Universidade Federal de Alagoas

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Acoustic radiation torque on particles in the Rayleigh limit.

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Maceió, Alagoas – Brasil

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Tiago Peixoto da Silva Lobo

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Abstract

In the present work, the acoustic radiation torque generated on a Rayleigh particle is investigated. A formalism for the scattering of Rayleigh particles based on the partial wave expansion of the pressure field is presented, and, from it, we derive a very elegant and simple formula for the acoustic radiation torque upon a Rayleigh particle that can be used for harmonic waves with arbitrary wave front. In particular, we studied the acoustic radiation torque generated by Bessel-beams of arbitrary order.

Keywords: Radiation torque. Rayleigh particles. Scattering. Acoustic. Bessel beams.

Resumo

Neste trabalho, o torque acústico de radiação sofrido por uma partícula de Rayleigh é estudado. Um formalismo matémático para o espalhamento de partículas de Rayleigh baseado na expanção de ondas parciais do campo de pressão é apresentado, o que nos permite derivar uma fórmula simples e elegante para o torque gerado sobre uma partícula de Rayleigh por ondas harmônicas com frente de onda arbitrária. Em particular, nós estudamos o torque acústico de radiação gerado por feixes de Bessel de ordem arbitrária.

Palavras chave: Torque de radiação. Partículas de Rayleigh. Espalhamento. Acústica. Feixes de Bessel.

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Chapter 1

Introduction

The interaction between an acoustic source with a suspended object may induce forces that changes the dynamics of an object. This interaction can either set the object to spin through a phenomenon called acoustic radiation torque, or make it move through a phenomenon called acoustic radiation force. These phenomena are being used to accelerate, trap, levitate or even stretch the particle itself. Despite these phenomenons were first observed in the 19th century the lack of a theoretical model for the acoustic radiation torque exerted by an arbitrary wave upon an object of arbitrary size and shape provides the impetus to tackle this problem. Here, the Cartesian components of the acoustic radiation torque produced by an arbitrary wave in a nonviscous fluid are derived. In particular, we provide an analytical formula for the acoustic radiation torque upon particles which size is much smaller than the wavelength that can be used as a tool for acoustic levitation, acoustic manipulation and acoustic trapping.

1.1 Historical perspective

This section gives the reader a good background of the area by making an overview that goes from the pioneer paper of Lord Rayleigh to the latest publications on experimental observation concerning transfer of angular momentum and new insights in the mathematical description of this problem, including two articles that the author of this thesis participated in.

The first known observation of torque, due an interaction between an acoustic wave and a suspended object, was made by Lord Rayleigh [17] in 1882. In his paper, Lord Rayleigh used an instrument capable of measuring the intensity of aerial vibrations and predicted that a suspended object interacting with an acoustic wave could make an object spin. M. Kotani [11] and King [10] calculated the acoustic radiation torque upon a Rayleigh disk, while Keller [9] extended their formulation for infinitively long thin strips and rigid disks of various shapes all used Lord Rayleigh's observations. However in 1958, Maidanik [12] derived a formula where the torque was calculated for any incident beam upon any arbitrarily shaped object.

Here, it is important to introduce the reader to a more detailed description of the acoustic radiation torque phenomenon. The torque studied in the previous publications were caused by an uneven pressure field on the surface of the object, therefore causing it to spin. However, this is not the only physical process that causes torque on an scatterer [22, 23]. The other physical process transfers momentum from the fluid particles in the vicinity of the scatterer to the scatterer. In 1981 Busse [4] published a paper showing the mathematical formalism of this new kind of torque, now known as viscous torque.

1.2 State of the Art

After a period without advances in this field, the scientific community turned its attention back to the acoustic radiation torque again when in 2003, Thomas [20] published a paper where a comparison between acoustic vortices and optical vortices was made. In his paper, he derived a relation between the angular momentum and the linear pseudo-momentum that enabled Marston [26] to deduce that the torque exerted by a monochromatic nonparaxial acoustic vortex beam, on an on-axis configuration, which is proportional to the power absorbed by the object. Later, Volke-Sepulveda [21] showed in his paper that acoustic beams with angular moment transfer part of it to the scatterer, predicting, that vortex-beams, like the acoustic Bessel-beam, would produce a torque in spherical objects through angular momentum transference.

Besides these advances, it was not yet possible to calculate the torque due to an arbitrary acoustic wave upon an arbitrary shaped scatterer analytically. The work published by Maidanik was at that time the most used way to calculate torque, but to solve the integrals that he proposed, a numerical scheme was often used. In 2008, Fan [5] analytically calculated the torque of non-absorptive irregularly shaped small particles, which actually provides a way to calculate the torque upon rigid arbitrarily shaped bodies, but did not take into account momentum transfered through absorption. The main focus of our work is to give an analytical formula that calculates the torque generated by an arbitrary acoustic wave when scattered by a compressional particle which size is much smaller than the wavelength. Although not being the main focus of this thesis, we will also outline a formula obtained by Farid, Lobo and Silva [18] that calculates the torque of an arbitrary acoustic field upon an object of any geometrical shape and size through a semi-analytical algorithm, and the results of [14], in which the same authors derived a closed formula for the axial radiation torque of a progressive, standing and quasi-standing Bessel beam of any order upon an object of any geometrical shape and size.

1.3 Contributions

In this work we develop a closed formula for the torque of an arbitrary acoustic beam upon a compressional particle which size is much smaller than the wavelength, i.e. a Rayleigh spherical particle. In this thesis the author shows that spherical particles have zero torque unless they are made of an absorptive material. The presented formula gives the torque of a spherical Rayleigh particle as a function of the velocity field of the incident wave at the origin of the coordinate system, the wave number (k), and the acoustical parameters of the fluid medium and of the fluid scatterer. To illustrate the method the author of this thesis derived the torque formula for the case in which the incident beam is an off-axis Bessel beam of any order and, unexpectedly, only the first-order Bessel beam will cause the scatterer to spin in the on-axial configuration.

It is important to emphasize the difference between our formulation and the one already published by Fan [5]. Although both researches are focused in torque calculations in the Rayleigh limit, this thesis considers absorptive particles, thus, it takes into account angular moment transferences between the incident wave field and the scatterer, while Fan's work only takes into account the torque due to Bernoulli pressure on the surface of the scatterer. This enables us to predict the torque of an absorptive sphere, while Fans paper is focused on non-absorptive asymmetric objects.

Chapter 2

Acoustic wave propagation

In this chapter the basic equations of linear acoustics are described. We also introduce the equations of the acoustic radiation torque, which is the subject of the present investigation. In order to derive the basic equations of acoustics we introduce the reader to some useful mathematical tools: material derivative and Reynolds transport theorem.

2.1 Mathematical tools

In fluid dynamics the position of a fluid element changes with time, following an external perturbation. Thus, it is important to define an operator that corresponds the rate in which certain quantities vary within the fluid. Here it's worth saying that a fluid element is not a microscopic entity, but rather a volume so small (infinitesimal) and in local thermodynamical equilibrium. Therefore, the field variations that describe the fluid, such as particle velocity $\mathbf{u}$, density ρ and pressure p, can be neglected inside a fluid element .

Let $\mathbf{x}(t)$ be a trajectory of a fluid element, as shown in figure 2.1, where t is time of the measure. We would like to measure the time rate in which a quantity changes along the trajectory, i.e., we want to be able to take time derivatives along trajectories. Considering $f(\mathbf{x}(t), t)$ a smooth function (functions of class $\mathbf{C}^\infty$), of the position $\mathbf{x}(t)$ at the time t, we define

$$f_{\mathbf{x}}(t) \equiv f(\mathbf{x}(t), t), \tag{2.1}$$

$$f'_{\mathbf{x}}(t) = \nabla f(\mathbf{x}(t), t) \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial f(\mathbf{x}(t), t)}{\partial t}. \tag{2.2}$$

where the symbol $\cdot$ means scalar product. Defining the particle velocity as $\mathbf{u} = d\mathbf{x}/dt$, we have

$$f'_{\mathbf{x}}(t) = \left(\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f\right) \tag{2.3}$$

Therefore, we can define a time differentiation operator which acts along fluid element trajectories, the material derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla. \tag{2.4}$$

For functions that depends only on the position $\mathbf{x}(t)$ and the time t the operator material derivative is equal to the total derivative in t of the function.

It's important to notice that while partial derivatives acts in a fixed point of space, the material derivative acts along the trajectory of an element. For example, the acceleration $\mathbf{a}(\mathbf{x}, t)$ of a fluid element $\mathbf{x}$ is not given by $\partial \mathbf{u}/\partial t$, but

$$\mathbf{a}(\mathbf{x}(t), t) = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{2.5}$$

Another important mathematical tool is the Reynold's transport theorem. Let $f(\mathbf{x}(t), t)$ be a smooth function, Ω_t be a region in the Euclidean space in a given

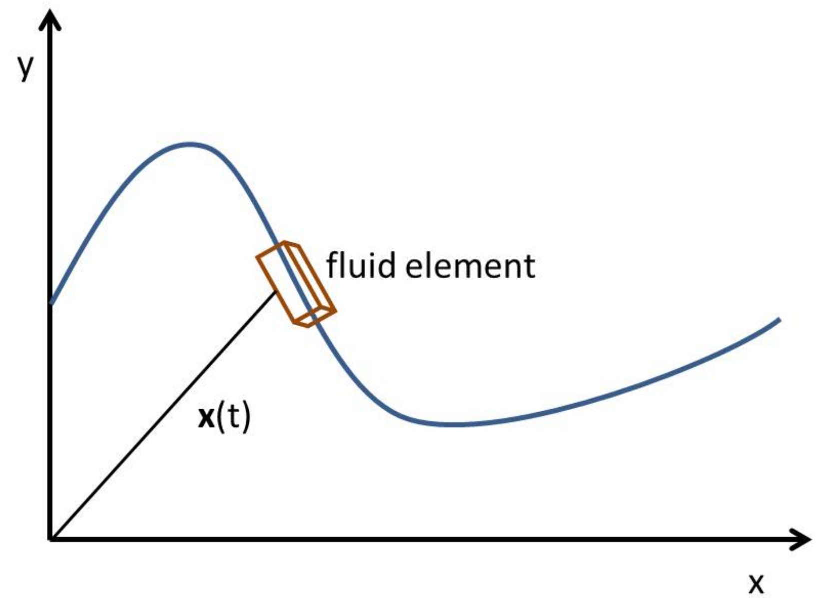

Figure 2.1: The trajectory of a fluid element (a brown box) is shown. $\mathbf{x}(t)$ denotes the position of the fluid element at a time t in his given trajectory (denoted by a blue curve).

time t, and $\mathbf{u}$ a fluid element velocity. The Reynold's transport theorem asserts that

$$\frac{d}{dt}\int_{\Omega_t} f dV = \int_{\Omega_t}\left(\frac{Df}{Dt} + f\nabla\cdot\mathbf{u}\right)dV, \tag{2.6}$$

$$= \int_{\Omega_t}\frac{\partial f}{\partial t}dV + \int_{\partial\Omega_t} f\mathbf{u}\cdot\mathbf{n}dA, \tag{2.7}$$

where $\partial\Omega_t$ is the bounding surface of Ω_t and $\mathbf{n}$ is the outward normal of $\partial\Omega_t$. The first term of the above equation gives the local variation of f while the second term takes into account the flux of f across the boundary of Ω_t. A detailed deduction of the Reynolds transport theorem is shown in [3].

