UNIVERSIDADE FEDERAL DE ALAGOAS
INSTITUTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA UFAL-UFBA

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SOME GEOMETRIC AND ANALYTICAL RESULTS ON WEIGHTED RIEMANNIAN MANIFOLDS

DOCTORAL THESIS

Maceió
2016

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Thesis presented to the Post-graduate Program in Mathematics at Instituto de Matemática da Universidade Federal de Alagoas as partial fulfillment of the requirements for the degree of Doctor in Philosophy in Mathematics.

Advisor: Prof. PhD. Márcio Henrique B. da Silva

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2016

Catalogação na fonte
Universidade Federal de Alagoas
Biblioteca Central
Divisão de Tratamento Técnico
Bibliotecário Responsável: Valter dos Santos Andrade

S237s Santos, José Ivan da Silva.
Some geometric and analytical results on weighted Riemannian manifolds / José Ivan da Silva Santos. – 2016.
73 f.

Orientador: Márcio Henrique B. da Silva.
Tese (Doutorado em Matemática) – Doutorado Interinstitucional UFBA/UFAL. Universidade Federal de Alagoas. Instituto de Matemática. Programa de Pós-Graduação em Matemática. Maceió, 2016.

Bibliografia: f. 70-73.

1. Variedades Reimannianas ponderadas. 2. Tensor de Bakry-Émery Ricci. 3. Operador de Jacobi ponderado. 4. Estabilidade. 5. Auto valores de Stekloff. 6. Estimativas de auto valores. 7. Teorema splitting ponderado. I. Título.

CDU: 514.764.2

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For my grandparents, Artur dos Santos and Maria Jovelina, my parents, Antônio Artur and Djesima Maria, and my wife Jaaresias Nascimento.

ACKNOWLEGMENTS

Como as pessoas a quem dedico as palavras seguintes têm como primeiro idioma o português, este será o idioma que escreverei meus agradecimentos.

Primeiramente agradeço a Deus por me conceder saúde e entusiasmo no decorrer de todo o curso e por trazer todo o necessário para a conclusão desse trabalho.

Sou imensamente grato ao Prof. Dr. Márcio Henrique Batista, orientador desssa tese, que aceitou o desafio de me conduzir e preparar para o meio científico. Quero aqui destacar suas qualidades como profissional e pessoa as quais levarei como exemplo, compromisso, seriedade e respeito com tudo o que fazemos e com todos que convivemos.

Sou imensamente grato a todos os professores do programa de pós-graduação em matemática aos quais devo todo o conhecimento aprendido.

Sou grato a todos os colegas do curso de pós-graduação em matemática pela amizade e contribuições.

Devo meus agradecimentos a CAPES por conceder o apoio finaceiro, sem o qual nada do que foi feito seria possível.

Agradeço e dedico esta tese a todos os meus familiares que torceram e apoiaram-me em todos os momentos, fossem eles "fáceis" ou difíceis.

ABSTRACT

Let $(M, \langle\, , \rangle, f)$ be a weighted Riemannian manifold. In this thesis we obtain some geometric and analytical results in $(M, \langle\, , \rangle, f)$ assuming that Bakry-Emery Ricci tensor is non-negative in some results in other results we assuming that the weighted mean curvature is bounded from below. Moreover, assuming that the radial generalized sectional curvature is bounded from below we obtain a comparison theorem for the Hessian of the distance function and some consequences of it. Let Σ be a closed surface in M, assuming that the Perelman scalar curvature is bounded from below, we obtain an upper bound for the first non-zero eigenvalue of the weighted Jacobi operator for surfaces $\Sigma \subset M$ and we generalize a result of Shoen and Yau about stable minimal surfaces, see [45]. We also obtained, for surfaces with boundary, a sharp estimate from below for the first non-zero Stekloff's eigenvalue. For surfaces we also obtain an upper bound for the first non-zero eigenvalue of the weighted Jacobi operator and some consequences of it, for instance, we show that in $\mathbb{R}^3$ there exist no closed stable self-shrinker. In higher dimension we obtain upper bound and lower bound for the first non-zero Stekloff's eigenvalue on suitable hypotheses. We conclude our work with a weighted splitting theorem.

Keywords: Weighted Riemannian manifolds. Bakry-Émery Ricci tensor. Weighted Jacobi operator. Stability. Stekloff's eigenvalue. Eigenvalue estimates. weighted splitting theorem.

RESUMO

Seja $(M, \langle\, , \rangle, f)$ uma variedade Riemanniana ponderada. Nesta tese obtemos resultados geométricos e analíticos em $(M, \langle\, , \rangle, f)$ assumindo que o tensor de Bakry-Émery Ricci é não negativo em alguns resultados e assumindo que a curvatura média ponderada é limitada inferiormente em outros. Além disso, assumindo que a curvatura seccional generalizada radial é limitada inferiormente obtemos um teorema de comparação para a Hessiana da função distância e algumas consequências. Seja Σ uma superfície fechada em M, assumindo que a curvatura escalar de Perelman é limitada inferiormente, obtemos um limite superior para o primeiro autovalor não nulo do operador de Jacobi ponderado da superfície $\Sigma \subset M$ e generalizamos um resultado de Schoen e Yau sobre superfícies mínimas estáveis, veja [45]. Também obtemos, para superfícies com fronteira, uma estimativa sharp inferiormente para o primeiro autovalor não nulo de Stekloff. Para superfícies também obtemos um limite superior para o primeiro autovalor não nulo do operador de Jacobi ponderado e algumas consequências, por exemplo, mostramos que em $\mathbb{R}^3$ não existe self-shrinker fechado e estável. Em dimensão alta obtemos limites superiores e inferiores para o primeiro autovalor não nulo de Stekloff sobre hipóteses apropriadas. Concluímos nosso trabalho com um teorema splitting.

Palavras chave: Variedades Riemannianas ponderadas. Tensor de Bakry-Émery Ricci. Operador de Jacobi ponderado. Estabilidade. Autovalores de Stekloff. Estimativas de autovalores. Teorema splitting ponderado.

CONTENTS

CHAPTER 1 INTRODUCTION

The study of Riemannian manifolds endowed with a smooth density function has flourished in last few years, and a much better understanding of their geometrical and analytical structure has evolved. We point out, for instance, the solution of Poincaré conjecture, the relaxation of the conditions for solve the Monge's problem for mass transportation, the behavior of singularities of the Ricci flow, the mean curvature flow and others, see [10, 17, 33, 36, 37, 39, 51, 55] and references therein. Moreover, the theory of these spaces and the generalized curvatures go back to Lichnerowich [31, 32] and more recently by Bakry and Émery [7], in context of diffusion process, and it has been a very active area in recent years.

In this thesis we give contributions to the study of weighted Riemannian manifolds. We obtain some geometric and analytic results on weighted Riemanniann manifolds, and these generalize some results in [2, 14, 57, 58] for the weighted context. More specifically, inspired in the work of Impera and Rimold in [28], and using the concept of weighted sectional curvature defined by Wylie in [53], we obtain a comparison theorem for the weighted Hessian of the distance function and some consequences to the weighted Laplace-Beltrame operator. In particular we recover the following estimates to f-Laplacian of the distance function

$$\Delta_f r(x) \leqslant (m-1)\frac{h'(r(x))}{h(r(x))} + \theta(r(x)),$$

obtained in [43], see theorem 3.2. The comparison theorem for the weighted Hessian of the distance function is the main result of the Chapter 3.

In the Chapter 4, encouraged by ideas in [2, 3, 4, 41], we study some analytical aspects of surfaces with constant weighted mean curvature. More specifically, we study upper estimates of the first eigenvalue of the weighted Jacobi operator on closed surfaces. Moreover, we characterize the equality cases. This study allow us to generalize a result obtained by Schoen and Yau on stable minimal surfaces in 3-Riemannian manifolds with nonnegative scalar curvature for the setting of weighted Riemannian manifolds. Some consequences of that results are obtained, in which we show that all closed λ-surfaces in the Gaussian space are unstable, in particular, there exist no closed stable self-shrinker surfaces in $\mathbb{R}^3$.

In the Chapter 5, motivated by works [14, 57, 58], we study the Stekloff eigenvalue problems on weighted Riemannian manifolds.

The Classical Stekloff's eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega, \end{cases}$$

was introduced by self in [49] for bounded domains Ω of the plane and later it was studied by Payne in [42] for bounded domains in the plane with non-negative curvature. This problem has a physical interest because the eigenfunctions represent the steady state temperature on a domain and the flux on the boundary is proportional to the temperature, see [49] for more details. After that many authors studied this subject and many results were obtained, see for instance [6, 14, 15, 16, 24, 30, 34, 42, 46, 49, 52, 56, 57] and references therein. More specifically, many authors studied ways to estimate or determine exactly the eigenvalues associated with the Stekloff problem and modifications of the latter, see [14, 56, 57]. By following this way, we prove some upper and lower bounds for the Stekloff eigenvalues, see Theorem 5.1, 5.2, 5.3, 5.4, 5.5. We point out that in the inequalities obtained in the Theorem 5.1, 5.2, 5.3 , 5.4 we characterize the equality cases. Moreover, in the 2-dimensional case, we obtain a sharp result for weighted Stekloff eigenvalue problem, see Theorem 5.5.

In the setting of weighted Riemannian manifolds we study the following weighted Stekloff eigenvalue problems:

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ \frac{\partial u}{\partial \nu} = p_1 u & \text{on } \partial M; \end{cases}$$

$$\begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \Delta_f u - q_1 \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M; \end{cases}$$

$$\begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \frac{\partial^2 u}{\partial \nu^2} - q_1 \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}$$

where ν denotes the outward unit normal on ∂M. The first non-zero eigenvalue of the above problems will be denoted by p_1 and q_1, respectively. We will use the same letter for the first non-zero eigenvalues for last two problems because whenever the weighted mean curvature of ∂M is constant then the problems are equivalents.

Finally, in the Chapter 6, we obtain a weighted splitting theorem. Our inspiration to study splitting theorems are the articles [11, 18], this latter using techniques from overdetermined problems to obtain geometric restrictions over the space.

Our result says that, if M is a complete non-compact weighted Riemannian manifold with $\mathrm{Ric}_f \geqslant 0$ under suitable conditions, then $M = N \times \mathbb{R}$, where N is complete, totally geodesic, and f-parabolic. Moreover, if $u, g \in C^\infty(M)$ satisfy

$$\Delta_f u + g(u) = 0,$$

where the function $|\nabla u|$ satisfies

$$\int_{B_R} |\nabla u|^2 d\mathrm{Vol}_f = o(R^2 \log R) \quad \text{as} \quad R \to +\infty,$$

then

$$\mathrm{Vol}_f(B_R^N) = o(R^2 \log R) \qquad\qquad \text{as } R \to +\infty,$$

and

$$\int_{-R}^{R} |y'(t)|^2 dt = o\left(\frac{R^2 \log R}{\mathrm{Vol}(B_R^N)}\right) \quad \text{as } R \to +\infty,$$

see Theorem 6.1.

CHAPTER 2

PRELIMINARIES

In this first chapter we establish the concepts and definitions that we will use along this thesis, as well as we also fix some notations.

Given any smooth positive function φ on a Riemanniann manifold $(M, \langle\, , \rangle)$ we can consider a new measure μ on M by formula $d\mu = \varphi d\nu$, where ν is the Riemanniann measure. The function φ is called the density function with respect to μ. For instance, the density function of the Riemanniann measure ν is 1.

