H_f \Theta\right)(p_k).$$

Moreover, from equation (1.5) we also have that

$$0 = \lim_{k\to\infty} |\nabla h|(p_k) = 1 - \lim_{k\to\infty} \Theta^2(p_k).$$

Thus, if we suppose, for instance, that $\Theta$ is not adhere to $-1$, we get

$$\lim_{k\to\infty}\Theta(p_k)=1.$$

Consequently,

$$0\geq\limsup_{k\to\infty}\Delta_f h(p_k)=n\limsup_{k\to\infty}H_f(p_k)\geq 0$$

and, hence, we conclude that

$$\limsup_{k\to\infty}H_f(p_k)=0.$$

If $H_f\leq 0$, from Lemma 3.2.1 and (1.5), we can apply once more Omori-Yau's generalized maximum principle in order to obtain a sequence $q_k\in\Sigma^n$ such that

$$0\leq\liminf_{k\to\infty}\Delta_f h(q_k)=n\liminf_{k\to\infty}\left(H_f\Theta\right)(q_k)$$

and, supposing once more that $\Theta$ is not adhere to $-1$, we get

$$0\leq\liminf_{k\to\infty}\Delta_f h(p_k)=n\liminf_{k\to\infty}H_f(p_k)\leq 0.$$

Consequently, from the above inequality we have that $\liminf_{k\to\infty}H_f(p_k)=0$. Hence, in this case, we also conclude that $\inf_\Sigma H_f=0$.

When $H_f$ is constant, we have that, in fact, $H_f$ vanishes identically on $\Sigma^n$. Thus, since $\Sigma^n$ is contained in a slab of $\mathbb{R}\times\mathbb{P}^n_f$, there exists a constant $\beta$ such that $h-\beta$ is a positive harmonic function of the $f$-Laplacian on $\Sigma^n$.

On the other hand, since $\nabla f=\overline{\nabla}f-\langle\overline{\nabla}f,N^*\rangle N$ and $|N^*|=|\nabla h|$, we obtain

$$|\nabla f|^2\leq|\overline{\nabla}f|^2(1-|\nabla h|^2).$$

Consequently, since we are assuming that $\overline{\nabla}f$ is bounded, we have that $\nabla f$ is also bounded on $\Sigma^n$. Hence, if $\mathrm{Ric}_f$ is nonnegative, then we can apply Corollary 1.4 of [45] to conclude that $h$ is constant on $\Sigma^n$, that is, $\Sigma^n$ is a slice $\{t\}\times\mathbb{P}^n$. □

## 3.4 Uniqueness results in weighted GRW spacetimes

We observe that it follows from a splitting theorem due to Case (cf. [21], Theorem 1.2) that a weighted GRW spacetime whose weight function $f$ is bounded and such that $\overline{\mathrm{Ric}}_f(V,V)\geq 0$

for all timelike vector field $V$, then $f$ must be constant along $\mathbb{R}$. So, motivated by this result, along this work we will consider weighted GRW spacetimes $-I\times_\rho \mathbb{P}$ whose weight function $f$ does not depend on the parameter $t\in I$, that is $\langle \overline{\nabla} f, \partial_t\rangle = 0$. For simplicity, we will denote them by $-I\times_\rho \mathbb{P}_f$.

In order to prove our Calabi-Bernstein's type results in weighted GRW spacetimes of the type $-I\times_\rho \mathbb{P}_f$, we will need some auxiliary lemmas. The first one is an extension of Lemma 4.1 of [4], and its demonstration follows the same steps of 3.2.1.

**Lemma 3.4.1.** *Let $\Sigma^n$ be a spacelike hypersurface immersed in a weighted GRW spacetime $-I\times_\rho \mathbb{P}_f$, with height function $h$. Then,*

*(i)* $\Delta_f h = -(\log\rho)'(h)(n+|\nabla h|^2) - nH_f\Theta$;

*(ii)* $\Delta_f \sigma(h) = -n(\rho'(h)+\rho(h)\Theta H_f)$,

*where* $\sigma(t) = \int_{t_0}^t \rho(s)ds$.

In what follows, a slab $[t_1,t_2]\times M^n = \{(t,q)\in -I\times_\rho \mathbb{P}^n : t_1\le t\le t_2\}$ is called a *timelike bounded region* of the weighted GRW spacetime $-I\times_\rho \mathbb{P}^n_f$. Now, we are in position to state and prove our first result.