2.2 Mass conservation equation

At a given time t consider a fluid region Ω_t, the fluid mass m is given by

$$m = \int_{\Omega_t} \rho(\mathbf{x}(t), t) dV; \tag{2.8}$$

where $\rho(\mathbf{x}(t), t)$ is the mass density. In the absence of sinks or sources, the fluid mass inside Ω_t is conserved. Hence, the time derivative of m vanish, witch yields

$$\frac{d}{dt} m = \frac{d}{dt} \int_{\Omega_t} \rho(\mathbf{x}(t), t) dV = 0. \tag{2.9}$$

Using the Reynold's transport (2.6) theorem in the above equation leads us to

$$\int_{\Omega_t} \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} \right) dV = 0. \tag{2.10}$$

Assuming that ρ is a smooth we find

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \tag{2.11}$$

This is the continuity equation in the differential form. This equation guarantees that the mass is conserved inside an element of the fluid.

2.3 Euler's equation of motion

In this section we use the Newton's law of motion to deduce the equation for balance of momentum. The linear momentum of a fluid region Ω_t is defined as

$$\mathbf{P} \equiv \int_{\Omega_t} \rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t) dV. \tag{2.12}$$

Thus, the Newton's second law of motion can be stated as

$$\mathbf{F} = \frac{d}{dt} \int_{\Omega_t} \rho \mathbf{u} dV = \int_{\Omega_t} \rho \frac{D\mathbf{u}}{Dt} dV, \tag{2.13}$$

where the second equality follows from the Reynold's transport theorem.

The net force on Ω_t is due to external forces (also known as body-forces) and internal forces (surface stress forces). External forces acts per unit of volume, thus it can be expressed as the product $\rho\mathbf{b}$, where $\mathbf{b}$ is a vector quantity associated with the external field. Internal forces have short range, i.e., they rapidly decay with distance. Furthermore, they are responsible for momentum exchange between particles. Internal forces can be written as a product between a stress tensor $\mathbf{T}(\mathbf{x}(t), t)$, with the normal $\mathbf{n}$ at the point $\mathbf{x}$ on the boundary $\partial\Omega_t$.

Using the Cauchy's stress theorem [2]

$$\mathbf{F} = \int_{\Omega_t} \rho \frac{D\mathbf{u}}{Dt} dV = \int_{\partial\Omega_t} \mathbf{T}(\mathbf{x}, t) \cdot \mathbf{n} dA + \int_{\Omega_t} \rho(\mathbf{x}, t)\mathbf{b}(\mathbf{x}, t) dV; \tag{2.14}$$

$$= \int_{\Omega_t} (\nabla\mathbf{T} + \rho\mathbf{b}) dV. \tag{2.15}$$

In this work we assume an ideal fluid, i.e. the fluid does not support tangential stress. For ideal fluids $\mathbf{T} = -p\mathbf{I}$, where $\mathbf{I}$ is the identity tensor of second rank, and p is the pressure. Therefore, one can obtain the Euler's equation of motion:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{b}. \tag{2.16}$$

2.4 Equations of linear acoustics

A small-amplitude acoustic perturbation can be usually described in terms of deviations from the ambient state of pressure, density and, particle velocity (p_0,ρ_0,$\mathbf{u}_0$). The corresponding acoustic disturbances are denoted by the functions (p',ρ',$\mathbf{u}'$). Considering that the perturbations are small, we can neglect higher-order terms, i.e, just the linear terms will be considered. Hence, the total acoustic fields read

$$p = p_0 + p', \tag{2.17}$$

$$\rho = \rho_0 + \rho', \tag{2.18}$$

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'. \tag{2.19}$$

We consider a homogeneous quiescent media, i.e. the ambient variables are independent of position and time, and the reference particle velocity is zero. Substituting equations 2.17, 2.18 and 2.19 into the mass and momentum conservation equations, we obtain

$$\frac{\partial}{\partial t}(\rho_0 + \rho') + \nabla\cdot [(\rho_0 + \rho')\mathbf{u}'] = 0, \tag{2.20}$$

$$(\rho_0 + \rho')\left(\frac{\partial}{\partial t} + \mathbf{u}'\cdot\nabla\right)\mathbf{u}' = -\nabla(p_0 + p'), \tag{2.21}$$

$$p_0 + p' = p(\rho_0 + \rho'). \tag{2.22}$$

In the last equation it was assumed that the acoustic process is adiabatic, i.e. constant entropy. Therefore the pressure is a function of density only.

Now we expand the excess pressure p' in a Taylor series, as follow

$$p' = \left(\frac{\partial p}{\partial \rho}\right)\bigg|_0 \rho' + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho^2}\right)\bigg|_0 (\rho')^2 + ... \tag{2.23}$$

where the subscript 0 denotes that the derivatives are evaluated at constant entropy and ambient density ρ_0. Thus, the linear acoustic equations take the form:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla\cdot\mathbf{u}' = 0; \tag{2.24}$$

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} = -\nabla p'; \tag{2.25}$$

$$p' = c^2 \rho', \quad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_0, \tag{2.26}$$

where c is the adiabatic speed of sound.

Now we are able to derive a linear wave equation from equations 2.24-2.26. In doing so, we eliminate ρ' from equation 2.24 and take the time derivative of the resulting equation. Taking the divergent of equation 2.25 and substituting into equation 2.24 we obtain

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0. \tag{2.27}$$

This is the linear wave equation for acoustic pertubations in terms of the pressure.

From equation 2.25 we notice $\frac{\partial}{\partial t}(\nabla \times \mathbf{u}) = 0$, a result that only holds for ideal fluids. Then one can write $\mathbf{u}$ as a gradient of a scalar potential ϕ. This way we can express the pressure and the velocity field as a function of the same potential:

$$\mathbf{u} = -\nabla\phi; \tag{2.28}$$

$$p = \rho_0 \frac{\partial \phi}{\partial t}. \tag{2.29}$$

The scalar potential ϕ also satisfies the linear wave equation

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0. \tag{2.30}$$

Our interest is how time harmonic waves can induce torque upon suspended particles in the fluid. Hereafter, the position vector $\mathbf{x}$ does not depend on time. Thus, the pressure is described by $p(\mathbf{x}, t) = \hat{p}(\mathbf{x})e^{-i\omega t}$. where ω is the angular frequency and, $\hat{p}$ is the complex amplitude, which depends only in the position vector. Substituting $p(\mathbf{x}, t)$ into the linear wave equation for the pressure we obtain the Helmholts wave equation.

$$\left(\nabla^2 + k^2\right)\hat{p}(\mathbf{x}) = 0, \tag{2.31}$$

where $k = \omega/c$ is the wavenumber.

2.5 Acoustic energy intensity

The acoustic energy w is defined as [16]:

$$w \equiv \frac{1}{2}\rho_0 u^2 + \frac{1}{2}\frac{p^2}{\rho_0 c^2}. \tag{2.32}$$

The term $\rho_0 u^2/2$ is called acoustic kinetic energy density, while the second therm $p^2/(2\rho_0 c^2)$ is the acoustic potential energy. The Lagrangian density function is defined as

$$L \equiv \frac{1}{2}\rho_0 u^2 - \frac{1}{2}\frac{p^2}{\rho_0 c^2}. \tag{2.33}$$

Other useful quantity is the acoustic intensity I, defined as the product of the sound pressure and the particle velocity [16]:

$$I = p\mathbf{u}. \tag{2.34}$$

The first term of the right hand side of equation 4.21 is zero is because $\mathbf{n}$ is colinear with $\mathbf{x}$.

Chapter 3

Acoustic scattering theory

The acoustic radiation torque is a second-order acoustic phenomenon that occurs from the interaction between an arbitrary wave field upon a suspended object. Thus, in order to obtain both the incident and scattered acoustic fields. This chapter presents the solution of the Helmholtz equation in spherical coordinates given in terms of the partial wave expansion for the incident, scattered and transmitted waves. Thereafter, the solutions are obtained for an arbitrary sized sphere and then simplified to small compressible particles.

3.1 Solutions of the Helmholtz equation in spherical coordinates

In spherical coordinates (r, θ, ϕ) this equation takes the form [1]

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \hat{p}}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \hat{p}}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \hat{p}}{\partial \phi^2} + k^2 p = 0 \tag{3.1}$$

where ϕ is the azimuthal angle, r is the radial distance and θ the polar angle. The solution of this equation can be obtained through the method of separation of variables. We write the pressure amplitude as

$$\hat{p}(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi), \tag{3.2}$$

where $R(r)$, $\Theta(\theta)$ and $\Phi(\phi)$ are the radial, polar and azimuthal functions. Substituting this equation into equation 3.1 result

$$\begin{aligned} \frac{1}{\Phi}\frac{d^2\Phi(\phi)}{d\phi^2} &= -m^2; && (3.3)\\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + l(l+1)\Theta &= 0; && (3.4)\\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{l(l+1)R}{r^2} &= 0, && (3.5) \end{aligned}$$

where m and l are integers yet to be determined.

The solution to equation 3.3 is given by

$$\Phi(\phi) = \Phi_1 e^{im\phi} + \Phi_2 e^{-im\phi}, \tag{3.6}$$

where Φ_1 and Φ_2 are constants.