The triple $(M, \langle\, , \rangle, \varphi)$ is called a weighted Riemanniann manifold, if $(M, \langle\, , \rangle)$ is a Riemanniann manifold and ν is a measure on M with density function φ. More generally, given any $f \in C^\infty(M)$ we can consider the density function $\varphi := e^{-f}$, and thus $d\nu_f = e^{-f} d\nu$, and we will write $(M, \langle\, , \rangle, f)$; some authors called $(M, \langle\, , \rangle, \varphi)$ by Bakry-Émery manifold. That concept is directly related to Ricci flow, mean curvature flow, theory of optimal transportation, see [17, 33] for a good overview of this subject. An important example of weighted Riemannian manifold is the Euclidean space endowed with the Gaussian density $e^{-\pi|x|^2}$, with applications in probability and statistics.

In a weighted Riemannian manifolds there are natural generalizations for sectional, Ricci, and scalar curvatures. With respect to the sectional curvature, William W. purpose in [53] two new concepts of weighted sectional curvature as follow: given X, Y, unit orthogonal vectors in T_pM him defined

$$\begin{aligned} \sec_f^X(Y) &= \sec_f(X,Y) = \sec(X,Y) + \mathrm{Hess} f(X,X) \\ \overline{\sec}_f^X(Y) &= \overline{\sec}_f(X,Y) = \sec(X,Y) + \mathrm{Hess} f(X,X) + (df(X))^2 \end{aligned} \tag{2.1}$$

where $\sec(X, Y)$ is the usual sectional curvature of the plane spanned by X, Y, and $\mathrm{Hess} f$ is the Hessian of f. We point out that $\sec_f$ and $\overline{\sec}_f$ are asymmetrical, that is, $\sec_f(X, Y) \neq \sec_f(Y, X)$ and $\overline{\sec}_f(X, Y) \neq \overline{\sec}_f(Y, X)$. In [53], we can see that these notions of sectional curvature come naturally from at least three places: the radial curvature equation, the second variation of energy formula, and formula for Killing fields. Among the several interesting results obtained by William W. we highlight, if $(M^n, \langle\, , \rangle, f)$ is a simply connected weighted

Riemannian manifold of dimension $n > 2$, and $\sec_f = h$ or $\overline{\sec}_f = h$ for some function h, then (M^n, g) has constant sectional curvature.

The concept of weighted Ricci tensor on a weighted Riemannian manifold $(M, \langle\, , \rangle, f)$ was defined by Bakry and Émery in [7] as follow

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f,$$

where Ric denotes the Ricci tensor on the Riemannian manifold $(M, \langle\, , \rangle)$. The tensor Ric_f is also known as Bakry-Émery Ricci tensor, and more generally the N-Bakry-Émery Ricci tensor is

$$\mathrm{Ric}_f^N = \mathrm{Ric}_f - \frac{df \otimes df}{N}, \quad \text{for } N > 0. \tag{2.2}$$

Finally, the natural generalization para the scalar curvature S, was introduced by Perelman in [39] as follows

$$S_\infty = S + 2\Delta_M f - |\nabla f|^2$$

known as *Perelman's scalar curvature.*

Now we explain a concept related with the extrinsic geometry of a submanifold. Consider an oriented hypersurface Σ. Let ν be a unit normal vector field and let A be the second fundamental form of Σ w.r.t N. In [23] M. Gromov introduced the weighted mean curvature as

$$H_f = H + \langle \nu, \nabla f \rangle,$$

where ∇f denote the gradient of f in M, and H is the trace of the second fundamental form A.

On a weighted Riemannian manifold $(M, \langle\, , \rangle, f)$ we can to define the f-Laplacian operator $\Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle$, that is a natural generalization of the Laplace-Beltrami operator Δ. In a complete weighted Riemannian manifold, we know that Δ_f is essentially a self-adjoint operator with respect to the measure $d\nu_f = e^{-f} d\nu$. The operator Δ_f is also known as diffusion operator and Drift Laplacian, and by simplicity, we will call it f-Laplacian. The operator Δ_f arises in probability theory, potential theory and harmonic analysis on complete and non-compact weighted Riemannian manifolds. Moreover, the f-Laplacian appear in the Ornstein-Uhlenbeck equation.

CHAPTER 3

THE HESSIAN OF THE DISTANCE FUNCTION

3.1 Introduction

Let $(M, \langle\,,\rangle)$ be a Riemannian manifold and $p_0 \in M$. The distance function in M with reference point p_0 is the function $r : M \to \mathbb{R}$ defined by $r(x) = d(p_0, x)$. The classical comparison result to the Hessian of the distance function state that if the radial sectional curvature has a lower (resp. upper) bound of the form

$$\sec_{rad} \geqslant -G(r(x)) \quad (\text{resp. } \sec_{rad} \leqslant -G(r(x)))$$

then the Hessian of the distance function satisfies

$$\operatorname{Hess} r \leqslant \frac{h'(r)}{h(r)}(\langle\cdot,\cdot\rangle - dr \otimes dr) \quad \left(\text{resp. } \operatorname{Hess} r \geqslant \frac{h'(r)}{h(r)}(\langle\cdot,\cdot\rangle - dr \otimes dr),\right) \tag{3.1}$$

for some appropriate function h, see Theorem 2.3 in [40]. Of course, by taking the tracing in (3.1) we obtain comparison results to the laplacian of the distance function.

In the setting of weighted Riemannian manifolds, Impera and Rimoldi obtained in [28] a result that generalizes the classical comparison theorem of the distance function, (Theorem 2.3 in [40]).

We remember below the notation of little o and big O

Definition 1 *We say that $f(x) = O(g(x))$ as $x \to a$ if there exists a constant C such that $|f(x)| \leqslant C|g(x)|$ in some punctured neighborhood of a, that is for $x \in (a-\delta, a+\delta)\backslash\{a\}$ for some value of δ.*

We say $f(x) = o(g(x))$ as $x \to a$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$. This implies that there exists a punctured neighborhood of $\underline{a}$ on which g does not vanish.

For more details and properties for notation of $\underline{o}$ and $\underline{O}$ see [20, page 391].

The following result play a important role in the proof of the Theorem 3.2:

Proposition 3.1 *Let G be a continuous function on $[0,+\infty)$ and let $g_i \in AC(0,T_i)$ (Absolutely Continuous) be solutions of the Riccati differential inequalities*

$$g_1' + \frac{g_1^2}{a} - aG \leqslant 0 \quad g_2' + \frac{g_2^2}{a} - aG \geqslant 0 \quad a.e.in\ (0,T_i)$$

satisfying the asymptotic condition

$$g_i(t) = \frac{a}{t} + O(1) \quad as\ t \to 0^+,$$

for some $a > 0$. Then $T_1 \leqslant T_2$ and $g_1(t) \leqslant g_2(t)$ in $(0,T_1)$.

For a proof of the result above see [40, page 29].

3.2 Comparison Theorem to the Hessian of the Distance Function

Using the same technique that [28] with suitable adaptations, and the concept of sectional curvature given by William W.

$$\sec_f^X(Y) = \sec_f(X,Y) = \sec(X,Y) + \mathrm{Hess}f(X,X)$$

we show that:

Theorem 3.2 *Let $(M^m, \langle\,,\rangle, f)$ be a complete m-dimensional weighted Riemannian manifold. Having fixed a reference point $p_0 \in M$, let $r(x) = \mathrm{dist}_M(x,p_0)$ and let $D_{p_0} = M \backslash \mathrm{Cut}(p_0)$ be the domain of the normal geodesic coordinates centered at p_0. Given a smooth even function G on $\mathbb{R}$, let h be the solution of the Cauchy problem*

$$\begin{cases} h'' - (c+G)h = 0 \qquad c \in \mathbb{R} \\ h(0) = 0,\ h'(0) = 1, \end{cases} \tag{3.2}$$

and let $I = [0,r_0) \subset [0,+\infty)$ be the maximal interval where h is positive. Suppose that the radial curvature

$$\sec_f \geqslant -G(r(x)) \ \ (resp. \leqslant) \ \ on \quad B_{r_0}(p_0). \tag{3.3}$$

Furthermore, assume that

$$\eta(r) := \langle \nabla r, \nabla f \rangle \geqslant -\theta(r) \ \ (resp. \leqslant) \tag{3.4}$$

for some $\theta \in C^0(\mathbb{R}_0^+)$ and $\eta(s) = o(1)$ as $s \to 0^+$. Let

$$\mathrm{Hess}_f r(\cdot\,,\cdot) := \mathrm{Hess}\, r(\cdot\,,\cdot) - \frac{1}{m}\langle \nabla r, \nabla f\rangle\langle\cdot\,,\cdot\rangle,$$

if $\mathrm{Hess}f \leqslant c\langle\cdot\,,\cdot\rangle$ *(resp. $\geqslant$), then*

$$\mathrm{Hess}_f r(\cdot\,,\cdot) \leqslant \frac{h'}{h}\{\langle\cdot\,,\cdot\rangle - dr \otimes dr(\cdot\,,\cdot)\} + \frac{1}{m}\theta(r)\langle\cdot\,,\cdot\rangle \ \ (resp. \geqslant).$$

Proof. Firstly, we observed that $\mathrm{Hess}r(\nabla r, X) = 0$ for all $X \in T_xM$ and $x \in D_{p_0}\backslash\{p_0\}$. In fact, let γ be the geodesic parametrized by arch length issuing from p_0 with $\gamma(s_0) = x$, then γ is an integral curve of ∇r so that $\gamma'(s) = \nabla r(\gamma(s))$, this imply $\nabla_{\nabla r}\nabla r(x) = \nabla_{\gamma'(s_0)}\gamma' = 0$ consequently

$$\mathrm{Hess}r(\nabla r, X) = \langle X, \nabla_{\nabla r}\nabla r\rangle = 0.$$

Since $\mathrm{Hess}_f r$ is symmetric, T_xM has an orthonormal base consisting of eigenvectors of the $\mathrm{Hess}_f r$. Denoting $\lambda_{\max}(x)$, and $\lambda_{\min}(x)$, respectively, the greatest and smallest eigenvalues of the $\mathrm{Hess}_f r$ in the orthonormal complement of $\nabla r(x)$ (the $\nabla r(x)$ is an eigenvector of the $\mathrm{Hess}\, r$ associated to the eigenvalue 0), the theorem is equivalent to show that on $(D_{p_0}\backslash\{p_0\}) \cap B_{r_0}(p_0)$

- **(i)** if (3.3) and (3.4) hold with $\geqslant$, then $\lambda_{\max} \leqslant \frac{h'}{h}(r(x)) + \frac{1}{m}\theta(r)$,
- **(ii)** if (3.3) and (3.4) hold with $\leqslant$, then $\lambda_{\min} \geqslant \frac{h'}{h}(r(x)) + \frac{1}{m}\theta(r)$.