**Theorem 3.4.1** (Theorem 1 [25]). *Let $\Sigma^n$ be a complete spacelike hypersurface which lies in a timelike bounded region of a weighted GRW spacetime $-I\times_\rho \mathbb{P}^n_f$. Suppose that the $f$-mean curvature $H_f$ of $\Sigma^n$ satisfies the following inequality*

$$H_f \ge \sup_\Sigma(\log\rho)'(h) > 0. \tag{3.49}$$

*If $|\nabla h|\in \mathscr{L}^1_f(\Sigma)$, then $\Sigma^n$ is a slice $\{t\}\times\mathbb{P}$.*

*Proof.* From Lemma 3.4.1, for $\sigma(t)=\int_{t_0}^t\rho(s)ds$, we get

$$\begin{aligned}\Delta_f\sigma(h) &= -n\rho(h)\left((\log\rho)'(h)+\Theta H_f\right) \\ &\ge n\rho(h)\left(H_f-(\log\rho)'(h)\right) \\ &\ge n\rho(h)\left(H_f-\sup_\Sigma(\log\rho)'(h)\right).\end{aligned}\tag{3.50}$$

Thus, taking into account our hypothesis (3.49), from (3.50) we have that $\Delta_f\sigma(h)\ge 0$ on $\Sigma^n$.

On the other hand, since $\Sigma^n$ is contained in a slab of $-I\times_\rho\mathbb{P}^n_f$, we have that exists a positive constant $C$ such that

$$|\nabla\sigma(h)| = \rho(h)|\nabla h| \le C|\nabla h|.$$

Consequently, the hypothesis $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$ implies that $|\nabla\sigma(h)| \in \mathscr{L}^1_f(\Sigma)$.

So, we can apply Lemma 3.2.2 to assure that $\Delta_f \sigma(h) = 0$ on $\Sigma^n$. Thus, returning to (3.50) we get

$$(\log\rho)'(h) = -\Theta H_f.$$

Consequently, we have that

$$H_f = \sup_{\Sigma}(\log\rho)'(h) \geq (\log\rho)'(h) = -\Theta H_f.$$

Therefore, $\Theta = -1$ on $\Sigma^n$ and, hence, we conclude that $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$. □

From the proof of Theorem 3.4.1 we also get the following.

**Corollary 3.4.1.** *Let $\Sigma^n$ be a complete spacelike hypersurface immersed in a timelike bounded region of a weighted GRW spacetime $-I \times_\rho \mathbb{P}^n_f$. Suppose that $\Sigma^n$ has constant $f$-mean curvature $H_f$ and that it holds the following inequality*

$$H_f \geq \sup_{\Sigma}(\log\rho)'(h) \geq 0.$$

*If $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$, then $\Sigma^n$ is either a $f$-maximal hypersurface or a slice $\{t\} \times \mathbb{P}$.*

According to the terminology established by Alías and Colares in [4], we say that a GRW spacetime $-I \times_\rho \mathbb{P}^n$ obeys the *strong null convergence condition* when the sectional curvature $K_{\mathbb{P}}$ of its Riemannian fiber $\mathbb{P}$ satisfies the following inequality

$$K_{\mathbb{P}} \geq \sup_{I}(f^2(\log f)''). \tag{3.51}$$

**Theorem 3.4.2** (Theorem 2 [25])**.** *Let $-I \times_\rho \mathbb{P}^n$ be a GRW spacetime obeying* (3.51)*. Let $\Sigma^n$ be a complete spacelike hypersurface which lies in a timelike bounded region of the weighted GRW spacetime $-I \times_\rho \mathbb{P}^n_f$. Suppose that $|\overline{\nabla} f|$ is bounded on $\Sigma^n$ and that the $f$-mean curvature $H_f$ of $\Sigma^n$ satisfies*

$$(\log\rho)'(h) \leq H_f \leq \alpha, \tag{3.52}$$

*for some constant $\alpha$. If*

$$|\nabla h| \leq \inf_{\Sigma}\left(H_f - (\log\rho)'(h)\right), \tag{3.53}$$

*then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*Proof.* From Lemma 3.4.1 we have

$$\Delta_f h = -(\log\rho)'(h)(n+|\nabla h|^2) - nH_f\Theta.$$

Since $N$ is a future-directed timelike vector field, we get

$$\Delta_f h \geq n(H_f - (\log\rho)'(h)) - (\log\rho)'(h)|\nabla h|^2. \tag{3.54}$$

We claim that the mean curvature of $\Sigma^n$ is bounded. Indeed, we have that

$$\begin{aligned} n|H| &= n|H_f| + |\langle\overline{\nabla} f, N\rangle| \\ &= n|H_f| + |\langle\overline{\nabla} f, N^*\rangle|. \end{aligned} \tag{3.55}$$

On the other hand, taking into account that $N^* = N + \Theta\partial_t$, we easily verify that $|N^*| = |\nabla h|$. Thus, from (3.55) we get

$$n|H| \leq n|H_f| + |\overline{\nabla} f||\nabla h|. \tag{3.56}$$

Consequently, since $H_f$, $\overline{\nabla} f$ and $\nabla h$ are supposed to be bounded, it follows from (3.56) that $H$ is also bounded on $\Sigma^n$.