The radial equation 3.5 is the spherical Bessel equation [1], which has the solution

$$R(r) = R_1 j_l(kr) + R_2 y_l(kr), \tag{3.7}$$

where j_l and y_l are the spherical Bessel and spherical Neumann functions of lth-order, respectively, and, R_1 and R_2 are constants. Alternatively, this solution can be written as [1]

$$R(r) = R_3 h_l^{(1)}(kr) + R_4 h_l^{(2)}(kr) \tag{3.8}$$

where $h_l^{(1)}$ and $h_l^{(2)}$ are the spherical Hankel functions of the first and second kind of lth-order, respectively, and, R_3 and R_4 are constants.

To solve the equation 3.4 for the polar angle θ we introduce $\eta = \cos\theta$. Thus, the equation becomes

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d\Theta}{d\eta}\right] + \left[l(l+1) - \frac{m^2}{1-\eta^2}\right]\Theta = 0. \tag{3.9}$$

The solutions are the associated Legendre functions of the first- and second-kinds. The second-kind Legendre function diverges when $\eta = \pm 1$ [1], thus they are discarded here. Furthermore, when l is an integer then $P_l^m(\eta) = 0$ when $m > l$. Now, the solution of 3.9 assumes the form

$$\Theta(\theta) = \Theta_1 P_l^m(\cos\theta), \tag{3.10}$$

where $P_l^m(\cos\theta)$ is the associated Legendre function and Θ_1 is a constant. The angular functions Φ and Θ can be combined to form the spherical harmonics as follows [1]

$$Y_l^m(\theta,\phi) \equiv \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta)e^{im\phi}. \tag{3.11}$$

Now the general solution of the Helmholtz equation is:

$$\hat{p}(kr,\theta,\phi) = p_0 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(a_l^m j_l(kr) + b_l^m y_l(kr))Y_l^m(\theta,\phi), \tag{3.12}$$

or, alternatively:

$$\hat{p}(kr,\theta,\phi) = p_0 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(s_l^m h_l^{(1)}(kr) + c_l^m h_l^{(2)}(kr))Y_l^m(\theta,\phi), \tag{3.13}$$

where $\hat{p}_0$ is the pressure magnitude and a_l^m, b_l^m, s_l^m and c_l^m are constants to be determined for a specific acoustic problem.

3.2 Scattering by a compressional fluid sphere

Consider that a fluid sphere with density ρ_1 and compressional speed of sound c_1 and radius a is suspended at the origin of the coordinate system in a host fluid. In turn, the host fluid is characterized by its density ρ_0 and compressional speed of sound c_0. A harmonic wave with arbitrary wave front hits the suspended sphere.

The incident pressure field $\hat{p}_i$ has to be finite at the origin of the coordinate system. Hence, the coefficients (b_l^m) multiplying the spherical Neuman functions are set to zero because the Neumann function diverges at the origin of the coordinate system. Thus, the incident pressure field can be expressed as

$$\hat{p}_i(kr, \theta, \phi) = p_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m j_l(kr) Y_l^m(\theta, \phi), \tag{3.14}$$

where the a_l^m is called beam-shape coefficient (BSC) of the incident wave, to be determined for a specific acoustic beam. Considering the orthogonal relation of the spherical harmonics

$$\int_{4\pi} Y_{n_1}^{m_1 *}(\theta, \phi) Y_{n_2}^{m_2 *}(\theta, \phi) d\Omega = \delta_{n_1 n_2} \delta_{m_1 m_2}, \tag{3.15}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle, the integration is taken over the whole solid angle (4π), the symbol * denotes complex conjugation and $\delta_{n_1 n_2}$ is the Kronicker delta function. Multiplying equation 3.14 for $Y_{l_1}^{m_1 *}$ and using the orthogonality relation 3.15 the BSC is obtained

$$a_l^m = \frac{1}{p_0 j_l(kR)} \int_{4\pi} \hat{p}_i(kR, \theta, \phi) Y_l^{m*}(\theta, \phi) d\Omega \tag{3.16}$$

where R is the radius of a control sphere where $\hat{p}_i$ will be evaluated. This equation correspond to a spherical harmonics transform (SHT) and allows the reconstruction of a full 3D field just by knowing its values in the surface of the a control sphere, i.e, we can reconstruct a 3D field using 2D information.

The incoming acoustic energy is conserved, i.e. part of it is scattered by the fluid sphere, and part is transmitted into it. Thus, the total pressure field for this problem is composed of three components, the incident $\hat{p}_i$, the scattered $\hat{p}_s$ and the transmitted $\hat{p}_r$ pressure fields.

The scattered pressure should satisfy the Sommerfeld radiation condition [7]

$$\lim_{r\to\infty} r\left(\frac{\partial \hat{p}_s}{\partial r} - ikp_s\right) = 0. \tag{3.17}$$

This condition states that the scatterer wave is an outgoing spherical wave at infinity. Therefore, no energy should be reflected from infinity into the domain. The solution that satisfies the Sommerfeld radiation condition is

$$\hat{p}_s(kr,\theta,\phi) = p_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} s_l^m h_l^{(1)}(kr) Y_l^m(\theta,\phi), \tag{3.18}$$

where s_l^m is known as the scatterer coefficient.

The transmitted pressure field into the spherical should be regular at the origin of the coordinate system. Thus the transmitted pressure can be expressed as

$$\hat{p}_r(kr,\theta,\phi) = p_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} t_l^m j_l(kr) Y_l^m(\theta,\phi). \tag{3.19}$$

where t_l^m is the transmitted BSC.

In order to obtain a unique solution throughout the wave propagation domain both pressure and particle velocity should be continuous across the fluid sphere surface. Accordingly,

$$\hat{p}_i\big|_{r=a} + \hat{p}_s\big|_{r=a} = \hat{p}_r\big|_{r=a} \tag{3.20}$$

$$\hat{\mathbf{u}}_i\big|_{r=a} + \hat{\mathbf{u}}_s\big|_{r=a} = \hat{\mathbf{u}}_r\big|_{r=a}, \tag{3.21}$$

where $\hat{\mathbf{u}}_i$, $\hat{\mathbf{u}}_s$ and $\hat{\mathbf{u}}_t$ are the velocity acoustic fields of the incident, scattered and transmitted waves respectively. The particle velocity can be written in terms of the pressure field as

$$\hat{\mathbf{u}} = \frac{1}{ik\rho_0 c_0}\nabla\hat{p}. \tag{3.22}$$

Substituting equations 3.14, 3.18, 3.19 into equations 3.20 and 3.21, using equation 3.22 and noting that the spherical harmonics are linearly independent, we can match them to find the transmitted and scattered coefficients to be

$$s_l^m = a_l^m \frac{\gamma j_l'(ka) j_l(k_1 a) - j_l(ka) j_l'(k_1 a)}{\gamma h_l^{(1)'}(k_0 a) j_l(k_1 a) - h_l^{(1)}(ka) j_l'(k_1 a)}; \tag{3.23}$$

$$t_l^m = -\frac{a_l^m j_l(ka) + s_l^m h_l^{(1)}(ka)}{j_l(k_1 a)}; \tag{3.24}$$

where the symbol $'$ denotes differentiation with respect to the functions argument, $\gamma = k\rho_1/(k_1\rho_0)$ is the impedance index and k_1 the wavenumber inside the scatterer sphere.

3.3 Rigid sphere scattering

A rigid sphere is a sphere that reflects all the incoming energy. This behavior can be seen as a limiting case of the fluid sphere with $\rho_1 \to \infty$ and $c_1 \to \infty$. Applying this limit in equation 3.23 we obtain the rigid sphere scattering coefficient as

$$s_l^m = -\frac{j_l'(ka)}{h_l^{(1)'}(ka)} a_l^m. \tag{3.25}$$

3.4 Numerical calculation of the BSCs

In this section an algorithm to solve equation 3.16 numerically will be presented. These algorithm will be used to aid with the validation of the analytical torque formula presented in this thesis.

This algorithm will enable the calculation of the a_l^m for an arbitrary shaped beam only by knowing its value on a virtual spherical surface. In particular, this algorithm has been used in reference [18].

Substituting equation 3.11 into equation 3.16

$$\begin{aligned} a_l^m &= \sqrt{\frac{(2l+1)(l-m)!}{4\pi p_0^2 j_l(kR)^2(l+m)!}} \int_{-1}^{1} \left(P_l^m(\cos\theta) \right. \\ &\times \left. \int_0^{2\pi} \hat{p}\left(kR,\theta,\varphi\right) e^{-im\varphi} d\varphi \right) d(\cos\theta). \end{aligned} \tag{3.26}$$

The Gauss-Legendre quadrature is used to solve the polar integration, and a simple left point rule is used for the azimuthal integration. Defining $\theta_n = \arccos t_n$, with t_n $(n = 1, 2...N)$ being the nth root of the Legendre polynomial, and $\varphi_j = 2\pi j/M$, one can find

$$\begin{aligned} a_l^m &= \sqrt{\frac{\pi(2l+1)(l-m)!}{M^2 p_0^2 j_l(kR)^2(l+m)!}} \sum_{n=1}^{N} \omega_n P_l^m(t_n) \\ &\times \left[\sum_{j=0}^{M-1} \hat{p}\left(kR,\theta_n,\varphi_j\right) e^{-2i\pi mj/N} \right]. \end{aligned} \tag{3.27}$$

where ω_n is the weight coefficient in the Gauss-Legendre quadrature method. The term in brackets in the above equation can be solved by the FFT algorithm, thus the algorithm to compute equation 3.27 consumes time $O(NM \ln M)$.

3.5 Rayleigh scattering

We turn our attention to the scattering problem of acoustic waves by small particles compared to the incident wavelength (Rayleigh limit). In particular, rigid particles are shown in this section while the fluid case will be shown latter. According to equation 3.23 to obtain the scattering coefficients one has to calculate the BSC of the incident waves.