We proof the item **(i)**, and the item **(ii)** is analogous. Let $x \in D_{p_0}\backslash\{p_0\}$, and γ be the minimizing geodesic joining p_0 to x. We claim that, if (3.3) holds, then the function $\psi = (\lambda_{\max} + \frac{\eta}{m}) \circ \gamma$ satisfies

$$\begin{cases} \psi' + \psi^2 \leqslant c + G & \text{for a.e. } s > 0, \\ \psi(s) = \frac{1}{s} + o(1), & \text{as } s \to 0^+. \end{cases}$$

Let $\phi := h'/h$, we have

$$\phi' + \phi^2 = \frac{hh'' - h'^2}{h^2} + \frac{h'^2}{h^2} = \frac{h''}{h} = c + G,$$

and by L'Hospital rule

$$\begin{aligned} \lim_{s\to 0^+}\left(\phi(s) - \frac{1}{s}\right) &= \lim_{s\to 0^+}\left(\frac{h'}{h}(s) - \frac{1}{s}\right) = \lim_{s\to 0^+}\frac{sh'(s) - h(s)}{sh(s)} \\ &= \lim_{s\to 0^+}\left(\frac{h'(s) + sh''(s) - h'(s)}{h(s) + sh'(s)}\right) \\ &= \lim_{s\to 0^+}\left(\frac{s(c+G)h(s)}{h(s) + sh'(s)}\right) \\ &= \lim_{s\to 0^+}\frac{s(c+G)}{2} = 0., \end{aligned} \tag{3.5}$$

that is,

$$\phi(s) - \frac{1}{s} = o(1).$$

Therefore, ϕ satisfies the following system

$$\begin{cases} \phi' + \phi^2 = c + G & \text{on} \quad (0, r_0) \text{ and} \\ \phi(s) = \dfrac{1}{s} + o(1) & \quad \text{as } s \to 0^+ \end{cases} \tag{3.6}$$

follow by proposition 3.1 that

$$\psi \leqslant \phi = \frac{h'}{h}.$$

We will to show that (3.6) holds. Indeed,

and observe that, Therefore,

$$\phi(s) - \frac{1}{s} = o(1).$$

Now, we will show that $\lambda_{\max}$ have the required properties. To this end, given a smooth real function u, denote by $\mathrm{hess}_f u$ the $(1,1)$ symmetric tensor field defined by

$$\mathrm{hess}_f u(X) = \mathrm{hess} u(X) - \frac{\langle \nabla f, \nabla u\rangle X}{m},$$

where

$$\mathrm{hess} u(X) = \nabla_X \nabla u$$

consequently, we have

$$\begin{aligned}\langle \mathrm{hess}_f u(X), Y\rangle &= \left\langle \mathrm{hess} u(X) - \frac{\langle \nabla f, \nabla u\rangle X}{m}, Y\right\rangle \\ &= \langle \mathrm{hess} u(X), Y\rangle - \frac{\langle \nabla f, \nabla u\rangle}{m}\langle X, Y\rangle \\ &= \mathrm{Hess} u(X,Y) - \frac{\langle \nabla f, \nabla u\rangle}{m}\langle X, Y\rangle \\ &= \mathrm{Hess}_f u(X,Y).\end{aligned}$$

By definition of covariant derivative of tensors

$$\nabla_X(\mathrm{hess}_f u)(Y) = \nabla_X[\mathrm{hess}_f u(Y)] - \mathrm{hess}_f u(\nabla_X Y)$$

and

$$\nabla_Y(\mathrm{hess}_f u)(X) = \nabla_Y[\mathrm{hess}_f u(X)] - \mathrm{hess}_f u(\nabla_Y X).$$

Hence

$$\begin{aligned}\nabla_X(\mathrm{hess}_f u)(Y) - \nabla_Y(\mathrm{hess}_f u)(X) &= \nabla_X[\mathrm{hess}_f u(Y)] - \mathrm{hess}_f u(\nabla_X Y) - \\ &\quad - \nabla_Y[\mathrm{hess}_f u(X)] + \mathrm{hess}_f u(\nabla_Y X)\end{aligned}$$

$$\begin{aligned}&= \nabla_X\left[\mathrm{hess} u(Y) - \frac{\langle \nabla f, \nabla u\rangle Y}{m}\right] - \mathrm{hess} u(\nabla_X Y) + \\ &\quad + \frac{\langle \nabla f, \nabla u\rangle \nabla_X Y}{m} - \nabla_Y\left[\mathrm{hess} u(X) - \frac{\langle \nabla f, \nabla u\rangle X}{m}\right] + \\ &\quad + \mathrm{hess} u(\nabla_Y X) - \frac{\langle \nabla f, \nabla u\rangle \nabla_Y X}{m} \\ &= \nabla_X \nabla_Y \nabla u - \frac{\langle \nabla f, \nabla u\rangle}{m}\nabla_X Y - X\left(\frac{\langle \nabla f, \nabla u\rangle}{m}\right) Y - \\ &\quad - \nabla_{\nabla_X Y}\nabla u + \frac{\langle \nabla f, \nabla u\rangle}{m}[X,Y] - \nabla_Y \nabla_X \nabla u + \\ &\quad + \frac{\langle \nabla f, \nabla u\rangle}{m}\nabla_Y X + Y\left(\frac{\langle \nabla f, \nabla u\rangle}{m}\right) X + \nabla_{\nabla_Y X}\nabla u\end{aligned}$$

$$\nabla_X(\mathrm{hess}_f u)(Y) - \nabla_Y(\mathrm{hess}_f u)(X) = \nabla_X\nabla_Y\nabla u - \nabla_Y\nabla_X\nabla u - \nabla_{[X,Y]}\nabla u - \frac{\langle \nabla f, \nabla u\rangle}{m}[X,Y] + \\ + \frac{\langle \nabla f, \nabla u\rangle}{m}[X,Y] - X\left(\frac{\langle \nabla f, \nabla u\rangle}{m}\right)Y + \\ + Y\left(\frac{\langle \nabla f, \nabla u\rangle}{m}\right)X,$$

from where

$$\nabla_X(\mathrm{hess}_f u)(Y) - \nabla_Y(\mathrm{hess}_f u)(X) = R(X,Y)\nabla u - \\ - X\left(\frac{\langle \nabla f, \nabla u\rangle}{m}\right)Y + Y\left(\frac{\langle \nabla f, \nabla u\rangle}{m}\right)X.$$

Now, choose $u = r(x)$, $X = \nabla r$ and let γ be the minimizing geodesic joining p_0 to $x \in D_{p_0}\backslash\{p_0\}$. For every unit vector $Y \in T_xM$ such that $Y \perp \gamma'(s_0)$, where $\gamma(s_0) = x$, define a vector field $Y \perp \gamma'$, by parallel translation along γ. Since, as noted above, $\mathrm{hess}r(\nabla r) \equiv 0$, so

$$\begin{aligned}
&\nabla_{\gamma'(s_0)}[\mathrm{hess}_f r(Y)] = \nabla_{\gamma'(s_0)}(\mathrm{hess}_f r)(Y) + \mathrm{hess}_f r(\nabla_{\gamma'(s_0)}Y)\\
&= \nabla_{\nabla r}(\mathrm{hess}_f r)(Y) = \nabla_Y(\mathrm{hess}_f r)(\nabla r) + R(\nabla r, Y)\nabla r -\\
&\quad - \nabla r\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)Y + Y\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)\nabla r\\
&= \nabla_Y[\mathrm{hess}_f r(\nabla r)] - \mathrm{hess}_f r(\nabla_Y\nabla r) - R(Y,\nabla r)\nabla r -\\
&\quad - \nabla r\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)Y + Y\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)\nabla r\\
&= \nabla_Y\left[\mathrm{hess}r(\nabla r) - \frac{\langle \nabla f, \nabla r\rangle\nabla r}{m}\right] - \mathrm{hess}r(\nabla_Y\nabla r) +\\
&\quad + \frac{\langle \nabla f, \nabla r\rangle\nabla_Y\nabla r}{m} - R(Y,\nabla r)\nabla r -\\
&\quad - \nabla r\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)Y + Y\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)\nabla r\\
&= -\frac{\langle \nabla f, \nabla r\rangle}{m}\nabla_Y\nabla r - Y\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)\nabla r -\\
&\quad - \mathrm{hess}r(\nabla_Y\nabla r) + \frac{\langle \nabla f, \nabla r\rangle\nabla_Y\nabla r}{m} - R(Y,\nabla r)\nabla r -\\
&\quad - \nabla r\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)Y + Y\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)\nabla r\\
&= -\mathrm{hess}r(\nabla_Y\nabla r) - R(Y,\nabla r)\nabla r - \nabla r\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)Y\\
&= -\mathrm{hess}r(\mathrm{hess}r(Y)) - R(Y,\nabla r)\nabla r - \nabla r\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right)Y,
\end{aligned}$$

Since Y is parallel, we have

$$\frac{d}{ds}\langle \mathrm{hess}_f r(Y), Y\rangle = \langle \nabla_{\gamma'}[\mathrm{hess}_f r(Y)], Y\rangle,$$

and consequently

$$\begin{aligned}
&\frac{d}{ds}(\mathrm{Hess}_f r(\gamma)(Y,Y)) + \langle \mathrm{hess}_f r(\gamma)(Y), \mathrm{hess}_f r(\gamma)(Y)\rangle = \\
&= \Big\langle -\mathrm{hess}r(\mathrm{hess}r(Y)) - R(Y,\nabla r)\nabla r - \\
&\quad - \nabla r\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right) Y, Y \Big\rangle + \\
&\quad + \Big\langle \mathrm{hess}r(Y) - \frac{\langle \nabla f, \nabla r\rangle}{m} Y, \mathrm{hess}r(Y) - \frac{\langle \nabla f, \nabla r\rangle}{m} Y \Big\rangle \\
&= -\langle \mathrm{hess}r(\mathrm{hess}r(Y)), Y\rangle - \langle R(Y,\nabla r)\nabla r, Y\rangle - \\
&\quad - \nabla r\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right) + \langle \mathrm{hess}r(Y), \mathrm{hess}r(Y)\rangle - \\
&\quad - \frac{2\langle \nabla f, \nabla r\rangle}{m}\langle \mathrm{hess}r(Y), Y\rangle + \frac{\langle \nabla f, \nabla r\rangle^2}{m^2} \\
&= -\langle R(Y,\nabla r)\nabla r, Y\rangle - \nabla r\left(\frac{\langle \nabla f, \nabla r\rangle}{m}\right) - \\
&\quad - \frac{2\langle \nabla f, \nabla r\rangle}{m}\langle \mathrm{hess}r(Y), Y\rangle + \frac{\langle \nabla f, \nabla r\rangle^2}{m^2} \\
&= -\sec(Y,\gamma') = \mathrm{Hess}f(Y,Y) - \sec_f(Y,\gamma') \qquad (3.7) \\
&\leqslant \mathrm{Hess}f(Y,Y) + G(r) \\
&\leqslant c + G(r) \qquad (3.8)
\end{aligned}$$

By other hand,

$$\begin{aligned}
\frac{d}{ds}[\mathrm{Hess}_f r(\gamma)(Y,Y)] &= \frac{d}{ds}[\mathrm{Hess}\, r(\gamma)(Y,Y)] - \frac{1}{m}\frac{d}{ds}\langle \nabla\, r, \nabla f\rangle \circ \gamma \\
&= \frac{d}{ds}[\mathrm{Hess}\, r(\gamma)(Y,Y)] - \frac{1}{m}\frac{d}{ds}\eta \circ \gamma.
\end{aligned}$$