Hence, since we are assuming that $\overline{\nabla} f$ is bounded then we can apply Proposition 3.1 of [18] jointly with generalized maximum principle of Omori and Yau to guarantee that there exists a sequence $\{p_k\}$ in $\Sigma^n$ such that

$$\lim_k h(p_k) = \sup_\Sigma h, \ \lim_k |\nabla h(p_k)| = 0 \ \text{and} \ \limsup_k \Delta_f h(p_k) \leq 0.$$

Thus, from inequality (3.54) we get

$$0 \geq \limsup_k \Delta_f h(p_k) \geq \lim_k (H_f - (\log\rho)'(h))(p_k) \geq 0.$$

Therefore, we have that $\lim_k (H_f - (\log\rho)'(h))(p_k) = 0$ and, taking into account our hypothesis (3.53), we conclude the proof. □

## 3.5 Weighted static GRW spacetimes

Along this section, we treat the case when the ambient space is a *static* GRW spacetime, that is, its warping function is constant, which, without loss of generality, can be supposed equal to

1. In this context, we will need of the following formula

**Lemma 3.5.1.** *Let $\Sigma^n$ be a spacelike hypersurface with constant $f$-mean curvature $H_f$ in a weighted static GRW spacetime $-I \times \mathbb{P}_f$. Then,*

$$\Delta_f \Theta = (|A|^2 + \widetilde{\mathrm{Ric}}_f(N^*, N^*))\Theta, \tag{3.57}$$

*where $A$ denotes the Weingarten operator of $\Sigma^n$ with respect to the future-pointing Gauss map $N$ of $\Sigma^n$ and $\widetilde{\mathrm{Ric}}_f$ stands for the Bakry-Émery Ricci tensor of the fiber $\mathbb{P}$.*

The proof of Lemma 3.5.1 follows the sames steps of Lemma 3.3.1 and, therefore, it will be omitted. Now, we can return to our uniqueness results.

**Theorem 3.5.1** (Theorem 3 [25])**.** *Let $\Sigma^n$ be a complete spacelike hypersurface in a weighted static GRW spacetime $-I \times \mathbb{P}_f$, such that its $f$-mean curvature $H_f$ does not change sign. If $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$, then $\Sigma^n$ is $f$-maximal. In addition, we also have the following:*

*(i) If $A$ is bounded and $\widetilde{\mathrm{Ric}}_f \geq 0$, then $\Sigma^n$ is totally geodesic. Moreover, if $\widetilde{\mathrm{Ric}}_f > 0$, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*(ii) If $\Sigma^n$ lies in a timelike bounded region of $-I \times \mathbb{P}_f$, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*Proof.* From Lemma 3.4.1 we have

$$\Delta_f h = -nH_f \langle N, \partial_t \rangle. \tag{3.58}$$

Since we are supposing that $H_f$ does not change sign on $\Sigma^n$, from equation (3.58) we get that $\Delta_f h$ also does not change sign on $\Sigma^n$. Thus, since $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$, we can apply Lemma 3.2.2 to conclude that $\Delta_f h$ vanishes identically on $\Sigma^n$. Hence, returning to equation (3.58) we obtain that $\Sigma^n$ is $f$-maximal.

Assuming the hypothesis of item $(i)$, we have

$$|\nabla \Theta| \leq |A||\nabla h| \in \mathscr{L}^1_f(\Sigma).$$

So, if $\widetilde{\mathrm{Ric}}_f \geq 0$, then from Lemmas 3.2.2 and equation (3.57) we get that $\Delta_f \Theta = 0$ on $\Sigma^n$. Hence, from (3.57) we conclude that $\Sigma^n$ is totally geodesic. Moreover, if $\widetilde{\mathrm{Ric}}_f > 0$, then (3.57) also gives that $N^* = 0$ on $\Sigma^n$, that is, $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.

On the other hand, since $\Sigma^n$ is $f$-maximal, we have

$$\Delta_f h^2 = 2h\Delta_f h + 2|\nabla h|^2 = 2|\nabla h|^2.$$

Now, since $|\nabla h| \in \mathscr{L}^1_f(\Sigma)$ and assuming that $\Sigma^n$ lies in a timelike bounded region of $-I \times \mathbb{P}_f$, then from Lemma 3.2.2 we get that $\Delta_f h^2 = 0$ on $\Sigma^n$. Hence, we obtain that $|\nabla h| = 0$ on $\Sigma^n$, which proves item $(ii)$. $\square$

When the fiber of the ambient spacetime is compact, we have

**Theorem 3.5.2** (Theorem 4 [25]). *Let $-\mathbb{R} \times \mathbb{P}^n_f$ be a static weighted GRW spacetime, whose fiber $\mathbb{P}^n$ is compact with positive sectional curvature and such that the weighted function $f$ is convex. Let $\Sigma^n$ be a complete spacelike hypersurface with constant $f$-mean curvature $H_f$ in $-\mathbb{R} \times \mathbb{P}^n_f$. If $|\nabla h|$ is bounded, then $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}$.*