To obtain the scattering coefficients wave we notice from equation 3.23 that these coefficients can be decomposed, thus $s_l^m = s_l a_l^m$, thus $s_l = -j_l^{'}(ka)/h_l^{(1)'}(ka)$. Now considering the asymptotic expansion of the Bessel and Hankel functions

$$j_n^{'}(x) = \frac{2^n n x^{n-1} n!}{(2n+1)!} - \frac{2^n(n+2)x^{n+1}}{(2n+3)!} + O(x^{n+3}), \tag{3.28}$$

$$h_n^{'(1)}(x) \approx i(n+1)\frac{(2n)!}{2^n n!}x^{-n-2}. \tag{3.29}$$

Substituting these equations into the s_l definition will reveal that the dominant term is proportional to $(ka)^{2l+1}$. In the Rayleigh limit $ka \ll 1$, we consider only the monopole ($l = 0$) and dipole ($l = 1$) terms

$$s_0 = i\frac{(ka)^3}{3} - i\frac{(ka)^5}{5} + \frac{(ka)^6}{9} + O((ka)^7); \tag{3.30}$$

$$s_1 = -i\frac{(ka)^3}{6} + i\frac{(ka)^5}{20} + \frac{(ka)^6}{36} + O((ka)^7). \tag{3.31}$$

In order to calculate the BSC of incident wave we expand pressure field in a Taylor series around the particle position (the origin of the coordinate system $\mathbf{x} = \mathbf{0} = (0, 0, 0)$

$$\hat{p}(\mathbf{x}) = \hat{p}(\mathbf{0}) + R\hat{\mathbf{x}} \cdot \nabla\hat{p}(\mathbf{0}) + \frac{R^2}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\hat{x}_i\hat{x}_j\frac{\partial\hat{p}(\mathbf{x})}{\partial x_i \partial x_j}\bigg|_{\mathbf{x}=\mathbf{0}} + O(R^3), \tag{3.32}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is the position vector in Cartesian coordinates and $\hat{\mathbf{x}} = \mathbf{x}/R$ with $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Substituting the spherical Bessel function expansion for small arguments [1]

$$j_n(x) \approx x^n \frac{2^n n!}{(2n+1)!}, \tag{3.33}$$

and equation 3.32 int equation 3.16 one finds

$$a_l^m = \frac{(2l+1)!!}{p_0(kR)^l}\left[\hat{p}(\mathbf{0})\int_{4\pi} Y_l^{m*}(\theta,\phi)d\Omega + ikR\rho_0 c_0 \hat{\mathbf{u}}\Big|_{\mathbf{x}=0}\cdot\int_{4\pi}\hat{\mathbf{x}}Y_l^{m*}(\theta,\phi)d\Omega + \frac{R^2}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial\hat{p}(\mathbf{x})}{\partial x_i\partial x_j}\Big|_{\mathbf{x}=0}\int_{4\pi}x_i x_j Y_l^{m*}(\theta,\phi)d\Omega + O(R^3)\right], \quad (3.34)$$

where R and $(2n)!! = 2^n n!$ and $(2n+1)!! = (2n+1)!/(2^n n!)$. Note that we used the equation 3.22 and assumed that the pressure field doesn't varies inside the control sphere in the limit $R \to 0$. To perform the integrals of equation 3.34 we used $x_1 = \sin\theta\cos\phi$, $x_2 = \sin\theta\sin\phi$ and $x_3 = \cos\theta$. Therefore, the monopole ($l = 0$) and dipole ($l = 1$) terms are found to be

$$a_0^0 = \frac{2\sqrt{\pi}}{p_0}\hat{p}(\mathbf{0}), \quad (3.35)$$

$$a_1^1 = \frac{\sqrt{6\pi}\rho_0 c_0}{p_0}(-i\hat{u}_x(\mathbf{0}) - \hat{u}_y(\mathbf{0})), \quad (3.36)$$

$$a_1^0 = \frac{2\sqrt{3\pi}i\rho_0 c_0}{p_0}\hat{u}_z(\mathbf{0}), \quad (3.37)$$

$$a_1^{-1} = \frac{\sqrt{6\pi}\rho_0 c_0}{p_0}(i\hat{u}_x(\mathbf{0}) - \hat{u}_y(\mathbf{0})), \quad (3.38)$$

where $\hat{u}_x, \hat{u}_y, \hat{u}_z$ are the Cartesian components of the particle velocity of the incident wave.

Chapter 4

Acoustic radiation torque

In this chapter, we derive a general formula for the Cartesian components of the acoustic radiation torque produced by a single frequency incident beam of arbitrary wavefront on an object of any geometrical shape in a nonviscous fluid. Subsequently the obtained formula will be applied to a small scatterer in the Rayleigh limit. This consideration will allow the acoustic radiation torque to be calculated as a function of the incident beam and the properties of the scatterer (density, speed of sound, radius, and wavenumber). Part of the results outlined here were recently published in references [18, 14].

4.1 Acoustic radiation torque

In order to calculate the acoustic radiation torque we need to perform some operations to the the mass conservation equation 2.11. Multiplying it by $\mathbf{u}$ and using Euler's equation 2.16 we obtain

$$\frac{(\partial \rho \mathbf{u})}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u}[\nabla \cdot (\rho \mathbf{u})] = -\nabla p. \tag{4.1}$$

From equation 4.1 it follows that

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \rho\nabla\mathbf{u}\mathbf{u} = -\nabla p, \tag{4.2}$$

where $\mathbf{u}\mathbf{u}$ is a dyad formed by the tensorial product between $\mathbf{u}$ with itself, and the following identity was used [1]

$$\nabla\mathbf{u}\mathbf{u} = \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\nabla\cdot\mathbf{u}. \tag{4.3}$$

Here, we will carry our calculations up to second order-terms, simplifying the above equation to

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \rho_0\nabla\mathbf{u}\mathbf{u} = -\nabla p', \tag{4.4}$$

where p' is the excess of pressure $p - p_0$. Defining the stress tensor of linear momentum as

$$\mathbf{T} \equiv p'\mathbf{I} + \rho_0\mathbf{u}\mathbf{u}, \tag{4.5}$$

the conservation of linear momentum for an ideal fluid takes the form

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot\mathbf{T} = 0, \tag{4.6}$$

note that the divergence of the stress tensor $\mathbf{T}$ is the contraction of a 2nd-order tensor which results in a vector. Now we need to find the excess of pressure p' up to second order in the acoustic fields. Using $\mathbf{u} = -\nabla\phi$ in Euler's equation one can obtain the following relation

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right] = -\frac{\nabla p}{\rho}. \tag{4.7}$$

From thermodynamics we know that the enthalpy per unit mass w obeys the relation $dw = Tds + dp/\rho$, where T is the temperature and s are the entropy per unit mass.

Here we assume that the wave propagation in the fluid is an adiabatic process $ds = 0$. Therefore $\nabla w = \nabla p/\rho$, which leads to

$$w = -\frac{\partial \phi}{\partial t} - \frac{1}{2}|\nabla \phi|^2 + C, \tag{4.8}$$

where C is a constant. We can expand p in a Taylor series in w as follows:

$$p' = \left(\frac{\partial p}{\partial w}\right)\bigg|_0 w + \frac{1}{2}\left(\frac{\partial^2 p}{\partial w^2}\right)\bigg|_0 w^2 + ... \tag{4.9}$$

Using thermodynamic relations $(\partial w/\partial p)\big|_{s,0} = 1/\rho$ and $(\partial^2 p/\partial w^2)\big|_{s,0} = \rho/c^2$ we obtain

$$p' = \frac{1}{2}\frac{\rho_0}{c_0^2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}|\nabla \phi|^2, \tag{4.10}$$

where $C = 0$ because we have an infinite medium and no constrain has to be satisfied. Replacing $\mathbf{u} = -\nabla \phi$ and $\partial \phi/\partial t = p/\rho_0$ we obtain

$$p' = \frac{1}{2\rho_0 c_0^2}p^2 - \frac{1}{2}\rho_0 u^2. = -L \tag{4.11}$$

where L is the Lagrangian density. Thus, the stress-tensor $\mathbf{T}$ can be written as

$$\mathbf{T} \equiv -L\mathbf{I} + \rho_0 \mathbf{u}\mathbf{u}. \tag{4.12}$$

Applying the vector product $(\mathbf{x}\times)$ in equation 4.13 results

$$\frac{\partial}{\partial t}(\mathbf{x} \times \rho \mathbf{u}) + \nabla \cdot \mathbf{T}' = 0, \tag{4.13}$$

where

$$\mathbf{T}^{'} = \mathbf{x} \times \mathbf{T} \tag{4.14}$$

is the angular stress tensor. Equation 4.13 represents the angular momentum conservation in the fluid.

The acoustic radiation torque is defined in terms of the time-average of the angular stress-tensor on an object. Thus, we define the time-average of an oscillating single-frequency function $f(t)$ as

$$\langle f(t)\rangle \equiv \frac{1}{\delta t}\int_0^{\delta t} f(t)dt, \tag{4.15}$$

where δt is the oscillation period. It follows that the time-average of the product of two complex functions $f(t) = \hat{f}e^{-i\omega t}$ and $g(t) = \hat{g}e^{-i\omega t}$ is given by

$$\langle f(t)g(t)\rangle = \frac{1}{2}\mathrm{Re}\big[\hat{f}^{*}\hat{g}\big], \tag{4.16}$$

where the symbol $*$ stands for complex conjugation. By taking the time average of equation 4.13 one obtains

$$\nabla\cdot\langle\mathbf{T}'\rangle = 0. \tag{4.17}$$

Note that the time-average of the time-derivative operator vanishes.

Consider now an object with volume R' and surface denoted $\partial R'$ suspended in a fluid. The time-averaged torque, or the radiation torque, exerted by a wave upon the object is defined by

$$\langle\mathbf{N}\rangle \equiv \int_{\partial R'}\langle\mathbf{T}'\rangle\cdot \mathrm{d}\mathbf{A}, \tag{4.18}$$

where $\mathrm{d}\mathbf{A} = \mathbf{n}\mathrm{d}A$ is the vector surface element, with $\mathbf{n}$ being the outward unit normal vector. Assume that this same object is surrounded by a control sphere of radius R and surface ∂R as shown in figure 4.1, in the absence of any sources or sinks within the control sphere, we can use the Gauss divergence theorem in equation 4.17 to obtain

$$\int_{\partial R}\langle\mathbf{T}'\rangle\cdot\mathrm{d}\mathbf{A} + \int_{\partial R'}\langle\mathbf{T}'\rangle\cdot\mathrm{d}\mathbf{A} = 0, \tag{4.19}$$

where ∂R is the surface of the control sphere. Hence, the radiation torque on the object is given by

$$\langle \mathbf{N} \rangle = -\int_{\partial R} \langle \mathbf{T}' \rangle \cdot \mathrm{d}\mathbf{A}. \tag{4.20}$$

Therefore, the radiation torque on the object can be calculated over any surface that encloses the object. In particular, we assume that the control surface is in the far-field region (many wavelenghts away from the object). Note that instead of solving the integral at the surface of an arbitrary shaped object, we can evaluate it at a spherical surface in the far-field.