Observe that

$$\mathrm{hess}_f r(\gamma)(Y) = \mathrm{hess}\, r(\gamma)(Y) - \frac{1}{m}(\eta\circ\gamma)Y$$

from where, we have

$$\begin{aligned}
\langle \mathrm{hess}_f r(\gamma)(Y), \mathrm{hess}_f r(\gamma)(Y)\rangle &= \langle \mathrm{hess}\, r(\gamma)(Y), \mathrm{hess}\, r(\gamma)(Y)\rangle - \\
&\quad - \frac{2(\eta\circ\gamma)}{m}\mathrm{Hess}\, r(\gamma)(Y,Y) + \frac{1}{m^2}(\eta\circ\gamma)^2,
\end{aligned}$$

and consequently,

$$\begin{aligned}
\frac{d}{ds}[\mathrm{Hess}_f r(\gamma)(Y,Y)] + \|\mathrm{hess}_f r(Y)\|^2 &= \frac{d}{ds}[\mathrm{Hess}\, r(\gamma)(Y,Y)] - \frac{1}{m}\frac{d}{ds}\eta\circ\gamma + \\
&\quad + \langle \mathrm{hess}\, r(\gamma)(Y), \mathrm{hess}\, r(\gamma)(Y)\rangle - \frac{2(\eta\circ\gamma)}{m}\mathrm{Hess}\, r(\gamma)(Y,Y) + \frac{1}{m^2}(\eta\circ\gamma)^2 \\
&\leqslant \mathrm{Hess} f(Y,Y) + G(r) - \frac{1}{m}\frac{d}{ds}\eta\circ\gamma - \frac{2(\eta\circ\gamma)}{m}\mathrm{Hess}\, r(\gamma)(Y,Y) + \\
&\quad + \frac{1}{m^2}(\eta\circ\gamma)^2 \\
&= \mathrm{Hess} f(Y,Y) + G(r) - \frac{1}{m}\frac{d}{ds}\eta\circ\gamma - \frac{2(\eta\circ\gamma)}{m}\mathrm{Hess}_f r(\gamma)(Y,Y) - \\
&\quad - \frac{(\eta\circ\gamma)^2}{m^2}.
\end{aligned} \tag{3.9}$$

Note that, for all unit vector field $X \perp \nabla r$,

$$\mathrm{Hess}_f r(X,X) \leqslant \lambda_{\max}.$$

In fact, choosing a base $\{v_1, \ldots, v_{n-1}\}$ of $\{\gamma'\}^\perp$ formed by eigenvalues of $\mathrm{hess}_f r$, and writing

$$X = \sum_{i=1}^{n-1} a_i v_i$$

we obtain

$$\begin{aligned}
\mathrm{Hess}_f r(X,X) &= \langle \mathrm{hess}_f r(X), X\rangle \\
&= \left\langle \mathrm{hess}_f r\left(\sum_{i=1}^{n-1} a_i v_i\right), \sum_{i=1}^{n-1} a_i v_i \right\rangle \\
&= \left\langle \sum_{i=1}^{n-1} a_i\lambda_i v_i, \sum_{i=1}^{n-1} a_i v_i \right\rangle \\
&\leqslant \lambda_{\max}\left\langle \sum_{i=1}^{n-1} a_i v_i, \sum_{i=1}^{n-1} a_i v_i \right\rangle = \lambda_{\max}.
\end{aligned} \tag{3.10}$$

Then, if Y is chosen such that, in s_0

$$\mathrm{Hess}_f r(\gamma)(Y,Y) = \lambda_{\max}(\gamma(s_0)),$$

that is, Y is eigenvector of $\mathrm{hess}_f r$ in $\gamma(s_0)$, then the function

$$\mathrm{Hess}_f r(\gamma)(Y,Y) - \lambda_{\max}\circ\gamma$$

attains its maximum at $s = s_0$ and, if at this point $\lambda_{\max}$ is differentiable, then its derivative vanishes:

$$\left.\frac{d}{ds}\right|_{s_0} \mathrm{Hess}_f r(\gamma)(Y,Y) - \left.\frac{d}{ds}\right|_{s_0} \lambda_{\max}\circ\gamma = 0.$$

consequently, using (3.9), we obtain in s_0

$$\begin{aligned}\frac{d}{ds}(\lambda_{\max}\circ\gamma)+(\lambda_{\max}\circ\gamma)^2 &\leqslant \mathrm{Hess}f(Y,Y)+G(r)-\frac{1}{m}\frac{d}{ds}(\eta\circ\gamma)-\\ &-\frac{2(\eta\circ\gamma)}{m}\mathrm{Hess}_f r(\gamma)(Y,Y)-\frac{(\eta\circ\gamma)^2}{m^2}\\ &=\mathrm{Hess}f(Y,Y)+G(r)-\frac{d}{ds}\frac{\eta\circ\gamma}{m}-2(\lambda_{\max}\circ\gamma)\frac{(\eta\circ\gamma)}{m}-\frac{(\eta\circ\gamma)^2}{m^2}\end{aligned}$$

Now, let $\psi=(\lambda_{\max}+\frac{\eta}{m})\circ\gamma$, from where

$$\begin{aligned}\psi'+\psi^2 &= \frac{d}{ds}(\lambda_{\max}\circ\gamma)+\frac{d}{ds}\frac{(\eta\circ\gamma)}{m}+(\lambda_{\max}\circ\gamma)^2+2(\lambda_{\max}\circ\gamma)\frac{(\eta\circ\gamma)}{m}+\\ &+\frac{(\eta\circ\gamma)^2}{m^2}\\ &\leqslant \mathrm{Hess}f(Y,Y)+G(r)\\ &\leqslant c+G(r)\end{aligned}$$

this is the desired inequality. The asymptotic behavior of ψ near $s=0^+$ follows from the fact that

$$\mathrm{Hess}r=\frac{1}{r}(\langle\cdot\,,\cdot\rangle-dr\otimes dr)+o(1),\quad r\to 0^+, \tag{3.11}$$

and from the assumptions about η. In fact, since Y is unit and $Y\perp\gamma'=\nabla r$, we have

$$\begin{aligned}\psi=\lambda_{\max}\circ\gamma(s)+\frac{\eta}{m}\circ\gamma(s) &= \mathrm{Hess}r(\gamma(s))(Y,Y)+\frac{\eta}{m}\circ\gamma(s)\\ &=\frac{1}{s}(\langle Y,Y\rangle-\langle\nabla r,Y\rangle\langle\nabla r,Y\rangle)+o(1)+\frac{1}{m}o(1)\\ &=\frac{1}{s}+o(1).\end{aligned}$$

Therefore,

$$\lambda_{\max}\leqslant\frac{h'}{h}-\frac{\eta}{m}\leqslant\frac{h'}{h}+\frac{1}{m}\theta,$$

consequently, using (3.10),

$$\mathrm{Hess}_f r(X,X)\leqslant\lambda_{\max}\langle X,X\rangle\leqslant\frac{h'}{h}\langle X,X\rangle+\frac{1}{m}\theta(r)\langle X,X\rangle.$$

By other hand, if $X=X_1+X_2$ where $X_1/\!\!/\nabla r$, and $X_2\perp\nabla r$, we have

$$\begin{aligned}&\mathrm{Hess}_f r(X,X)=\mathrm{Hess}_f r(X_1+X_2,X_1+X_2)\\ &=\mathrm{Hess}_f r(X_1,X_1)+\underbrace{\mathrm{Hess}_f r(X_1,X_2)}_{=0}+\underbrace{\mathrm{Hess}_f r(X_2,X_1)}_{=0}+\mathrm{Hess}_f r(X_2,X_2)\\ &\leqslant\frac{h'}{h}\langle X_2,X_2\rangle+\frac{1}{m}\theta(r)\langle X_2,X_2\rangle-\frac{1}{m}\langle\nabla r,\nabla f\rangle\langle X_1,X_1\rangle\\ &\leqslant\frac{h'}{h}\{\langle X,X\rangle-\langle X_1,X_1\rangle\}+\frac{1}{m}\theta(r)\langle X_2,X_2\rangle+\frac{1}{m}\theta(r)\langle X_1,X_1\rangle\\ &=\frac{h'}{h}\{\langle X,X\rangle-dr\otimes dr(X,X)\}+\frac{1}{m}\theta(r)\langle X,X\rangle\end{aligned}$$

and this conclude the proof. ■

Remark 3.3 *Note that, if we assume* $\overline{\sec}_f \geqslant -G$ *(resp.* $\leqslant$*) in the place of* $\sec_f$*, we conclude from 3.7 that the theorem holds.*

Corollary 3.1 *In the same assumptions of the theorem, we have*

$$\Delta_f r \leqslant (m-1)\frac{h'}{h}(r) + \theta(r) \qquad (resp. \ \geqslant).$$

Remark 3.4 *The corollary 3.1 recover comparison results for weighted Riemannian manifolds with* $\mathrm{Ric}_f(\nabla r, \nabla r) \geqslant -(m-1)G(r)$ *and* f *satisfying 3.4 for some non-decreasing function* $\theta \in C^0[0,+\infty)$*, see [43, Theorem 3.1].*

Corollary 3.2 *Let* $\mathbb{R}^m$ *be the euclidean space m-dimensional with weighted* $f = \|x\|^2/2$*. Given a smooth even function* G *on* $\mathbb{R}$*, let* h *be the solution of the Cauchy problem*

$$\begin{cases} h'' - (1+G)h = 0 \\ h(0) = 0, \ h'(0) = 1, \end{cases}$$

and let $I = [0, r_0) \subset [0, +\infty)$ *be maximal interval where* h *is positive. Suppose that the radial curvature satisfies*

$$\sec_f \geqslant -G(r(x)) \ \ (resp. \leqslant) \ \ on \quad B_{r_0}(0). \tag{3.12}$$

Let

$$\mathrm{Hess}_f r(\cdot\,,\cdot) := \mathrm{Hess}\, r(\cdot\,,\cdot) - \frac{1}{m}\langle \nabla r, \nabla f\rangle\langle\cdot\,,\cdot\rangle,$$

then

$$\mathrm{Hess}_f r(x)(\cdot\,,\cdot) \leqslant \frac{h'}{h}\{\langle\cdot\,,\cdot\rangle - dr\otimes dr(\cdot\,,\cdot)\} + \frac{1}{2m}\|x\|\langle\cdot\,,\cdot\rangle \ \ (resp. \geqslant).$$

Proof. We have that $\eta(r(x)) = \langle \nabla r, \nabla f\rangle = \|x\|/2$, we choose $\theta(r(x)) = \|x\|/2$, and is clear that

$$\lim_{s\to 0^+} \eta(s) = 0 = o(1).$$

Since $\sec(V, U) = 0$, we have

$$\sec_f(V, U) = \mathrm{Hess} f(V, V) = \langle V, V\rangle = \|V\|^2,$$

choosing $G(s) = s^2$, the result follow by the Theorem 3.2. ■

Theorem 3.5 *Let* $(M^m, \langle\,,\rangle, f)$ *be a complete weighted Riemannian manifold, and* f *super harmonic. Assume that the radial Bakry-Émery-Ricci tensor of* M *satisfies*

$$\mathrm{Ric}_f(\nabla r, \nabla r) \geqslant -(m-1)G(r) \tag{3.13}$$

for some function $G \in C^0([0,+\infty))$, and that $\eta(r) = \langle \nabla r, \nabla f\rangle$ let such that $\eta(s) = o(1)$ as $s \to 0^+$. Let $h \in C^2([0,+\infty))$ a solution to the problem

$$\begin{cases} h'' - Gh \geqslant 0, \\ h(0) = 0, \;\; h'(0) = 1. \end{cases} \tag{3.14}$$

then the inequality

$$\Delta_f r(x) \leqslant (m-1)\frac{h'(r(x))}{h(r(x))} \tag{3.15}$$

hold pointwise on $M \backslash (\mathrm{Cut}(p_0) \cup \{p_0\})$.