*Proof.* Using the fact that $\mathbb{P}^n$ is compact with $K_{\mathbb{P}} > 0$, it follows from inequalities (3.3) and (3.4) of [32] that there exists a positive constant $\kappa$ such that

$$\begin{aligned} \mathrm{Ric}(X,X) \quad &\geq \quad \kappa\left((n-1)|X|^2 + |\nabla h|^2|X|^2 + (n-2)\langle X, \nabla h\rangle^2\right) \\ &\quad\; + nH\langle AX,X\rangle + |AX|^2. \end{aligned} \tag{3.59}$$

Since we are supposing that the weighted function $f$ is convex and taking into account that

$$\mathrm{Hess} f(X,X) = \overline{\mathrm{Hess}} f(X,X) - \langle \overline{\nabla} f, N\rangle\langle AX,X\rangle,$$

we have

$$\mathrm{Hess} f(X,X) \geq -\langle \overline{\nabla} f, N\rangle\langle AX,X\rangle. \tag{3.60}$$

From (3.59) and (3.60) we get the following lower bound for $\mathrm{Ric}_f$

$$\begin{aligned} \mathrm{Ric}_f(X,X) \quad &\geq \quad \kappa\left((n-1)|X|^2 + |\nabla h|^2|X|^2 + (n-2)\langle X, \nabla h\rangle^2\right) \\ &\quad\; + nH_f\langle AX,X\rangle + |AX|^2. \end{aligned} \tag{3.61}$$

Inequality (3.61) provides us

$$\begin{aligned} \mathrm{Ric}_f(\nabla h, \nabla h) \quad &\geq \quad (n-1)\kappa|\nabla h|^2(1+|\nabla h|^2) \\ &\quad\; + nH_f\langle A(\nabla h), \nabla h\rangle + |A(\nabla h)|^2. \end{aligned} \tag{3.62}$$

Since $H_f$ is constant, we have

$$\nabla \Delta_f h = -nH_f A(\nabla h). \tag{3.63}$$

Moreover, from Bochner's formula (1.1), we have again (3.32). Consequently, from (3.62), (3.63) and (3.32) we get

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq (n-1)\kappa|\nabla h|^2(1+|\nabla h|^2). \tag{3.64}$$

Now, we observe that we can write

$$nH_f\langle AX,X\rangle + |AX|^2 = \left|AX+\frac{nH_f}{2}X\right|^2 - \frac{n^2H_f^2}{4}|X|^2. \tag{3.65}$$

Thus, from (3.61) and (3.65) we obtain that

$$\mathrm{Ric}_f(X,X) \geq -\frac{n^2H_f^2}{4}|X|^2, \tag{3.66}$$

for all $X \in \mathfrak{X}(\Sigma)$.

Consequently, from equation (3.66) we conclude that the Bakry-Émery Ricci tensor of $\Sigma^n$ is bounded from below. Hence, from Theorem 1.1.1 we have that there exists a sequence of points $(p_k)_{k\geq 1}$ in $\Sigma^n$ such that

$$\lim_k |\nabla h|^2(p_k) = \sup_\Sigma |\nabla h|^2 \text{ and } \limsup_k \Delta_f|\nabla h|^2(p_k) \leq 0.$$

Now, returning to (3.64), we conclude that

$$0 \geq \limsup_k \Delta_f|\nabla h|^2(p_k) \geq \kappa \sup_\Sigma |\nabla h|^2 \geq 0.$$

Consequently, we obtain that $\sup_\Sigma |\nabla h|^2 = 0$ and, hence, $h$ is constant on $\Sigma^n$. Therefore, $\Sigma^n$ is a slice $\{t\}\times\mathbb{P}^n$. $\square$

Proceeding, we obtain the following

**Theorem 3.5.3** (Theorem 5 [25]). *Let $-\mathbb{R}\times\mathbb{P}^n_f$ be a weighted static GRW spacetime, whose sectional curvature $K_{\mathbb{P}}$ of its fiber $\mathbb{P}^n$ is such that $K_{\mathbb{P}} \geq -\kappa$ for some positive constant $\kappa$ and such that the weighted function $f$ is convex. Let $\Sigma^n$ be a complete spacelike hypersurface with constant $f$-mean curvature $H_f$ in $-\mathbb{R}\times\mathbb{P}^n_f$ and with bounded second fundamental form $A$. If*

$$|\nabla h|^2 \leq \frac{\alpha}{\kappa(n-1)}|A|^2, \tag{3.67}$$

*for some constant $0<\alpha<1$, then $\Sigma^n$ is a slice $\{t\}\times\mathbb{P}^n$.*

*Proof.* From equation (3.67) we get $\inf_{p\in\Sigma}\Theta(p)$ exists and it is negative.