Using equation 4.12 and 4.14 into equation 4.20 we have

$$\begin{aligned} \langle \mathbf{N} \rangle &= \int_{\partial R} \langle L(\mathbf{x} \times \mathbf{n}) \rangle \mathrm{d}A - \int_{\partial R} \rho_0 \langle (\mathbf{x} \times \mathbf{u})\mathbf{u} \rangle \cdot \mathrm{d}\mathbf{A}, & (4.21) \\ &= -\int_{\partial R} \rho_0 \langle (\mathbf{x} \times \mathbf{u})\mathbf{u} \rangle \cdot \mathrm{d}\mathbf{A}. & (4.22) \end{aligned}$$

The average value of the flux of momentum, $\rho_0 \langle \mathbf{uu} \rangle$, can be written in terms of the incident and the scattered field [24]:

$$\rho_0 \langle \mathbf{uu} \rangle = \rho_0 \langle \mathbf{u}_i \mathbf{u}_i \rangle + \rho_0 \langle \mathbf{u}_s \mathbf{u}_s \rangle + \rho_0 \langle \mathbf{u}_i \mathbf{u}_s \rangle + \rho_0 \langle \mathbf{u}_s \mathbf{u}_i \rangle. \tag{4.23}$$

Noting that the incident field will not cause any torque to the sphere by itself we can discard the first term of the right hand side of the equation 4.23. Substituting equation 4.23 and $\mathbf{u} = -\nabla \phi$ in equation 4.21 one finds

$$\mathbf{N} = -\frac{\rho_0 r^2}{2} \mathrm{Im} \left[\int_{4\pi} \left(\frac{\partial \phi_i^*}{\partial r} \hat{\boldsymbol{L}} \phi_s - \phi_i^* \hat{\boldsymbol{L}} \frac{\partial \phi_s}{\partial r} + \frac{\partial \phi_s^*}{\partial r} \hat{\boldsymbol{L}} \phi_s \right) d\Omega \right]; \tag{4.24}$$

where [8] $\hat{\boldsymbol{L}} = -i(\mathbf{r} \times \nabla)$ is the angular momentum operator. Now we are going to specify the scatterer as a fluid sphere. Using equations 3.14 and 3.18 one can find

$$\begin{aligned} \mathbf{N} &= -\frac{\rho_0 k \phi_0^2 r^2}{2} \mathrm{Im} \Bigg[\sum_{l,m,l_1,m_1} s_{l_1}^{m_1} \Bigg(a_l^{m*} j_l'(kr) h_{l_1}^{(1)}(kr) - a_l^{m*} j_l(kr) h_{l_1}^{(1)'}(kr) \\ &+ s_l^{m*} h_l^{(1)'}(kr) h_{l_1}^{(1)}(kr) \Bigg) \int_{4\pi} Y_l^{m*} \hat{\boldsymbol{L}} Y_{l_1}^{m_1} d\Omega \Bigg]. \end{aligned} \tag{4.25}$$

In the far-field region ($kr \to \infty$) the spherical functions can be written as [8]

$$j_n(x) \approx \frac{1}{kr}\sin\left(kr - \frac{n\pi}{2}\right); \quad (4.26)$$

$$h_n^{(1)}(x) \approx i^{-n}\frac{e^{ikr}}{ikr}. \quad (4.27)$$

Now we just have to solve $\hat{\boldsymbol{L}}Y_{l_1}^{m_1}$. Introducing the ladder operators $\hat{\boldsymbol{L}}_+ = \hat{\boldsymbol{L}}_x + i\hat{\boldsymbol{L}}_y$ and $\hat{\boldsymbol{L}}_- = \hat{\boldsymbol{L}}_x + i\hat{\boldsymbol{L}}_y$, one can find:

$$L_\pm = b_l^m Y_l^{m\pm1}; \quad (4.28)$$

$$L_z = mY_l^m, \quad (4.29)$$

where $b_l^m = \sqrt{(l \mp m)(l \pm m + 1)}$. Replacing these equations in 4.25 we can find

$$\mathbf{N} = \pi a^3 E_0 \boldsymbol{\tau}, \quad (4.30)$$

where $E_0 = \rho_0 k^2 \phi_0^2/2$ is the characteristic energy density. The dimensionless radiation torque $\boldsymbol{\tau}$ is

$$\tau_x = -\frac{1}{2\pi(ka)^3}\mathrm{Re}\sum_{l,m}(a_l^m + s_l^m)(b_l^m s_l^{m+1*} + b_l^{-m} s_l^{m-1*}), \quad (4.31)$$

$$\tau_y = -\frac{1}{2\pi(ka)^3}\mathrm{Im}\sum_{l,m}(a_l^m + s_l^m)(b_l^m s_l^{m+1*} - b_l^{-m} s_l^{m-1*}), \quad (4.32)$$

$$\tau_z = -\frac{1}{\pi(ka)^3}\mathrm{Re}\sum_{l,m} m(a_l^m + s_l^m)s_l^{m*}. \quad (4.33)$$

The torque upon any target object by a single frequency wave with arbitrary wavefront can be calculated by these formulas. One can simplify the formula of τ_x and τ_y by defining a new variable $\tau_\perp$:

$$\tau_\perp = -\frac{1}{2\pi(ka)^3}\sum_{l,m}\left[(a_l^m + s_l^m)b_l^m s_l^{m+1*} + (a_l^{m*} + s_l^{m*})b_l^{-m} s_l^{m-1}\right], \quad (4.34)$$

then we have a new definition for the x and y components of the torque: $\tau_x = \mathrm{Re}\,[\tau_\perp]$ and $\tau_y = \mathrm{Im}\,[\tau_\perp]$.

Now, we are able to calculate a simplified formula for the Rayleigh limit. We already shown that only the monopole $l = 0$ and the dipole $l = 1$ terms will be relevant for the case of a fluid sphere scattering. Thus, for a very small spherical scatterer the torque can be rewritten as

$$\tau_\perp = -\frac{\sqrt{2}\left(\mathrm{Re}\left[s_1^*\right] + |s_1|^2\right)}{\pi(ka)^3}\left(a_1^{-1}a_1^{0*} + a_1^0 a_1^{1*}\right), \tag{4.35}$$

$$\tau_z = -\frac{\mathrm{Re}\left[s_1^*\right] + |s_1|^2}{\pi(ka)^3}\left(|a_1^1|^2 - |a_1^{-1}|^2\right). \tag{4.36}$$

We already have the beam shape coefficients for the incident wave, thus, replacing equations 3.37 and 3.38 into the above equations we find:

$$\tau_\perp = -\frac{24\rho_0^2 c_0^2\left(\mathrm{Re}\left[s_1^*\right] + |s_1|^2\right)}{p_0^2(ka)^3}\left(i\mathrm{Im}\left[\hat{u}_x\hat{u}_z^*\right] - \mathrm{Im}\left[\hat{u}_y\hat{u}_z^*\right]\right) \tag{4.37}$$

$$\tau_z = -\frac{24\rho_0^2 c_0^2\left(\mathrm{Re}\left[s_1^*\right] + |s_1|^2\right)}{p_0^2(ka)^3}\mathrm{Im}\left[\hat{u}_y\hat{u}_x^*\right]. \tag{4.38}$$

The term $\mathrm{Re}\left[s_1^*\right] + |s_1|^2$ deserves more attention. For a non absorptive scatterer the object-shape coefficient can be written as [6] $s_l = (e^{i2\delta_l} - 1)/2$, where δ_l is the phase shift of the l-th partial wave. Replacing this definition into equations 4.37 and 4.38 we find that $\tau_z = \tau_x = \tau_y = 0$. A physical explanation for this is that as we are considering a time dependence of $e^{-i\omega t}$, there's no symmetry break in the axis of propagation, thus, there's no preferential direction for the torque. If we add absorption to the sphere, the field inside it will now have its symmetry broken, thus creating a preferential direction for the sphere to spin.

We have not yet introduced the reader to the concept of an absorptive medium. In this work we use the model described by Szabo [19], where the absorption varies linearly with the frequency. He shows that the wavenumber an absorptive sphere can be written as:

$$k_1 = kc_0 \left(\frac{1}{c_1} + i\frac{\alpha_0}{2\pi} \right) \tag{4.39}$$

where α_0 is the absorption coefficient.

Referring to equation 3.23 the dipole term for the fluid sphere is given by [25]

$$\begin{aligned} s_1 = -i &\left[\frac{4(\pi^2\rho_0\rho_1 c_1^2 + \pi^2\rho_1^2 c_1^2 - \pi^2\rho_0^2 c_1^2 - \pi^2\rho_0\rho_1 c_0^2) + \alpha_0^2\rho_0\rho_1 c_0^2 c_1^2}{20\pi^2 c_1^2(\rho_0 + 2\rho_1)^2} \right] (ka)^5 \\ &- \frac{\alpha_0\rho_0\rho_1 c_0^2}{5\pi(\rho_0 + 2\rho_1)^2 c_1}(ka)^5 - \frac{(\rho_0 - \rho_1)^2}{9(\rho_0 + 2\rho_1)^2}(ka)^6 - i\frac{(\rho_0 - \rho_1)}{3(2\rho_1 + \rho_0)}(ka)^3. \end{aligned} \tag{4.40}$$

In the term $|s_1|^2$ the dominant term in the Rayleigh limit is of order $(ka)^6$, therefore the radiation torque for absorptive fluid sphere is given by:

$$\tau_\perp = \frac{24\alpha_0\rho_0^3\rho_1 c_0^4}{5\pi p_0^2 c_1(\rho_0 + 2\rho_1)^2}(ka)^2 \left(i\mathrm{Im}\left[u_x u_z^*\right] - \mathrm{Im}\left[u_y u_z^*\right] \right), \tag{4.41}$$

$$\tau_z = \frac{24\alpha_0\rho_0^3\rho_1 c_0^4}{5\pi p_0^2 c_1(\rho_0 + 2\rho_1)^2}(ka)^2\mathrm{Im}\big[u_y u_x^*\big], \tag{4.42}$$

From this formula one can see that it is invariant under axis rotations. If we make $x \to y$, $y \to z$, $z \to x$ we can see that we have the exact same formula for the torque, as we should expect.