Proof. Let $[0, r_0) \subset [0,+\infty)$ be the maximal interval where h is positive. Let $D_{p_0} = M\backslash\mathrm{Cut}(p_0)$ and fix $x \in D_{p_0} \cap [B_{r_0}(p_0)\backslash\{p_0\}]$. Let $\gamma : [0,\ell] \to M$ be the minimizing geodesic joining p_0 to x parametrized by arch length. Define

$$\varphi(s) = (\Delta_f r)(\gamma(s)), \quad s \in (0,\ell].$$

We claim that φ satisfies

$$\begin{cases} (\mathbf{i}) \quad \varphi(s) = \frac{m-1}{s} + o(1), & \text{as } s \to 0^+, \\ (\mathbf{ii}) \quad \varphi' + \frac{1}{m-1}\varphi^2 \leqslant (m-1)G, & \text{on } (0,\ell]. \end{cases} \tag{3.16}$$

Indeed, as $\eta(s) = o(1)$ when $s \to 0^+$ we have, using (3.11), that

$$\Delta_f r = \frac{m-1}{r} + o(1), \quad \text{as } r \to 0^+.$$

that proof the item (**i**) of (3.16).

To the item (**ii**), of (3.16), we obtain in (3.7) that

$$\begin{aligned}\frac{d}{ds}(\mathrm{Hess}_f r(\gamma)(Y,Y)) + \langle \mathrm{hess}_f r(\gamma)(Y), \mathrm{hess}_f r(\gamma)(Y)\rangle = \\ = \mathrm{Hess} f(Y,Y) - Sect_f(Y,\gamma').\end{aligned} \tag{3.17}$$

Taking the trace of (3.17), we obtain

$$\frac{d}{ds}(\Delta_f r \circ \gamma) + \|\mathrm{Hess}_f r\|^2(\gamma) = \Delta f(\gamma) - \mathrm{Ric}_f(\nabla r, \nabla r)(\gamma).$$

Taking an orthonormal basis $\{\nabla r, \ldots, e_m\}$, and using Cauchy-Schwarz we obtain

$$\|\mathrm{Hess}_f r\|^2 \geqslant \frac{(\Delta_f r)^2}{m-1},$$

therefore, using that $\Delta f \leqslant 0$, we obtain

$$\frac{d}{ds}(\Delta r \circ \gamma) + \frac{(\Delta r \circ \gamma)^2}{m-1} \leqslant -\mathrm{Ric}_f(\nabla r, \nabla r)(\gamma). \tag{3.18}$$

Since $\mathrm{Ric}_f(\nabla r, \nabla r) \geqslant -(m-1)G(r)$ we have

$$\varphi' + \frac{1}{m-1}\varphi^2 \leqslant (m-1)G(r).$$

Now, let $\psi = (m-1)\frac{h'}{h}$, so

$$\begin{aligned}
\psi' + \frac{1}{m-1}\psi^2 &= (m-1)\frac{hh'' - h'^2}{h^2} + \frac{(m-1)^2}{m-1}\frac{h'^2}{h^2} \\
&= (m-1)\left(\frac{hh'' - h'^2}{h^2} + \frac{h'^2}{h^2}\right) \\
&= (m-1)\frac{h''}{h^2} \\
&\geqslant (m-1)G.
\end{aligned}$$

Since $\psi = (m-1)\phi$, follow of (3.5) that

$$\psi(s) = \frac{m-1}{s} + o(1).$$

Therefore, by the Proposition 3.1, we have

$$\varphi \leqslant \psi \qquad \text{em } D_0 \cap (B_{r_0}(p_0)\backslash\{p_0\}),$$

that is,

$$\Delta_f r(x) \leqslant (m-1)\frac{h'(r(x))}{h(r(x))} \qquad \text{in } D_0 \cap (B_{r_0}(p_0)\backslash\{p_0\}).$$

■

Remark 3.6 *Since* $\mathrm{Ric}_f^N \leqslant \mathrm{Ric}_f$. *Then, if* $\mathrm{Ric}_f^N \geqslant -(m-1)G(r)$ *the result remains valid for any* $N > 0$.

CHAPTER 4

ON THE FIRST STABILITY EIGENVALUE OF SURFACES

4.1 Introduction

In this chapter we presented upper bounds for the first eigenvalue of the weighted Jacobi operator on surface with constant weighted mean curvature. In particular we generalize a result obtained by Schoen and Yau (see [45, Theorem 5.1]) on stable minimal surfaces in 3-Riemannian manifolds with non-negative scalar curvature for the setting of weighted manifolds. We also show that, all closed λ-surface in Gaussian space are stable, and consequently in $\mathbb{R}^3$ there is no closed stable self-shrinker.

Now, we introduce some objects related with the theory of surfaces in a weighted Riemannian manifold. Let $\Sigma \subset M^3$ be a two-sided surface of M^3 and consider N an unit normal vector field globally defined on Σ. We will denote by A its second fundamental form and by H the mean curvature of Σ, that is, the trace of A.

Following [53], we will use (2.1) slightly modified, as follow

$$\overline{\sec}_f^{2m}(X,Y) = \sec(X,Y) + \frac{1}{2}\left(\mathrm{Hess} f(X,X) - \frac{(df(X))^2}{2m}\right), \tag{4.1}$$

where X and Y are unit and orthogonal vectors fields tangents to M, and $\sec(X,Y)$ is the usual sectional curvature of the plane spanned by X and Y.

Taking $N = 2m$ in (2.2) we have

$$\mathrm{Ric}_f^{2m} = \mathrm{Ric}_f - \frac{df \otimes df}{2m}, \tag{4.2}$$

where $m > 0$.

We remember that, the natural generalization for the scalar curvature S of a weighted Riemannian manifold M, is

$$S_\infty = S + 2\Delta_M f - |\overline{\nabla} f|^2, \tag{4.3}$$

known as *Perelman's scalar curvature*, see [39] for a good overview. We point out that S_∞ is not the trace of Ric_f^{2m} for any $m > 0$, and is not the trace of Ric_f.

Throughout this chapter, $d\nu_f = e^{-f}d\nu$ will denote the weighted measure of the surface Σ, where $d\nu$ is the Riemannian measure of Σ, $|\Sigma|$ and $|\Sigma|_f$ denote the area of Σ with respect to the Riemannian measure and weighted measure of Σ, respectively. Furthermore, we will denote by K the Gaussian curvature of Σ and by $\sec_\Sigma$ the sectional curvature of M restricted to Σ.

It is a remarkable fact that, in the variational setting, surfaces with constant weighted mean curvature are stationary points of the weighted area functional under variations that preserves the weighted volume (see [8]). Moreover, the second variation of the weighted area gives rise the weighted Jacobi operator on Σ, see [17], which is defined by

$$J_f u = \Delta_f u + (|A|^2 + \mathrm{Ric}_f(N,N))u, \tag{4.4}$$

for any $u \in C^\infty(\Sigma)$ and $|A|^2$ is the square Hilbert-Schmidt norm of A.

Now we will introduced the notion of stability with the goal of present some consequences.

Definition 2 *Under the above notation. We say that a surface Σ is stable if the first eigenvalue λ_1 of the weighted Jacobi operator is nonnegative. Otherwise, we say that Σ is unstable.*

4.2 Estimates for the First Eigenvalue of the Weighted Jacobi Operator

In this section we present the main results of this chapter. Our first result reads as follows

Theorem 4.1 *Let $(M^3, \langle\,,\rangle, f)$ be a weighted Riemannian manifold with $S_\infty \geqslant 6c$, for some $c \in \mathbb{R}$. Let $\Sigma^2 \subset M^3$ be a closed surface with constant weighted mean curvature H_f. Denote by λ_1 the first eigenvalue of the weighted Jacobi operator. Then,*

$$\lambda_1 \leqslant -\frac{1}{2}(H_f^2 + 6c) - \frac{4\pi(g-1)}{|\Sigma|}.$$

Moreover, equality holds if and only if Σ is totally geodesic, f is constant, $S|_\Sigma = 6c$ and K is constant.

Remark 4.2 *In Riemannian case, $f = 0$, the estimate can be improved. See Corollary 4.3 in the subsection 4.4.1.*

The next result is a generalization of a result of Schoen and Yau on stable minimal surfaces (see [45]) and this technique allow us to give an improvement of Theorem 2.1 in [19].

The result is the following:

Corollary 4.1 *Let $(M^3, \langle\,,\rangle, f)$ be a weighted Riemannian manifold with nonnegative Perelman's scalar curvature. Let Σ be a closed stable surface with constant weighted mean curvature H_f. Then Σ is conformally equivalent to the sphere $\mathbb{S}^2$ or Σ is a totally geodesic flat torus $\mathbb{T}^2$. Moreover, if $S_\infty > 0$, then Σ is conformally equivalent to the sphere $\mathbb{S}^2$.*

The second result is the following:

Theorem 4.3 *Let $(M^3, \langle\, ,\rangle, f)$ be a weighted Riemannian manifold with $\overline{\sec}_f^{2m} \geqslant c$, for some $c \in \mathbb{R}$, and $Hess f \leqslant \sigma \cdot g$ for some real function σ on M. Let $\Sigma^2 \subset M^3$ be a closed surface with constant weighted mean curvature H_f. Denote by λ_1 the first eigenvalue of the weighted Jacobi operator. Then,*

(i) $\lambda_1 \leqslant -\frac{1}{2}\left(\frac{H_f^2}{1+m} + 4c\right)$, *with equality if and only if Σ is totally umbilical in M^3, $\mathrm{Ric}_f^{2m} = 2c$ and $df(N) = \dfrac{m}{1+m}H_f$ on Σ;*

(ii) $\lambda_1 \leqslant -\dfrac{H_f^2}{(1+2m)} - \left(4c - \dfrac{\int_\Sigma \sigma\, d\nu_f}{|\Sigma|_f}\right) + \dfrac{2}{|\Sigma|_f}\int_\Sigma K\, d\nu_f$.

Moreover, if equality holds, then $\overline{\sec}_f^{2m} = c$, $\mathrm{Ric}_f^{2m} = 2c$, $df(N) = \frac{2m}{1+2m}H_f$, and $|A|$ is a constant on Σ. Moreover, M^3 has constant sectional curvature k and e^{-f} is the restriction of a coordinate function from the appropriate canonical embedding of a space form $\mathbb{Q}_k^3$ in $\mathbb{E}^4$, where $\mathbb{E}^4$ is $\mathbb{R}^4$ or $\mathbb{L}^4$.