On the other hand, since the weighted function $f$ is convex, $|N^*| = |\nabla h|$ and taking a local orthonormal frame $\{E_1,\dots,E_n\}$ on $\mathbb{P}^n$, we have that

$$\begin{aligned}\widetilde{\mathrm{Ric}}_f(N^*,N^*) &\geq \widetilde{\mathrm{Ric}}(N^*,N^*)\\ &= \sum_i \langle R_{\mathbb{P}}(N^*,E_i)N^*,E_i\rangle_{\mathbb{P}}\\ &= \sum_i K_{\mathbb{P}}(N^*,E_i)\left(\langle N^*,N^*\rangle_{\mathbb{P}} - \langle N^*,E_i\rangle_{\mathbb{P}}^2\right)\\ &\geq -\kappa\sum_i\left(\langle N^*,N^*\rangle_{\mathbb{P}} - \langle N^*,E_i\rangle_{\mathbb{P}}^2\right)\\ &= -\kappa(n-1)|\nabla h|^2,\end{aligned} \tag{3.68}$$

where we also have used our restriction on the sectional curvature $K_{\mathbb{P}}$ of $\mathbb{P}^n$.

Thus, from equation (3.57) jointly with (3.67) and (3.68), we get

$$\begin{aligned}\Delta_f\Theta &\leq \left(|A|^2 - \kappa(n-1)|\nabla h|^2\right)\Theta\\ &\leq (1-\alpha)|A|^2\Theta \leq 0.\end{aligned}$$

On the other hand, following the same ideas of Theorem 3.5.2, we can verify that the Bakry-Émery Ricci tensor of $\Sigma^n$is bounded from below and, hence, from Theorem 1.1.1 there exists a sequence of points $p_k \in \Sigma^n$ such that

$$\liminf_k \Delta_f\Theta(p_k) \geq 0$$

and

$$\lim_k \Theta(p_k) = \inf_{p\in\Sigma}\Theta.$$

Consequently,

$$\lim_k \Theta^2(p_k) = \sup_{p\in\Sigma}\Theta^2.$$

Thus,

$$0 \leq \liminf_k \Delta_f\Theta(p_k) \leq (1-\alpha)\liminf_k |A|^2(p_k)\inf_{p\in\Sigma}\Theta \leq 0.$$

Up to a subsequence, it follows that $\lim_k |A|^2(p_k) = 0$. Now, by using hypothesis (3.67), we obtain that $\lim_k |\nabla h|^2(p_k) = 0$ and therefore $\sup_{p\in\Sigma}\Theta^2 = \lim_k \Theta^2(p_k) = 1$. But $\Theta^2 \geq 1$, hence, $\Theta^2 = 1$ on $\Sigma^n$ and, therefore, $\Sigma^n$ is a slice $\{t\}\times\mathbb{P}^n$. □

From the proof of Theorem 3.5.3 it is not difficult to see that we also get

**Corollary 3.5.1.** *Let $-\mathbb{R}\times\mathbb{P}^n_f$ be a weighted static GRW spacetime, such that $K_{\mathbb{P}} \geq -\kappa$ and $\overline{\mathrm{Hess}} f \geq -\gamma$ for some positive constants $\kappa$ and $\gamma$. Let $\Sigma^n$ be a complete spacelike hypersurface with constant $f$-mean curvature $H_f$ in $-\mathbb{R}\times\mathbb{P}^n_f$ and with bounded second fundamental form $A$. If*

$$|\nabla h|^2 \leq \frac{\alpha}{\kappa(n-1)+\gamma}|A|^2,$$

*for some constant $0<\alpha<1$, then $\Sigma^n$ is a slice $\{t\}\times\mathbb{P}^n$.*

We close our chapter, with the following result

**Theorem 3.5.4** (Theorem 6 [25])**.** *Let $-\mathbb{R}\times\mathbb{P}^n_f$ be a weighted static GRW spacetime, such that $K_{\mathbb{P}}\geq 0$ and $|\overline{\nabla} f|$ is bounded. Let $\Sigma^n$ be a complete spacelike hypersurface immersed in a timelike bounded region of $-\mathbb{R}\times\mathbb{P}^n_f$. If $|\nabla h|$ and $H_f$ are bounded and $H_f$ does not change sign on $\Sigma^n$, then $H_f$ is not globally bounded away from zero. In particular, if $f$ is convex and $H_f$ is constant, then $\Sigma^n$ is a slice $\{t\}\times\mathbb{P}^n$.*

*Proof.* Taking into account our restriction on the sectional curvature of the fiber $\mathbb{P}^n$ jointly with the hypothesis that $|\overline{\nabla} f|$, $|\nabla h|$ and $H_f$ are bounded on $\Sigma^n$, as in the proof of Theorem 3.4.2 we can apply Proposition 3.2 of [32] to guarantee that the Ricci curvature of $\Sigma^n$ is bounded from below.