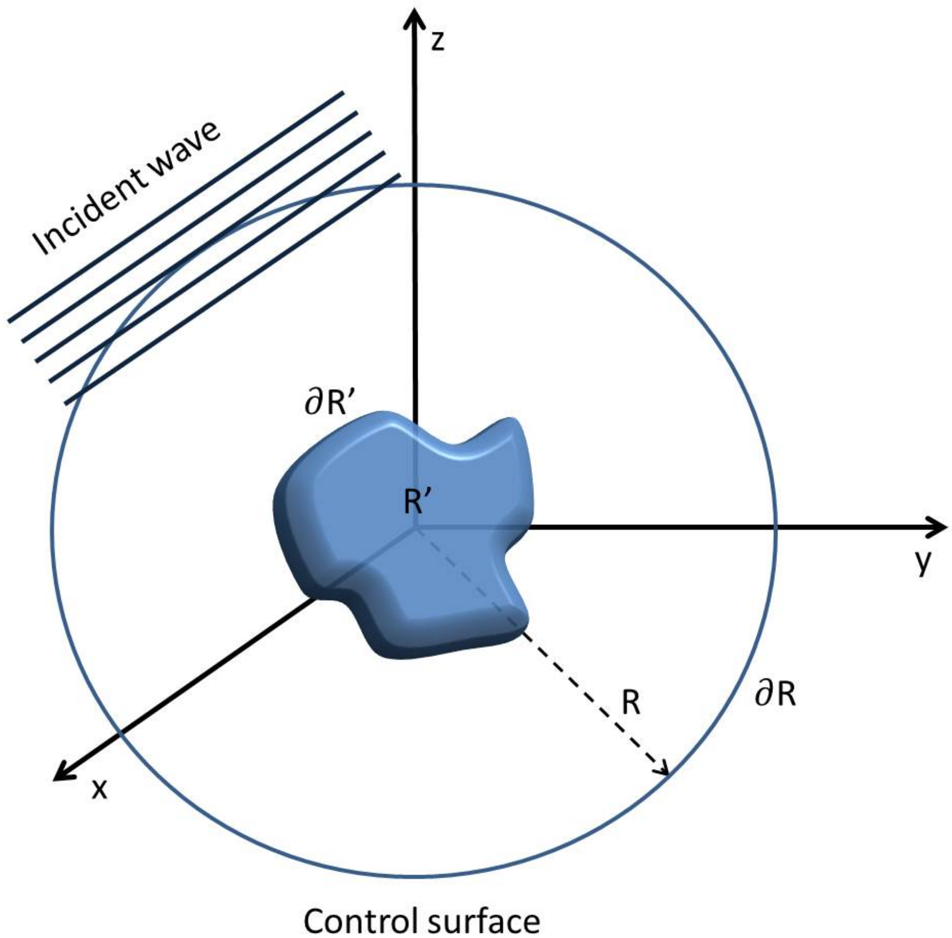

Figure 4.1: Description of the radiation torque problem produced by a single frequency wave on an arbitrarily shaped object. The object has volume R', surface $\partial R'$ and it is surrounded by a control sphere with radius R and surface ∂R.

Chapter 5

Results

In this chapter, we apply the developed acoustic radiation torque theory to acoustic Bessel beams. Moreover, we provide the validation of the acoustic radiation torque formulas in the Rayleigh limit. Part of the results outlined here were recently published in references [18, 14]

5.1 Acoustic Bessel beams

The case in point is the acoustic radiation torque produced by a Rayleigh particle by a Bessel beam. A class of waves known as Bessel beams has been used for microparticle manipulation. A hallmark feature of Bessel beams is that they are limited-diffraction beams. Such a feature makes them highly desirable in various applications for particle manipulation, because they possess a long depth of field, which may range over many wavelengths. Furthermore, an exquisite property of Bessel vortex beams is that they carry angular momentum along the propagation direction. The angular momentum of acoustic vortex beams can be transferred to a suspended object in the wavepath according to the object scattering and absorbing properties.

The pressure amplitude of the Bessel beam of order m propagating along the z-axis centered at (x_0, y_0) is given by [13]

$$p = p_0 e^{ik_z z} J_m(k_r \sqrt{(x-x_0)^2 + (y-y_0)^2}) e^{im\phi} \tag{5.1}$$

where $k_z = k \cos\beta$, $k_r = k \sin\beta$, J_m is the m-th order cylindrical Bessel function and β is the half-cone angle formed by the wavevector $\mathbf{k} = (k_r, k_z)$ to the beam axis of propagation. In figure 5.1 a first order bessel beam with $\beta = 70°$ is shown in the xy-plane.

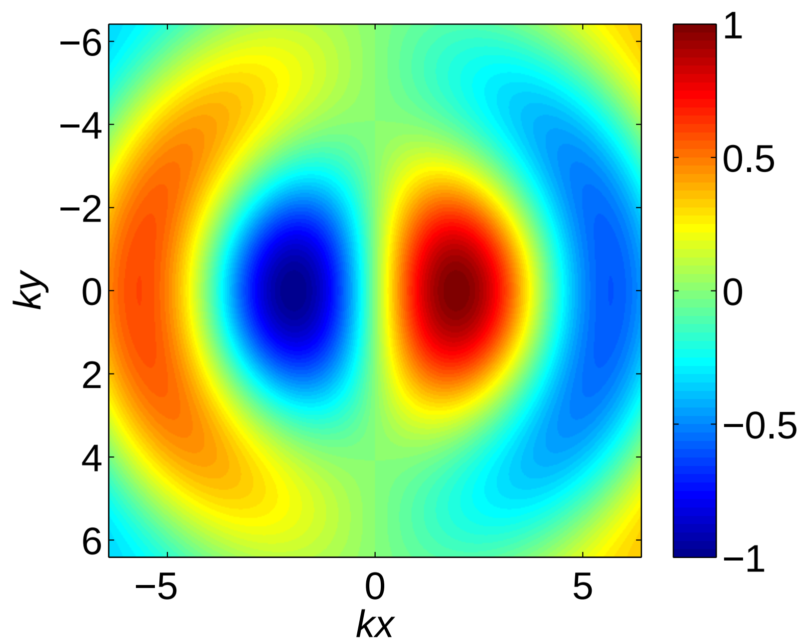

Figure 5.1: The real part of the first-order Bessel beam with $\beta = 70°$ in the xy-plane.

The beam-shape coefficient for the Bessel beam in the on-axis configuration

$(x_0 = 0, y_0 = 0)$ is given by [15]

$$a_l^m = \sqrt{4\pi(2l+1)\frac{(l-m)!}{(l+m)!}}i^{l-m}H(l-m)P_l^m(\cos\beta), \tag{5.2}$$

where H is the Heaviside step-function for which $H(x) = 1$, for $x > 0$ and $H(x) = 0$, otherwise.

5.2 Validation

It will be analyzed here the behavior of equation 4.42 with ka varying from 0.001 to 5. Our goal here is to show the accuracy of this equation in the Rayleigh limit. To do that we compare the result obtained with this equation to that from equation 4.33. For this purpose a polyurethane sphere ($\rho_1 = 1130$ kg/m^3, $c_1 = 1468$ m/s and $\alpha_0 = 1.49$ Np$\cdot$ MHz/m) immersed in water ($\rho_0 = 1000$ kg/m^3 and $c_0 = 1500$ m/s). The incident field is considered a first-order Bessel beam traveling along the z-axis. The incident beam frequency is set to 1 MHz.

Equation 4.33 has an infinite summation that has to be truncated for computational purposes, in this example we set $L = 10$ and the error in the reconstructed wave was smaller than 0.1%. We used equation 5.2 for the BSC instead of calculating them numerically. In figure 5.2 we present the dimensionless radiation torque produced by the first-order Bessel beam with $\beta = 15°$ on a polyurethane particle located at the beam axis of propagation. We have a very good agreement with both expressions (general formulation and the torque in the Rayleigh limit). When $ka = 1$ the expressions yield equivalent results, but as ka increases the results starts to be apart. While the torque calculated by the Rayleigh approximation increases monotonically the torque calculated by equation 4.38 appears to saturate at some point. At $ka = 1$ we have an error of 0.59%, and at $ka = 0.5$ we have an error of 0.09%. As we can see, the error keeps growing as ka increases. At $ka = 2$ the error is already 7% and at $ka = 3$ it is 15%. This happens because as ka increases the contribution from the terms that were left out of the torque formula, as the quadrupole term ($l = 2$), begin to have more influence on the the torque.

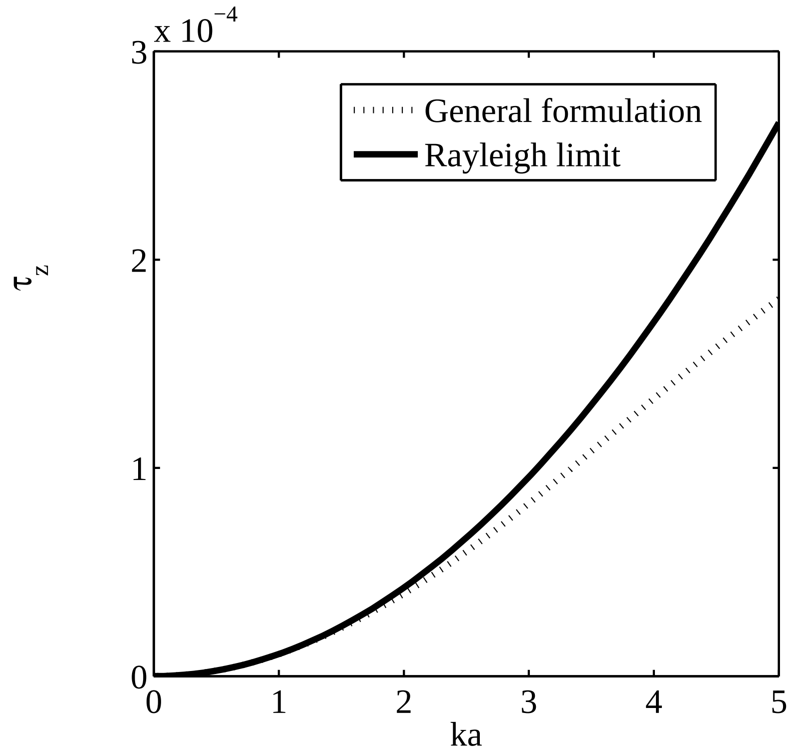

Figure 5.2: Comparison between the radiation torque in the Rayleigh limit and the general formula, given in equation 4.33, for the torque. As one should expect the formulas are equivalent for small ka, and they grow apart when ka increases.