Our third result reads as follows:

Theorem 4.4 *Let $(M^3, \langle\, ,\rangle, f)$ be a weighted Riemannian manifold with $\sec \geqslant c$, for some $c \in \mathbb{R}$, and $\mathrm{Hess} f \geqslant \dfrac{df \otimes df}{2m}$ (in the sense of quadratic forms). Let $\Sigma^2 \subset M^3$ be a closed surface with constant weighted mean curvature H_f. Denote by λ_1 the first eigenvalue of the weighted Jacobi operator. Then,*

(i) $\lambda_1 \leqslant -\dfrac{1}{2}\left(\frac{H_f^2}{1+m} + 4c\right)$, *with equality if and only if Σ is totally umbilical in M^3, $\mathrm{Ric}(N,N) = 2c$, $df(N) = \dfrac{m}{1+m}H_f$ on Σ and $\mathrm{Hess} f(N,N) = \dfrac{df(N)^2}{2m}$;*

(ii) $\lambda_1 \leqslant -\dfrac{H_f^2}{(1+2m)} - 4c + \dfrac{2}{|\Sigma|_f}\int_\Sigma K\, d\nu_f$. *Furthermore, equality holds if and only if K is constant, $\sec_\Sigma = c$, $df(N) = \dfrac{m}{1+m}H_f$ on Σ and $\mathrm{Hess} f(N,N) = \dfrac{df(N)^2}{2m}$.*

Remark 4.5 *We believe that the hypotheses on the function f in theorems 4.3 and 4.4 are natural, because we recovered the Riemannian case if the function is constant and also, for m large enough, we captured the Gaussian space, which is very important in literature.*

Now, we will give an application on the context of the mean curvature flow. For that, we recall that a *self-shrinker* of the mean curvature flow is an oriented surface $\Sigma \subset \mathbb{R}^3$ such that

$$H = -\frac{1}{2}\langle x, N\rangle,$$

where N is an unit normal vector field on Σ. The simplest examples of self-shrinkers in $\mathbb{R}^3$ are the plane $\mathbb{R}^2$, the sphere of radius 2, and the cylinder $\mathbb{S}^1 \times \mathbb{R}^1$, where the $\mathbb{S}^1$ has radius $\sqrt{2}$. So, if we consider $\mathbb{R}^3$ endowed with the function $f(x) = \frac{|x|^2}{4}$, then a self-shrinker is a f-minimal surface in the Euclidean space. More generally, the triple $(\mathbb{R}^3, \delta_{ij}, |x|^2/4)$ is known as *Gaussian space* and the surfaces with weighted mean curvature λ are know as λ*-surfaces*.

The next result is a consequence of the proof of the Theorem 4.4 and it reads as follows:

Corollary 4.2 *All closed λ-surfaces in the Gaussian space are unstable. In particular, there exist no closed stable self-shrinker surfaces in $\mathbb{R}^3$.*

This chapter is organized in this way: In section 4.3 we give a classification of weighted Riemannian manifolds with constant weighted sectional curvature, we present also a way to describe the first eigenvalue of the weighted Jacobi operator and, to conclude the section, we rewrite the terms of the weighted Jacobi operator in an appropriate manner. In section 4.4 we present the proof of the results and the other consequences of them.

4.3 Preliminaries

An important result for us is the classification of weighted Riemannian manifolds with constant weighted sectional curvature. The result below follows closely the one in [53], and we include the proof here for the sake of completeness.

Lemma 1 *Let $(M^3, \langle\,,\rangle, f)$ be a weighted Riemannian manifold. Assume that $\overline{\sec}_f^{2m} = c$, then M has constant sectional curvature k, for some $k \in \mathbb{R}$. Moreover, $c = -(m-1)k$ and if f is a non constant function, then $u = e^{-f/m}$ is the restriction of a coordinate function from the appropriate canonical embedding of a space form of curvature k, $\mathbb{Q}^3_k$, in $\mathbb{E}^4$, where $\mathbb{E}^4$ is $\mathbb{R}^4$ or $\mathbb{L}^4$.*

Proof. Let X and Y be an unit and orthogonal vectors on M. Then, by equation (4.1), we get

$$c = \sec(X,Y) + \frac{1}{2}\left(\mathrm{Hess}f(X,X) - \frac{(df(X))^2}{2m}\right)$$

and

$$c = \sec(Y,X) + \frac{1}{2}\left(\mathrm{Hess}f(Y,Y) - \frac{(df(Y))^2}{2m}\right).$$

So, there exists a smooth function $w : M \to \mathbb{R}$ such that

$$\mathrm{Hess}f - \frac{df \otimes df}{2m} = w \cdot g.$$

Then, letting $\{E_1, E_2, X\}$ be an orthonormal frame we have

$$2c = \sum_{i=1}^{2} \overline{\sec}_f^{2m}(X, E_i) = \mathrm{Ric}(X,X) + 2w.$$

Thus, by Schur's Lemma, w is a constant function and so M has constant sectional curvature, say k. Defining the function $u = e^{-f/m}$, we have that

$$\mathrm{Hess} u = -\frac{c-k}{m} u \cdot g. \tag{4.5}$$

So, by Lemma 1.2 in [50],

$$g = dt^2 + (u')^2 g_0, \tag{4.6}$$

where g_0 is a local metric on a surface orthogonal to ∇u (a level set of u) and u' denote the derived of u in the direction of the gradient of u.

Computing the radial sectional curvature of the metric (4.6), we have $(c+(m-1)k)u' = 0$. Since f is non constant, we have that $c = -(m-1)k$. Moreover, as u satisfies equations (4.5) and (4.6), u is the restriction of a coordinate function from the appropriate canonical embedding of $\mathbb{Q}^3_k$ in $\mathbb{E}^4$, where $\mathbb{E}^4$ is $\mathbb{R}^4$ or $\mathbb{L}^4$. ∎

Now we will describe the first stability eigenvalue in an appropriate manner. For this, consider a first eigenfunction $\rho \in C^\infty(\Sigma)$ of the weighted Jacobi operator J_f, that is, $J_f\rho = -\lambda_1\rho$; or equivalently,

$$\Delta_f \rho = -(\lambda_1 + |A|^2 + \mathrm{Ric}_f(N,N))\rho. \tag{4.7}$$

Furthermore, λ_1 is simple and it is characterized by

$$\lambda_1 = \inf\left\{ \frac{-\int_\Sigma u J_f u \, d\nu_f}{\int_\Sigma u^2 \, d\nu_f} : u \in C^\infty(\Sigma),\ u \neq 0 \right\}. \tag{4.8}$$

We observe that the first eigenfunction of an elliptic second-order differential operator has a sign. Therefore, without loss of generality, we can assume that $\rho > 0$.

Thus,

$$\begin{aligned}
\Delta_f \ln\rho &= \Delta \ln\rho - \langle \nabla f, \nabla \ln\rho\rangle \qquad (4.9)\\
&= \mathrm{div}_\Sigma(\nabla \ln\rho) - \langle \nabla f, \rho^{-1}\nabla\rho\rangle\\
&= \mathrm{div}_\Sigma(\rho^{-1}\nabla\rho) - \rho^{-1}\langle \nabla f, \nabla\rho\rangle\\
&= \rho^{-1}\mathrm{div}_\Sigma(\nabla\rho) + \langle \nabla\rho^{-1}, \nabla\rho\rangle - \rho^{-1}\langle \nabla f, \nabla\rho\rangle\\
&= \rho^{-1}(\Delta\rho - \langle \nabla f, \nabla\rho\rangle) - \rho^{-2}|\nabla\rho|^2\\
&= \rho^{-1}\Delta_f\rho - \rho^{-2}|\nabla\rho|^2\\
&= -(\lambda_1 + |A|^2 + \mathrm{Ric}_f(N,N)) - \rho^{-2}|\nabla\rho|^2.
\end{aligned}$$

Integrating the equality above on Σ with respect to the weighted measure $d\nu_f$ and using the divergence theorem we have that

$$0 = -\lambda_1|\Sigma|_f - \int_\Sigma (|A|^2 + \mathrm{Ric}_f(N,N))\, d\nu_f - \alpha,$$

where $\alpha := \int_\Sigma \rho^{-2}|\nabla\rho|^2 \, d\nu_f \geqslant 0$ defines a simple invariant that is independent of the choice of ρ, because λ_1 is simple. So,

$$\lambda_1 = -\frac{1}{|\Sigma|_f}(\alpha + \int_\Sigma (|A|^2 + \mathrm{Ric}_f(N,N))\, d\nu_f). \tag{4.10}$$

Let $\{E_i\}$ be an orthonormal frame in $T\Sigma$ and $\{a_{ij}\}$ the coefficients of A in the frame, using the Gauss equation

$$K = \sec_\Sigma - \langle A(X), Y\rangle^2 + \langle A(X), X\rangle\langle A(Y), Y\rangle,$$

we have that

$$K - \sec_\Sigma = a_{11}a_{22} - a_{12}^2 = \frac{1}{2}\left((a_{11}+a_{22})^2 - \sum_{i,j=1}^{2} a_{ij}^2\right) = \frac{1}{2}\left(H^2 - |A|^2\right),$$

hence

$$|A|^2 = H^2 + 2(\sec_\Sigma - K). \tag{4.11}$$

To complete this section, we recall the traceless of the second fundamental form of Σ, that is, the tensor ϕ defined by $\phi = A - \frac{H}{2}I$, where I denote the identity endomorphism on $T\Sigma$. We note that $tr(\phi) = 0$ and $|\phi|^2 = |A|^2 - \frac{H^2}{2} \geqslant 0$, with equality if and only if Σ is totally umbilical, where $|\phi|^2$ is the Hilbert-Schmidt norm.

In the literature, ϕ is know as the total umbilicity tensor of Σ. In terms of ϕ, the weighted Jacobi operator is rewritten as

$$J_f u = \Delta_f u + \left(|\phi|^2 + \frac{H^2}{2} + \mathrm{Ric}_f(N,N)\right)u. \tag{4.12}$$

We use exactly this expression in next section to obtain estimates of the first eigenvalue of the weighted Jacobi operator.