Now, suppose for instance that $H_f\geq 0$ on $\Sigma^n$. Thus, since $\Sigma^n$ lies between two slices of $-\mathbb{R}\times\mathbb{P}^n$, from Lemma 3.4.1 and the generalized maximum principle of Omori [57] and Yau [71] we obtain a sequence of points $p_k\in\Sigma^n$ such that

$$0\leq \liminf_k \Delta_f(-h)(p_k) = n\liminf_k \left(H_f\Theta\right)(p_k).$$

On the other hand, note that

$$0=\lim_k |\nabla h|(p_k) = \lim_k \Theta^2(p_k)-1.$$

Thus, since $\Theta\leq -1$,

$$\lim_k \Theta(p_k) = -1.$$

Consequently,

$$0\leq \liminf_k \Delta_f(-h)(p_k) = -n\liminf_k H_f(p_k)\leq 0$$

and, hence, we conclude that

$$\liminf_k H_f(p_k) = 0.$$

If $H_f \leq 0$, with the aid of Lemma 3.4.1 and applying once more the generalized maximum principle of Omori [57] and Yau [71], we get a sequence $q_k \in \Sigma^n$ such that

$$0 \leq \liminf_k \Delta_f h(q_k) = -n \liminf_k \left(H_f \Theta\right)(q_k)$$

and

$$\lim_k \Theta(q_k) = -1.$$

Therefore, we conclude again that $H_f$ is not globally bounded away from zero.

When $H_f$ is constant, we have that, in fact, $H_f$ vanishes identically on $\Sigma^n$. Thus, since $\Sigma^n$ is contained into a timelike bounded region of $-\mathbb{R} \times \mathbb{P}^n_f$, there exists a constant $\beta$ such that $h - \beta$ is a positive harmonic function of the drifting Laplacian on $\Sigma^n$. Moreover, since $f$ is supposed convex, from inequality (3.61) to get that $\mathrm{Ric}_f \geq 0$.

On the other hand, since $\nabla f = \overline{\nabla} f + \langle \overline{\nabla} f, N^* \rangle N$ and $|N^*| = |\nabla h|$, we obtain

$$|\nabla f|^2 \leq |\overline{\nabla} f|^2 (1 + |\nabla h|^2).$$

Consequently, since we are assuming that $|\overline{\nabla} f|$ and $|\nabla h|$ are bounded, we have that $\nabla f$ is also bounded on $\Sigma^n$.

Therefore, we are in position to apply Corollary 1.4 of [45] and conclude that $h$ is constant on $\Sigma^n$, that is, $\Sigma^n$ is a slice $\{t\} \times \mathbb{P}^n$. $\square$

CHAPTER 4

# Height Estimate for weighted semi-Riemannian manifolds

The results of this chapter are part of [23].

## 4.1 Introduction

In 1954 Heinz [42] proved that a compact graph of positive constant mean curvature $H$ in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with boundary on a hyperplane can reach at most height $\frac{1}{H}$ from the hyperplane. A hemisphere in $\mathbb{R}^{n+1}$ of radius $\frac{1}{H}$ shows that this estimate is optimal. In particular, Heinz's result motivated several authors to approach the problem of obtain a priori estimates for the height function of a compact hypersurface whose boundary is contained into a slice of a Riemannian product space (see, for instance, [2, 3, 26, 39, 44, 53]).

Concerning the Lorentzian setting, López [52] obtained a sharp estimate for the height of compact constant mean curvature spacelike surfaces with boundary contained in a spacelike plane of the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$. Later on, de Lima [30] established a height estimate for compact spacelike hypersurfaces with some positive constant higher order mean curvature and whose boundary is contained in a spacelike hyperplane of the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{L}^{n+1}$. As in [52], through the computation of the height of the hyperbolic caps of $\mathbb{L}^{n+1}$, he showed that his estimate is sharp. Afterwards, he jointly with Colares [27] generalized the results of [30] to the context of the Lorentzian product spaces $-\mathbb{R}\times\mathbb{P}^n$.

In this chapter, we prove height estimates concerning compact hypersurfaces with nonzero constant weighted mean curvature and whose boundary is contained into a slice of $\varepsilon I\times\mathbb{P}$.

## 4.2 The Riemannian setting

Now, we present our first height estimate.