5.3 On-axial radiation torque

Now we obtain the axial radiation torque τ_z in an on-axis configuration on a sphere. Substituting equation 5.2 on equation 4.33 one finds the axial dimensionless torque to be [14]

$$\tau_z = -\frac{4m}{(ka)^3}\sum_{n=m}^{\infty}\left\{P_n^m(\cos\beta)^2(2n+1)\frac{(n-m)!}{(n+m)!}(\mathrm{Re}\big[s_l\big]+\mathrm{Re}\big[s_l\big]^2+\mathrm{Im}\big[s_l\big]^2\right\}. \quad (5.3)$$

This formula allows the calculation of the torque produced by a m-th order Bessel beam on an spherical absorptive target with any size.

If we consider that the object in the Rayleigh limit equation 5.3 can be further simplified. In this equation we notice that only the first-order Bessel beam will produce torque. The radiation torque is proportional to the order of Bessel beam, thus the zero-order Bessel beam does not produces axial torque. On the other hand the dominant term of equation 5.3 is proportional to s_m, where m is the order of the beam. Thus, in the Rayleigh limit only the first order Bessel beam will produce torque.

To find the torque of a first-order Bessel beam upon a Rayleigh particle, using equation 4.42, we only need to calculate the velocity field of the first-order Bessel beam at the origin of the coordinate system. With the linearized relation, $\hat{\mathbf{u}} = \nabla\hat{p}/(ik\rho_0 c_0)$, we can see that we only need to calculate the gradient of equation 5.1, thus we find

$$\hat{u}_x(\mathbf{0}) = -\frac{i\sin\beta}{2c_0\rho_0}; \quad (5.4)$$

$$\hat{u}_y(\mathbf{0}) = \frac{\sin\beta}{2c_0\rho_0}; \quad (5.5)$$

$$\hat{u}_z(\mathbf{0}) = 0. \quad (5.6)$$

Using these equations on equation 4.42 we find:

$$\tau_z = \frac{6\alpha_0\rho_0\rho_1 c_0^2}{5\pi(\rho_0+2\rho_1)^2 c_1}(ka)^2\sin^2\beta. \tag{5.7}$$

This equation 5.7 shows that the torque induced by a on axis first-order Bessel beam on a Rayleigh particle has a quadratic dependence on the size factor of the sphere ka . It reaches it's maximum when $\beta = 90°$. Note also that axial torque caused by a Bessel beam in the Rayleigh limit is always positive.

Figure 5.3 shows the dimensionless radiation torque in the z-direction caused by a first-order Bessel beam upon a polyurethane sphere as a function of the azimuthal angle β and the sphere size factor ka. As we expected, the torque increases as ka increases, and we get the maximum value for a fixed ka when $\beta = \pi/2$.

5.4 Off-axial radiation torque

As the previous analysis was focused only in on-axis configurations, now we will derive the Rayleigh torque for off-axis Bessel beams of arbitrary order. According to equation 5.1 the particle velocity field for the m-th order Bessel beam is given by

$$\begin{aligned}
\hat{u}_x(\mathbf{0}) &= m\frac{ie^{im\theta_0}\sin\theta_0 J_m(k_r r_0)}{r_0}, \\
&\quad -\frac{e^{im\theta_0}k_r\cos\theta_0\left[J_{m-1}(k_r r_0)-J_{m+1}(k_r r_0)\right]}{2}, && (5.8)\\
\hat{u}_y(\mathbf{0}) &= -m\frac{ie^{im\theta_0}\cos\theta_0 J_m(k_r r_0)}{r_0} \\
&\quad -\frac{e^{im\theta_0}k_r\sin\theta_0\left[J_{m-1}(k_r r_0)-J_{m+1}(k_r r_0)\right]}{2}, && (5.9)\\
\hat{u}_z(\mathbf{0}) &= ie^{im\theta_0}k_r J_m(k_r r_0), && (5.10)
\end{aligned}$$

where $\theta_0 = \arctan(y_0/x_0)$ and $r_0 = \sqrt{x_0^2+y_0^2}$.

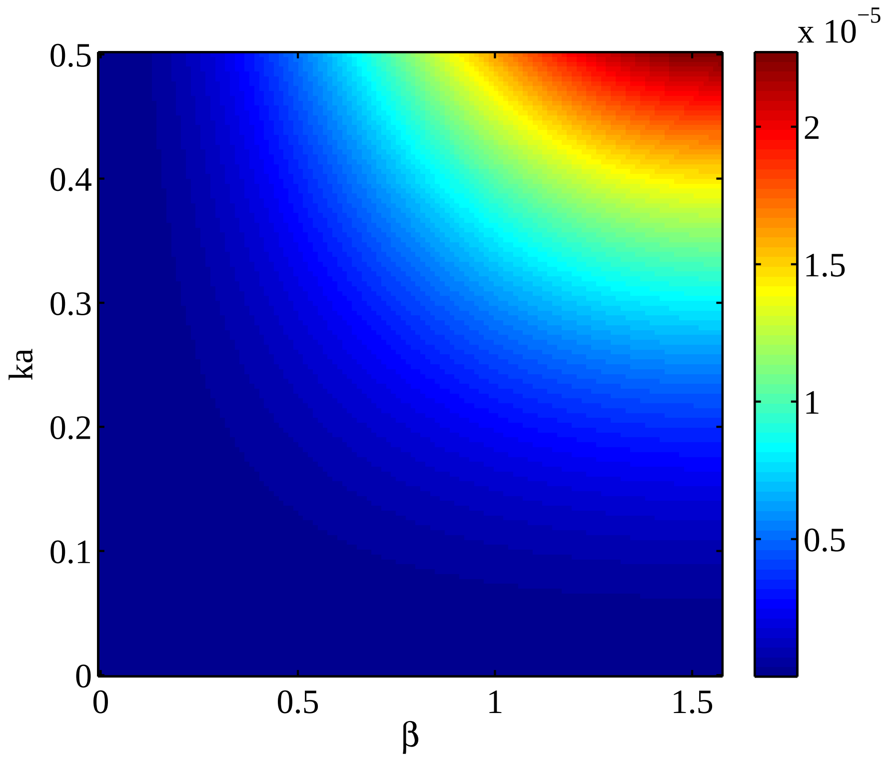

Figure 5.3: The axial radiation torque τ_z generated by the first order Bessel beam on the polyurethane sphere as a function of the scatterer size factor ka and the half cone-angle β.

For the sake of simplicity we will not present the radiation torque in terms of the particle velocity given by equations 5.8, 5.9 and 5.10. The resultant formula is too big and do not add new insights to the analysis of the problem. The obtained radiation torque formula was coded in MATLAB (Mathworks Inc.).

In the off-axis configuration the zero-order Bessel beam traveling along the z direction will break the symmetry in the xy-directions, giving rise to the transversal components of the acoustic radiation torque (τ_x, τ_y). However, this beam does not have angular momentum, hence the axial torque remains zero. Figure 5.4 shows the transversal torque caused by a zero-order Bessel beam, with $\beta = 70°$, upon a polyurethane sphere, with $ka = 0.1$, immersed in water. Here we varied the relative position between the center of the sphere and the center of the beam within the range $-6.4 \leq kx_0, ky_o \leq 6.4$.

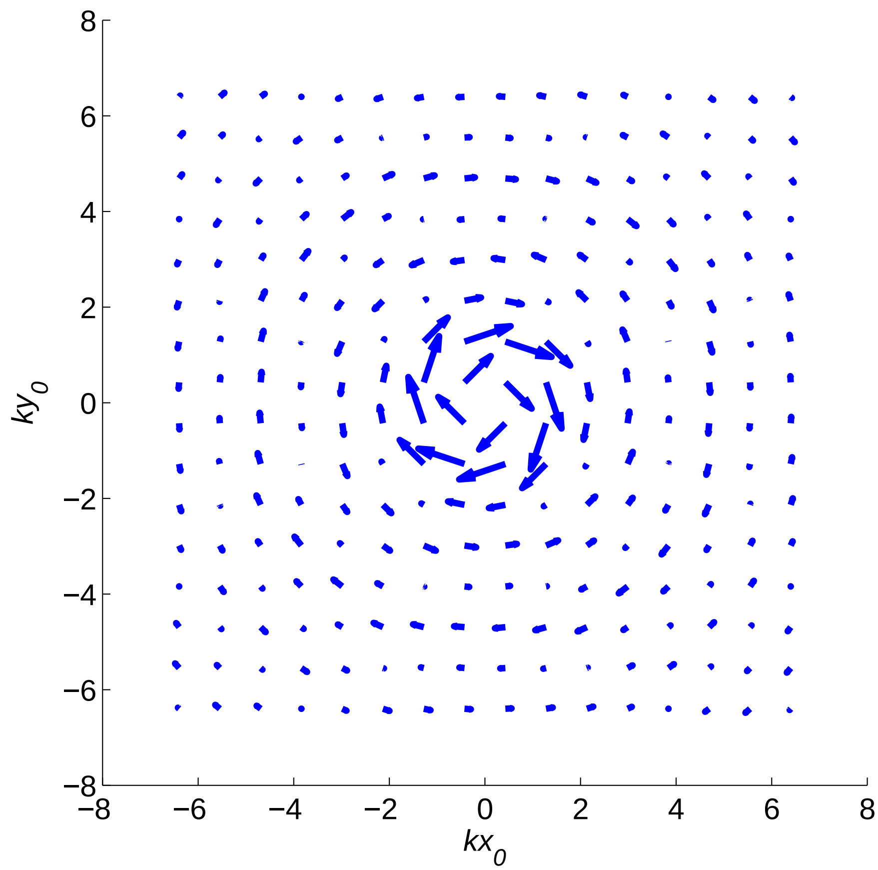

Figure 5.4: The transversal torque generated by the zero-order Bessel beam, with $\beta = 70°$, on the polyurethane sphere, with $ka = 0.1$, immersed in water as a function of the relative position between the scatterer and the center of the beam.