4.4 Proof of the Theorems 4.1, 4.3 and 4.4

4.4.1 Proof of the Theorem 4.1

We start doing a straightforward calculus. Let $\{e_1, e_2, e_3\}$ be a adapted referential of Σ to M. Lets rewrite the expression $|A|^2 + \mathrm{Ric}_f(N,N)$. We know that

$$\frac{S}{2} = \sec_\Sigma + \mathrm{Ric}(e_3),$$

where S is the scalar curvature of M. By Gauss equation (4.11), we have

$$\sec_\Sigma = K - \frac{H^2}{2} + \frac{|A|^2}{2}.$$

Thus,

$$
\begin{aligned}
|A|^2 + \mathrm{Ric}_f(N,N) &= \frac{S}{2} - K + \frac{H^2}{2} + \frac{|A|^2}{2} + \mathrm{Hess} f(e_3,e_3)\\
&= \frac{1}{2}S_\infty - \Delta_M f + \frac{1}{2}|\overline{\nabla} f|^2 - K + \frac{H^2}{2} + \frac{|A|^2}{2} + \mathrm{Hess} f(e_3,e_3)\\
&= \frac{1}{2}S_\infty - (\Delta_\Sigma f - H f_3 + \mathrm{Hess} f(e_3,e_3)) + \frac{1}{2}(|\nabla f|^2 + f_3^2)\\
&\qquad -K + \frac{H^2}{2} + \frac{|A|^2}{2} + \mathrm{Hess} f(e_3,e_3)\\
&= \frac{1}{2}S_\infty - K - \Delta_\Sigma f + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}H_f^2 + \frac{1}{2}|A|^2. \qquad (4.13)
\end{aligned}
$$

Integrating with respect to Riemannian measure $d\nu$, using the divergence theorem and Gauss-Bonnet theorem we obtain

$$\int_\Sigma |A|^2 + \mathrm{Ric}_f(N,N)\,d\nu = 4\pi(g-1) + \frac{1}{2}\int_\Sigma (S_\infty + H_f^2 + |A|^2 + |\nabla f|^2)\,d\nu.$$

By the other hand, integrating (4.9) with respect to $d\nu$ we obtain that

$$-\int_\Sigma \langle \frac{1}{\rho}\nabla\rho, \nabla f\rangle d\nu = -\lambda_1|\Sigma| - (\alpha + \int_\Sigma (|A|^2 + \mathrm{Ric}_f(N,N))d\nu),$$

and so,

$$-\int_\Sigma (\frac{|\nabla\rho|^2}{2\rho^2} + \frac{|\nabla f|^2}{2})d\nu \leqslant -\lambda_1|\Sigma| - (\alpha + \int_\Sigma (|A|^2 + \mathrm{Ric}_f(N,N))d\nu).$$

After a straightforward computation we have that

$$\lambda_1 \leqslant -\frac{1}{|\Sigma|}(\frac{\alpha}{2} + 4\pi(g-1) + \frac{1}{2}\int_\Sigma (S_\infty + H_f^2 + |A|^2)d\nu).$$

By our hypothesis,

$$\lambda_1 \leqslant -\frac{1}{2}(H_f^2 + 6c) - \frac{4\pi(g-1)}{|\Sigma|}.$$

Moreover, if equality holds then $\alpha = 0$ and thus ρ and f are constants, Σ is totally geodesic, $S|_\Sigma = 6c$ and K is constant. The reciprocal is immediate.

In Riemannian case, $f = 0$, we can improve the estimate in theorem 4.1. The result is the following:

Corollary 4.3 *Let $(M^3, \langle\,,\rangle)$ be a Riemannian manifold with $S \geqslant 6c$, for some $c \in \mathbb{R}$. Let $\Sigma^2 \subset M^3$ be a closed surface with constant mean curvature H. Then,*

$$\lambda_1 \leqslant -\frac{3}{4}(H^2 + 4c) - \frac{4\pi(g-1)}{|\Sigma|}.$$

Moreover, equality holds if and only if Σ is totally umbilical, $S|_\Sigma = 6c$ and K is constant.

Proof. The equation (4.13) can be rewrite, with $f = 0$, in the following way

$$|A|^2 + \mathrm{Ric}(N,N) = \frac{1}{2}S - K + \frac{3}{4}H^2 + \frac{1}{2}|\phi|^2.$$

After a straightforward computation we have that

$$\lambda_1 \leqslant -\frac{1}{|\Sigma|}(\alpha + 4\pi(g-1) + \frac{1}{2}\int_\Sigma (S + \frac{3}{2}H^2 + |\phi|^2)d\nu),$$

and so

$$\lambda_1 \leqslant -\frac{3}{4}(H^2 + 4c) - \frac{4\pi(g-1)}{|\Sigma|}.$$

Moreover, if equality holds then $\alpha = 0$ and thus ρ is constant, Σ is totally umbilical, $S|_\Sigma = 6c$ and K is constant. The reciprocal is immediate. ■

4.4.2 Proof of the Theorem 4.3

Before to initiate the proof, we will recall the generalized sectional curvature

$$\overline{Sect}_f^{2m}(X,Y) = \sec(X,Y) + \frac{1}{2}\left(\mathrm{Hess}f(X,X) - \frac{(df(X))^2}{2m}\right),$$

where X, Y are unit and orthogonal vectors fields on M.

Moreover,

$$\mathrm{Ric}_f^{2m}(X,X) = \sum_{i=1}^{2}\overline{Sect}_f^{2m}(X,Y_i).$$

Since

$$\mathrm{Ric}_f^{2m}(N,N) + 2\sec_\Sigma \geqslant \mathrm{Ric}_f^{2m}(N,N) + 2\overline{\sec}_f^{2m}|_\Sigma - \mathrm{Hess}f(X,X),$$

where X is a vector field on Σ. So, if $\mathrm{Hess}f \leqslant \sigma g$, using (4.15) we rewrite the expression (4.10) by

$$\lambda_1 \leqslant -\frac{H_f^2}{1+2m} - \frac{1}{|\Sigma|_f}\left\{\alpha - 2\int_\Sigma K\,d\nu_f + \int_\Sigma \left(\mathrm{Ric}_f^{2m}(N,N) + 2\overline{\sec}_f^{2m} - \sigma\right)d\nu_f\right\}. \tag{4.14}$$

Now, we are able to prove our result.

Proof. The item (i) is a consequence of Theorem 4.4 (i). To second item, we using the expression in (4.14) and our hypotheses.

Now, if equality holds, then $\alpha = 0$, $\mathrm{Ric}_f^{2m} = 2c$ and $\overline{Sect}_f^{2m} = c$. By equality in the inequality (4.15), we obtain

$$df(N) = \frac{2m}{1+2m}H_f,$$

and so

$$H = H_f - \frac{2m}{1+2m} H_f = \frac{1}{1+2m} H_f.$$

Moreover, $\alpha = 0$ imply that ρ is constant and of the equation (4.7) we have that $|A|^2$ is also a constant.

To conclude, we using the Lemma 1 to obtain that M^3 has constant sectional curvature and e^{-f} has the property enunciate in case of the equality. ■

In the next subsection we will provide the prove of Theorem 4.4 and some consequences.

4.4.3 Proof of the Theorem 4.4

Using (4.11) in (4.10) we obtain that

$$\lambda_1 = -\frac{1}{|\Sigma|_f}\left\{\alpha - 2\int_\Sigma K\,d\nu_f + \int_\Sigma [H^2 + 2\sec_\Sigma + \mathrm{Ric}_f(N,N)]d\nu_f\right\}.$$

So, using the definition of weighted mean curvature we have

$$\lambda_1 = -\frac{1}{|\Sigma|_f}\left\{\alpha - 2\int_\Sigma K\,d\nu_f + \int_\Sigma (H_f - \langle N, \nabla f\rangle)^2 d\nu_f + \int_\Sigma [2\sec_\Sigma + \mathrm{Ric}_f(N,N)]d\nu_f\right\}.$$

Moreover, we know that for all $a, b \in \mathbb{R}$ and $k > -1$, it holds that

$$(a+b)^2 \geqslant \frac{a^2}{1+k} - \frac{b^2}{k}, \tag{4.15}$$

with equality if and only if $b = -\frac{k}{1+k}a$. Applying that inequality with $k = 2m$, using the definition of Ric_f^{2m} and a straightforward computation, we obtain that

$$\lambda_1 \leqslant -\frac{H_f^2}{1+2m} - \frac{1}{|\Sigma|_f}\left\{\alpha - 2\int_\Sigma K\,d\nu_f + \int_\Sigma \left(\mathrm{Ric}_f^{2m}(N,N) + 2\sec_\Sigma\right)d\nu_f\right\}.$$

Using the hypotheses we obtain

$$\lambda_1 \leqslant -\frac{H_f^2}{1+2m} - 4c + \frac{2}{|\Sigma|_f}\int_\Sigma K\,d\nu_f. \tag{4.16}$$

Proof. (i) Choosing the constant function $u = 1$ to be the test function in (4.8) to estimate λ_1, and using the expression in (4.12), we obtain that

$$\begin{aligned}
\lambda_1 &\leqslant \frac{-\int_\Sigma 1 J_f 1\,d\nu_f}{\int_\Sigma 1\,d\nu_f} = -\frac{1}{|\Sigma|_f}\left[\int_\Sigma |\phi|^2 d\nu_f + \frac{1}{2}\int_\Sigma H^2 d\nu_f + \int_\Sigma \mathrm{Ric}_f(N,N)d\nu_f\right] \\
&= -\frac{1}{|\Sigma|_f}\left[\int_\Sigma |\phi|^2 d\nu_f + \frac{1}{2}\int_\Sigma (H_f - \langle N,\nabla f\rangle)^2 d\nu_f + \int_\Sigma \mathrm{Ric}_f(N,N)d\nu_f\right] \\
&\leqslant -\frac{1}{|\Sigma|_f}\left[\int_\Sigma |\phi|^2 d\nu_f + \frac{1}{2}\int_\Sigma \left(\frac{H_f^2}{1+m} - \frac{\langle N,\nabla f\rangle^2}{m} d\nu_f\right) + \int_\Sigma \mathrm{Ric}_f(N,N)d\nu_f\right] \\
&\leqslant -\frac{H_f^2}{2(1+m)} - 2c - \frac{1}{|\Sigma|_f}\int_\Sigma |\phi|^2 d\nu_f \\
&\leqslant -\frac{1}{2}\left(\frac{H_f^2}{1+m} + 4c\right).
\end{aligned}$$

If $\lambda_1 = -\frac{1}{2}\left(\frac{H_f^2}{1+m} + 4c\right)$, then all the inequalities above becomes equalities and consequently Σ is totally umbilical, $\mathrm{Ric}(N,N) = 2c$, $df(N) = \dfrac{m}{1+m}H_f$ and $\mathrm{Hess}f(N,N) = \dfrac{df(N)^2}{2m}$.

On the other hand, if Σ is totally umbilical, $\mathrm{Ric}(N,N) = 2c$, $df(N) = \dfrac{m}{1+m}H_f$ and $\mathrm{Hess}f(N,N) = \dfrac{df(N)^2}{2m}$, we have

$$\begin{aligned} H &= H_f - df(N) \\ &= H_f - \frac{m}{1+m}H_f \\ &= \frac{1}{1+m}H_f, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ric}_f(N,N) &= 2c + \frac{1}{2m}(df(N))^2 \\ &= 2c + \frac{m}{2(1+m)^2}H_f^2. \end{aligned}$$

Hence,

$$\begin{aligned} J_f &= \Delta_f + \frac{H^2}{2} + 2c + \frac{m}{2(1+m)^2}H_f^2 \\ &= \Delta_f + \frac{1}{2(1+m)^2}H_f^2 + 2c + \frac{m}{2(1+m)^2}H_f^2 \\ &= \Delta_f + \frac{1}{2(1+m)}H_f^2 + 2c, \end{aligned}$$

and thus,

$$\lambda_1 = -\frac{1}{2}\left(\frac{H_f^2}{1+m} + 4c\right),$$

as desired.

(ii) Using our hypotheses, we have by (4.16) that

$$\lambda_1 \leqslant -\frac{H_f^2}{1+2m} - 4c - \frac{2}{|\Sigma|_f}\int_\Sigma K\, d\nu_f.$$

If equality holds, then $\alpha = 0$, $\sec_\Sigma = c$, $\mathrm{Hess}f(N,N) = \frac{df(N)^2}{2m}$. Firstly, we obtain of the equation (4.15) that

$$df(N) = \frac{2m}{1+2m}H_f.$$

and so $H = \dfrac{1}{1+2m}H_f$. Moreover, $\alpha = 0$ implies $\nabla\rho = 0$ and thus using the equation (4.7) we have that $|A|^2$ is constant. Futhermore, by equation (4.11), we have that K is constant.