**Theorem 4.2.1** (Theorem 1 [23]). *Let $I\times\mathbb{P}^n_f$ be a weighted Riemannian product space with $\overline{\mathrm{Ric}}_f\geq 0$ and let $\Sigma^n$ be a compact hypersurface with boundary contained into the slice $\{s\}\times\mathbb{P}^n$, for some $s\in I$, and whose angle function $\Theta$ does not change sign. If $\Sigma^n$ has nonzero constant*

*$f$-mean curvature such that $nH_f^2 \leq |A|^2$, where $A$ denotes the Weingarten operator of $\Sigma^n$ with respect to its unit normal vector field $N$, then the height function $h$ of $\Sigma^n$ satisfies*

$$|h-s| \leq \frac{1}{|H_f|}. \tag{4.1}$$

*Proof.* Define on $\Sigma^n$ the function

$$\varphi = H_f h + \Theta. \tag{4.2}$$

From (3.14) and (3.10) we get that

$$\Delta_f \varphi = -\Theta(|A|^2 - nH_f^2 + \widetilde{\mathrm{Ric}}_f(N^*,N^*)). \tag{4.3}$$

Consequently, since $\widetilde{\mathrm{Ric}}_f(N^*,N^*) = \overline{\mathrm{Ric}}_f(N,N) \geq 0$, $nH_f^2 \leq |A|^2$ and choosing $N$ such that $-1 \leq \Theta \leq 0$, from (4.3) we get that $\Delta_f \varphi \geq 0$. Thus, we conclude from the maximum principle that $\varphi \leq \varphi_{|\partial\Sigma}$ and, hence, from (4.2) we have that

$$H_f h - 1 \leq H_f h + \Theta \leq H_f s. \tag{4.4}$$

We then consider the two possible cases. In the case that $H_f > 0$, from (3.10) we have $\Delta_f h \leq 0$ and, from the maximum principle, $h \geq s$ on $\Sigma^n$. Thus, from (4.4) we conclude that

$$h - s \leq \frac{1}{H_f}. \tag{4.5}$$

Finally, in the case that $H_f < 0$, from (3.10) we have $\Delta_f h \geq 0$ and, again from the maximum principle, $h \leq s$ on $\Sigma^n$. Thus, from (4.4) we must have

$$s - h \leq -\frac{1}{H_f}. \tag{4.6}$$

Therefore, estimate (4.1) follows from (4.5) and (4.6). □

**Remark 4.2.1.** *We point out that the hypothesis $nH_f^2 \leq |A|^2$ is automatically satisfied in the case that the weighted function $f$ is constant. Furthermore, taking into account Heinz's estimate [42] already commented in the introduction, we see that our estimate* (4.1) *is optimal.*

From Theorem 4.2.1 we obtain the following half-space type result

**Theorem 4.2.2** (Theorem 2 [23]). *Let $\mathbb{R} \times \mathbb{P}_f^n$ be a weighted Riemannian product space with $\overline{\mathrm{Ric}}_f \geq 0$ and $\mathbb{P}^n$ compact. Let $\Sigma^n$ be a complete noncompact two-sided hypersurface properly* *immersed in $\mathbb{R}\times\mathbb{P}^n_f$, whose angle function $\Theta$ does not change sign. If $\Sigma^n$ has nonzero constant $f$-mean curvature such that $nH_f^2 \leq |A|^2$, then $\Sigma^n$ cannot lie in a half-space of $\mathbb{R}\times\mathbb{P}$. In particular, $\Sigma^n$ must have at least one top and one bottom end.*

*Proof.* Suppose by contradiction that, for instance, $\Sigma^n \subset (-\infty,\tau]\times\mathbb{P}$, for some $\tau\in\mathbb{R}$. Thus, for each $s<\tau$ we define

$$\Sigma_s^+ = \{(t,x)\in\Sigma^n : t\geq s\}.$$

Since $\mathbb{P}$ is compact and $\Sigma^n$ is properly immersed in $\mathbb{R}\times\mathbb{P}^n_f$, we have that $\Sigma_s^+$ is a compact hypersurface contained in a slab of width $\tau-s$ and with boundary in $\{s\}\times\mathbb{P}$. Thus, we can apply Theorem 4.2.1 to get that $\Sigma_s^+$ is contained in a slab of width $\frac{1}{|H_f|}$, so that it must be $\tau-s\leq\frac{1}{|H_f|}$. Consequently, choosing $s$ sufficiently small we violate this estimate, reaching to a contradiction.

Analogously, if we suppose that $\Sigma^n\subset[\tau,+\infty)\times\mathbb{P}$ with $\tau\in\mathbb{R}$, for each $s>\tau$ we define $\Sigma_s^-$ by

$$\Sigma_s^- = \{(t,x)\in\Sigma; t\leq s\}.$$

Hence, since $\Sigma_s^-$ is a compact hypersurface with boundary in $\{s\}\times\mathbb{P}$, we can reason as in the previous case and obtain another contradiction. □

## 4.3 The Lorentzian setting

We proceeding with our second height estimate.