In figure 5.5, the transversal torque caused by a first-order Bessel beam ($\beta = 70°$), on a polyurethane sphere ($ka = 0.1$) is shown. Here we varied the relative position between the center of the sphere and the center of the beam within the range $-6.4/k \leq x_0, y_o \leq 6.4/k$. The associated vector field is plotted on top of the z component of the torque normalized to one, with the normalization factor being 3.009×10^{-5} Nm. We can see that the z component of the torque changes sign a few times. The main contribution for the axial component of the torque τ_z is given by transfer of angular momentum from the beam, thus if the beam rotates in the clockwise direction the scatterer will rotate in the same direction in the on-axis configuration. When the beam is in the off-axis configuration we may see the axial component of the torque changing its sign in a region that corresponds to the beam axis being almost tangent to the sphere surface. Moreover, the beam angular momentum might be transferred to the sphere similarly to the transferring process of two rotating gears in contact. The angular momenta of the two gears in contact have opposite directions.

Figures 5.6 and 5.7 are the transversal torque caused by a second- and third-order Bessel beam($\beta = 70°$), respectively, on a polyurethane sphere, with $ka = 0.1$. The associated vector field is plotted on top of the z component of the torque normalized to one, with the normalization factor being 9.8850×10^{-6} Nm for the second-order Bessel beam and 6.6064×10^{-6} Nm for the third-order Bessel beam. The relative position between the center of the sphere and the center of the beam was varied within the range $-6.4/k \leq kx_0, ky_o \leq 6.4/k$. Note that we still have the gear effect happening in these two cases. The axial torque τ_z in the region corresponding to the beam axis being almost tangent to the sphere surface have negative sign, as expected.

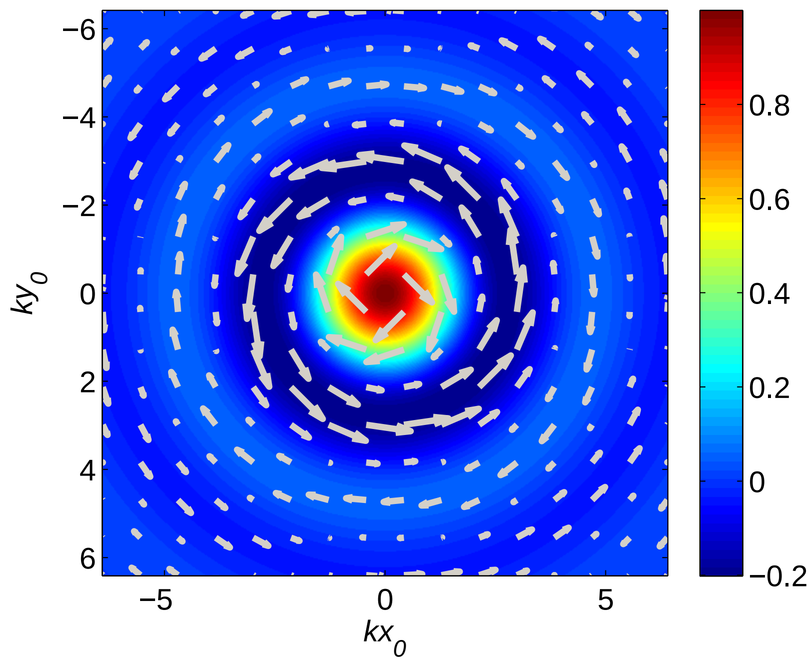

Figure 5.5: The transversal torque plotted upon the z component generated by a first-order Bessel beam ($\beta = 70°$) on a polyurethane sphere ($ka = 0.1$) immersed on water as a function of the relative position between the scatterer and the center of the beam. The vector field is plotted on the top of the axial radiation torque.

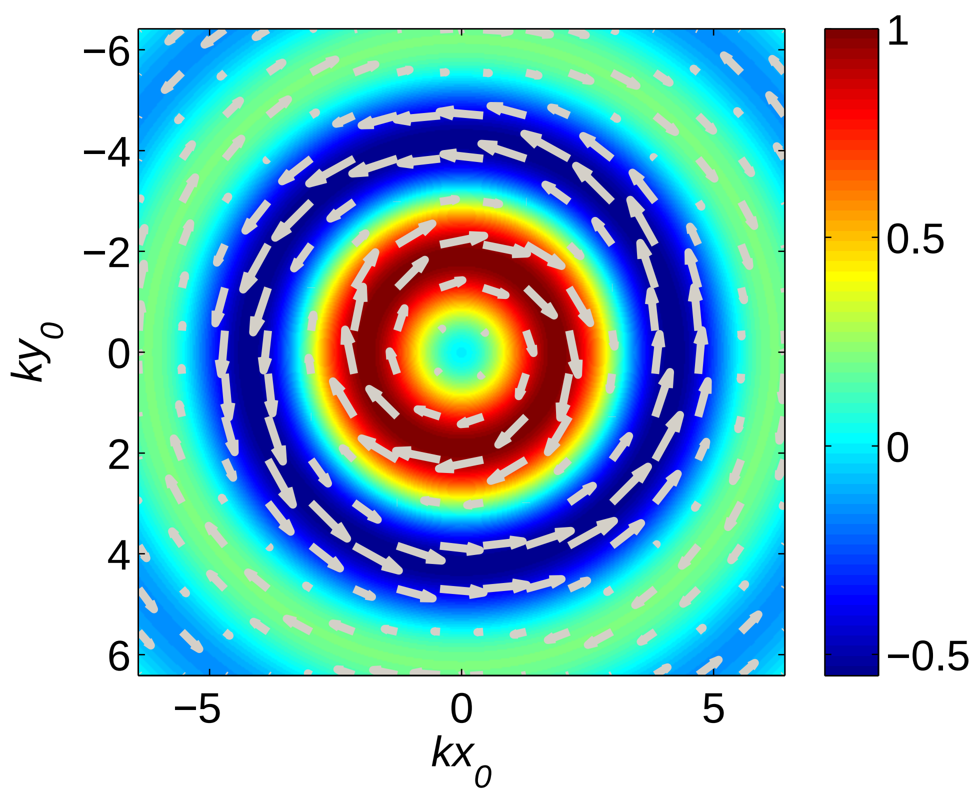

Figure 5.6: The transversal torque generated by a second-order Bessel beam ($\beta = 70°$) on a polyurethane sphere ($ka = 0.1$) as a function of the relative position between the scatterer and the center of the beam plotted upon the respective z components of the torque normalized to one.

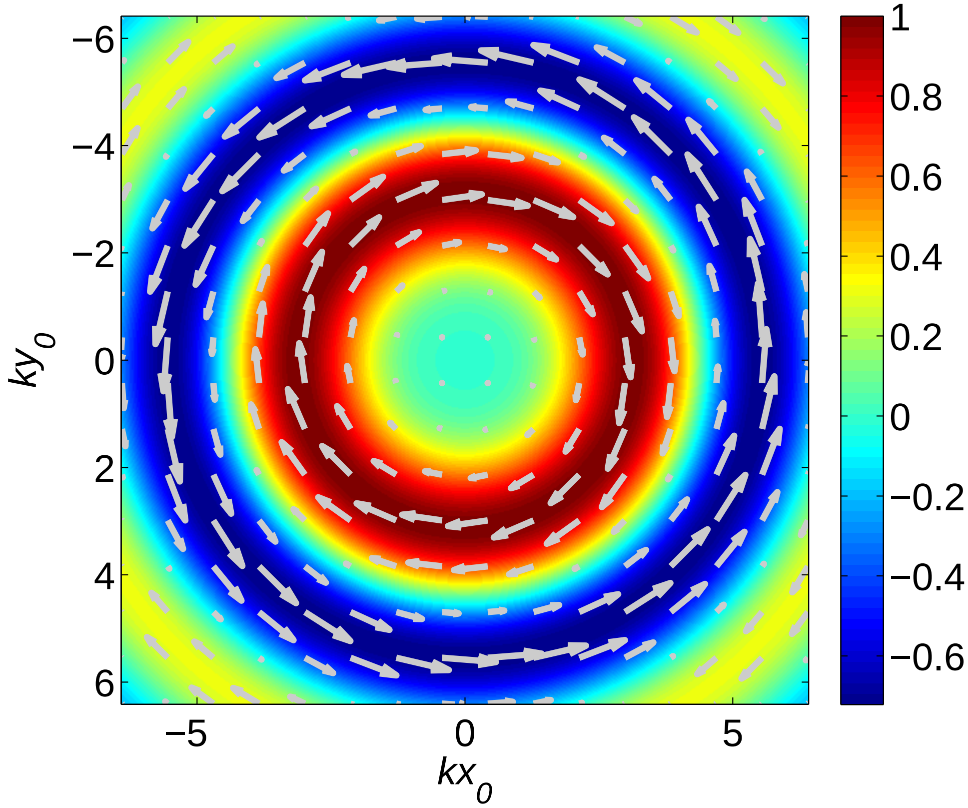

Figure 5.7: The transversal torque generated by a third-order Bessel beam ($\beta = 70°$) on a polyurethane sphere ($ka = 0.1$) as a function of the relative position between the scatterer and the center of the beam plotted upon the respective z components of the torque normalized to one.

Chapter 6

Conclusion

The acoustic torque produced on a spherical Rayleigh particle by an arbitrary acoustic wave was derived in terms of the parameters of the scatterer (the density ρ, the sound velocity in the scatterer fluid c and the radius of the scatterer a), parameters of the host fluid (its density ρ and the velocity of sound propagation c), and, the velocity field of the incident wave.

Specifically, this thesis derived a formula for the case in which the incident beam is a Bessel beam of any order, particularly analyzing the zero- and first-order Bessel beams in an on- and off-axis configurations. It shows that in the on-axis configuration only the first-order Bessel beam will cause the object to spin, but as the scatterer is taken out of the center of the beam, any order Bessem beam will cause a torque in the object. In addition, the off-axis configuration reverses the sign of the torque depending on the relative position between the object and the center of the beam.

If we calculate the acoustic radiation force acting on a Rayleigh particle we will obtain the full dynamics of the particle. A calculation very similar to the one presented here for the acoustic radiation torque can be carried out to find the acoustic radiation force on the particle, and it can be seen as a natural extension of this work.

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