**Theorem 4.3.1** (Theorem 3 [23]). *Let $-I\times\mathbb{P}^n_f$ be a weighted Lorentzian product space with $\overline{\mathrm{Ric}}_f\geq 0$ and let $\Sigma^n$ be a compact spacelike hypersurface with boundary contained into the slice $\{s\}\times\mathbb{P}^n$, for some $s\in I$. If $\Sigma^n$ has nonzero constant $f$-mean curvature such that $nH_f^2\leq|A|^2$, where $A$ denotes the Weingarten operator of $\Sigma^n$ with respect to its future-pointing unit normal vector field $N$, then the height function $h$ of $\Sigma^n$ satisfies*

$$|h-s|\leq\frac{\max_{\partial\Sigma}|\Theta|-1}{|H_f|}. \tag{4.7}$$

*Proof.* Analogously, we define on $\Sigma^n$ the function

$$\varphi = -H_f h-\Theta. \tag{4.8}$$

From equation 3.57 and Lemma 3.4.1 we get that

$$\Delta_f \varphi = -\Theta(|A|^2 - nH_f^2 + \widetilde{\mathrm{Ric}}_f(N^*, N^*)).$$

Consequently, since $\widetilde{\mathrm{Ric}}_f(N^*, N^*) = \overline{\mathrm{Ric}}_f(N, N) \geq 0$, $nH_f^2 \leq |A|^2$ and choosing $N$ future-pointing (that is, $\Theta \leq -1$), we get that $\Delta_f \varphi \geq 0$. Thus, we conclude from the maximum principle that $\varphi \leq \varphi_{|\partial\Sigma}$ and, hence, from (4.8) we have that

$$-H_f h + 1 \leq -H_f h - \Theta \leq -H_f s + \max_{\partial\Sigma} |\Theta|. \tag{4.9}$$

We then consider the two possible cases. In the case that $H_f > 0$, note that $\Delta_f h \geq 0$ and, from the maximum principle, $h \leq s$ on $\Sigma^n$. Thus, from (4.9) we conclude that

$$s - h \leq \frac{\max_{\partial\Sigma} |\Theta| - 1}{H_f}. \tag{4.10}$$

Finally, in the case that $H_f < 0$, we have $\Delta_f h \leq 0$ and, again from the maximum principle, $h \geq s$ on $\Sigma^n$. Thus, from (4.9) we must have

$$h - s \leq \frac{1 - \max_{\partial\Sigma} |\Theta|}{H_f}. \tag{4.11}$$

Therefore, estimate (4.7) follows from (4.10) and (4.11). □

**Remark 4.3.1.** *Taking into account the height estimate of [30] mentioned in the introduction, we see that our estimate* (4.7) *is also sharp.*

Finally, reasoning as in the proof of Theorem 4.2.2, from Theorem 4.3.1 we get the following

**Theorem 4.3.2** (Theorem 4 [23]). *Let $-\mathbb{R} \times \mathbb{P}_f^n$ be a weighted Lorentzian product space with $\overline{\mathrm{Ric}}_f \geq 0$ and $\mathbb{P}^n$ compact. Let $\Sigma^n$ be a complete noncompact spacelike hypersurface properly immersed in $-\mathbb{R} \times \mathbb{P}_f^n$, with bounded angle function $\Theta$. If $\Sigma^n$ has nonzero constant $f$-mean curvature such that $nH_f^2 \leq |A|^2$, then $\Sigma^n$ cannot lie in a half-space of $-\mathbb{R} \times \mathbb{P}$. In particular, $\Sigma^n$ must have at least one top and one bottom end.*

**Remark 4.3.2.** *We recall that an integral curve of the unit timelike vector field $\partial_t$ is called a comoving observer and, for a fixed point $p \in \Sigma^n$, $\partial_t(p)$ is called an instantaneous comoving observer. In this setting, among the instantaneous observers at $p$, $\partial_t(p)$ and $N(p)$ appear naturally. From the orthogonal decomposition $N(p) = N^*(p) - \Theta(p)\partial_t(p)$, we have that $|\Theta(p)|$*

*corresponds to the energy* $e(p)$ *that* $\partial_t(p)$ *measures for the normal observer* $N(p)$*. Furthermore, the speed* $|\upsilon(p)|$ *of the Newtonian velocity* $\upsilon(p) := e^{-1}(p)N^*(p)$ *that* $\partial_t(p)$ *measures for* $N(p)$ *satisfies the equation* $|\upsilon(p)|^2 = \tanh(\cosh^{-1}|\Theta(p)|)$*. Hence, the boundedness of the angle function* $\Theta$ *of the spacelike hypersurface* $\Sigma^n$ *means, physically, that the speed of the Newtonian velocity that the instantaneous comoving observer measures for the normal observer do not approach the speed of light* 1 *on* $\Sigma^n$ *(cf. [65], Sections* 2.1 *and* 3.1*). In this direction, as it was already observed by Latorre and Romero [47], the assumption of* $\Theta$ *be bounded on a complete spacelike hypersurface is a natural hypothesis to supply the noncompactness of it.*

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