On the other hand, if K is constant, $\sec_\Sigma = c$, $\text{Hess}f(N,N) = \dfrac{df(N)^2}{2m}$ and $df(N) = \dfrac{2m}{1+2m}H_f$, we have that

$$\text{Ric}_f(N,N) = 2c + \frac{2m}{(1+2m)^2}H_f^2,$$

and so

$$\begin{aligned} J_f &= \Delta_f + |A|^2 + \text{Ric}_f(N,N) \\ &= \Delta_f + H^2 + 2(c-K) + 2c + \frac{2m}{(1+2m)^2}H_f^2 \\ &= \Delta_f + 4c + \frac{1}{1+2m}H_f^2 - 2K, \end{aligned}$$

and this implies that

$$\lambda_1 = -4c - \frac{1}{1+2m}H_f^2 + 2K.$$

Now, using that K is constant,

$$\lambda_1 = -4c - \frac{H_f^2}{1+2m} + \frac{2}{|\Sigma|_f}\int_\Sigma K\, d\nu_f,$$

as desired. ■

Now considering that the ambient is a 3-dimensional simply connected space form with sectional curvature c, $\mathbb{Q}_c^3$. If c is positive, we assume that all surfaces are contained in a hemisphere. In that conditions we obtain the follows result:

Corollary 4.4 *Let $\Sigma \subset \mathbb{Q}_c^3$ be a closed and orientable surface with constant weighted mean curvature H_f, where f is one half of the square of the extrinsic distance function. Assume that Σ is contained in the geodesic ball center in the origin 0 and radius $\sqrt{2m}$. Then*

(i) $\lambda_1 \leqslant -\dfrac{1}{2}\left(\dfrac{H_f^2}{1+m} + 4c\right)$

(ii) $\lambda_1 \leqslant -\dfrac{H_f^2}{(1+2m)} - 4c + \dfrac{2}{|\Sigma|_f}\int_\Sigma K\, d\nu_f.$

The equalities holds if and only if Σ is the sphere center in the origin and radius $\sqrt{2m}$, provided that $\sqrt{2m} \leqslant \dfrac{\pi}{\sqrt{c}}$ in case $c > 0$.

Proof. We know that

$$\text{Ric}_f^{2m} r(N,N) = 2c + \text{Hess}r^2(N,N) - \frac{(dr^2(N))^2}{2m}$$

and

$$\begin{aligned} \text{Hess}r^2(N,N) &= \langle \nabla_N \nabla r^2, N\rangle \\ &= 2(dr(N))^2 + 2r\text{Hess}r(N,N). \end{aligned}$$

Now, using the expression of the Hessian of the distance function in a space form, we have that

$$\mathrm{Hess}r(N,N) = \cot_c(r)[1-(dr(N))^2],$$

where

$$cot_c(s) = \begin{cases} \sqrt{-c}\, coth(\sqrt{-c}s) & \text{if} \quad c<0, \\ \frac{1}{s} & \text{if} \quad c=0, \\ \sqrt{c}\, cot(\sqrt{c}s) & \text{if} \quad c>0. \end{cases}$$

So,

$$\mathrm{Hess}r^2(N,N) = 2(dr(N)^2) + 2r\cot_c(r)[1-(dr(N))^2].$$

Now, using that the surface is contained in the ball center in the origin and radius $\sqrt{m}$ and $(dr(N))^2 \leqslant 1$, we obtain that

$$\begin{aligned} \mathrm{Ric}_f^{2m}(N,N) &= 2c + 2(dr(N))^2 + 2r\ \cot_c(r)(1-(dr(N))^2) - \frac{4r^2(dr(N))^2}{2m} \\ &\geqslant 2c. \end{aligned}$$

Therefore, by Theorem 4.4, we conclude the inequalities enunciates. To conclude, if the equalities holds, then $dr(N)=1$ and $r^2=2m$. ∎

Corollary 4.5 *Let $(M^3, \langle\,,\rangle, f)$ be a weighted Riemannian manifold with $\mathrm{Ric}_f^{2m} \geqslant 2c$ and $\sec \geqslant c$.*

(i) There is no closed stable surface with

$$\frac{H_f^2}{1+m} + 4c > 0.$$

(ii) If Σ^2 is a closed and stable surface such that $\frac{H_f^2}{1+2m} + 4c < 0$, then

$$|\Sigma|_f \geqslant -2\Big(\int_\Sigma K\, d\nu_f\Big)\Big(\Big|\frac{H_f^2}{1+2m} + 4c\Big|\Big)^{-1}.$$

Proof. By definition, a surface is stable if and only if $\lambda_1 \geqslant 0$. Thus the item (i) follows from the Theorem 4.4 (i). For the item (ii), we using the definition of stability and the Theorem 4.4 (ii). So,

$$0 \leqslant \lambda_1 \leqslant -\frac{H_f^2}{1+2m} - 4c + \frac{2}{|\Sigma|_f}\int_\Sigma K\, d\nu_f,$$

and thus

$$|\Sigma|_f \left|\frac{H_f^2}{1+2m} + 4c\right| \geqslant -2\int_\Sigma K\, d\nu_f.$$

∎

Another consequence of the Theorem 4.4 is an improvement of the proposition 3.2 in [28] for the case in that Σ is not necessarily f-minimal.

Corollary 4.6 *Under the same assumptions of the Theorem 4.4.*

(i) If $c > 0$, then Σ cannot be stable;

(ii) If $c = 0$, but $H_f \neq 0$, then Σ cannot be stable;

(iii) If $c = 0$ and Σ is stable, then $H_f = 0$.

CHAPTER 5

STEKLOFF'S EIGENVALUES TO WEIGHTED RIEMANNIAN MANIFOLDS

5.1 Introduction

The Classical Stekloff's eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega, \end{cases}$$

was introduced by him in [49] for bounded domains Ω of the plane and afterward this was studied by Payne in [42] for bounded domains in the plane with non-negative curvature.

In this chapter we study Stekloff's eigenvalue problems in the weighted context. More specifically, if $(M, \langle\,,\rangle, f)$ is a weighted Riemannian manifold with boundary ∂M, we study the following weighted Stekloff's eigenvalue problems

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ \frac{\partial u}{\partial \nu} = pu & \text{on } \partial M; \end{cases} \tag{5.1}$$

$$\begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \Delta_f u - q\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M; \end{cases} \tag{5.2}$$

$$\begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \frac{\partial^2 u}{\partial \nu^2} - q\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{5.3}$$

where ν denote the outward unit normal on ∂M. The first non-zero eigenvalues of the above problems will be denoted by p_1 and q_1, respectively. We will use the same letter for the first non-zero eigenvalues of last two problems because whenever the weighted mean curvature of ∂M is constant then the problems are equivalents, in the sense that u is solution of (5.2) if, and only if, u it is solution of (5.3). Indeed, since in the boundary $\Delta_{\partial M} u = 0$, we can write

$$\Delta_M u = \frac{\partial^2 u}{\partial \nu^2} + nH \frac{\partial u}{\partial \nu},$$

being $u = 0$ on ∂M, and in this case $\nabla u = \frac{\partial u}{\partial \nu}\nu$, we have

$$\begin{aligned} \Delta_f u &= \Delta u - \langle \nabla u, \nabla f \rangle = \Delta u - \frac{\partial u}{\partial \nu}\langle \nu, \nabla f \rangle \\ &= \frac{\partial^2 u}{\partial \nu^2} + (nH - \langle \nu, \nabla f \rangle)\frac{\partial u}{\partial \nu} \\ &= \frac{\partial^2 u}{\partial \nu^2} + nH_f \frac{\partial u}{\partial \nu}. \end{aligned} \tag{5.4}$$

Consequently,

$$\Delta_f u - q_1 \frac{\partial u}{\partial \nu} = \frac{\partial^2 u}{\partial \nu^2} - (q_1 - nH_f)\frac{\partial u}{\partial \nu}$$

therefore if H_f is constant, u is solution of (5.2) if, and only if, u it is solution of (5.3). Note that, in this case, the difference between p_1 and q_1 is H_f.

In this chapter the N-Bakry-Émery Ricci tensor will be defined as

$$\mathrm{Ric}_f^k = \mathrm{Ric}_f - \frac{df \otimes df}{k - n - 1}, \tag{5.5}$$

where $k > n+1$ or $k = n+1$ and f a constant function. We will consider M^{n+1} a compact oriented Riemannian manifold with boundary ∂M. Let $i : \partial M \hookrightarrow M$ be the standard inclusion and ν the outward unit normal on ∂M. We will denote by II its second fundamental form associate to ν, $\langle \nabla_X \nu, Y \rangle = II(X, Y)$, and by H the mean curvature of ∂M, that is, the trace of II over n.

In this chapter we will denote the weighted mean curvature, introduced by Gromov in [23], of the inclusion i by

$$H_f = H - \frac{1}{n}\langle \nu, \nabla f \rangle.$$

This chapter is organized of the following way: in the section 5.2 we presented results about upper bound and lower bound for the first non-zero Stekloff's eigenvalue; in the section 5.3 we obtain the auxiliary results to proof the results of the previous section, in the section 5.4 we prove the four first results of the section 5.2, and finally in the section 5.5 we prove the last theorem of the section 2.2.

Lastly, for the sake of simplicity, we will omit the weighted volume element in the integrals in all text.

5.2 Weighted Stekloff's Eigenvalue Problems

In this section we presented our results. We point out that the Riemannian cases of following theorems was studied by Wang and Xia in [57, 58] and by Escobar in [14], respectively.

We start obtaining an upper bound for the first non-zero Stekloff's eigenvalue of 5.1.

Theorem 5.1 *Let M^{n+1} be a compact weighted Riemannian manifold with* $\mathrm{Ric}_f^k \geqslant 0$ *and boundary ∂M. Assume that the weighted mean curvature of ∂M satisfies $H_f \geqslant \frac{(k-1)c}{n}$, to some positive constant c, and that second fundamental form $II \geqslant cI$, in the quadratic form sense. Denote by λ_1 the first non-zero eigenvalue of the f-Laplacian acting on functions on ∂M. Let p_1 the first non-zero eigenvalue of the weighted Stekloff eigenvalue problem* (5.1)*. Then,*

$$p_1 \leqslant \frac{\sqrt{\lambda_1}}{(k-1)c}(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}) \tag{5.6}$$

with equality occurs if and only if M is isometric to an n-dimensional euclidean ball of radius $\frac{1}{c}$, f is constant and $k = n + 1$.

The second result is the following:

Theorem 5.2 *Let M^{n+1} be a compact connected weighted Riemannian manifold with* $\mathrm{Ric}_f^k \geqslant 0$ *and boundary ∂M. Assume that the weighted mean curvature of ∂M satisfies $H_f \geqslant \frac{k-1}{k}c$, to some positive constant c. Let q_1 the first eigenvalue of the weighted Stekloff eigenvalue problem* (5.2)*. Then*

$$q_1 \geqslant nc.$$

Moreover, equality occurs if and only if M is isometric to a euclidean ball of radius $\frac{1}{c}$ in $\mathbb{R}^{n+1}$, f is constant and $k = n + 1$.

The next results are

Theorem 5.3 *Let M^{n+1} be a compact connected weighted Riemannian manifold with boundary ∂M. Denote by A, V the weighted area of ∂M and the weighted volume of M, respectively. Let q_1 the first eigenvalue of the weighted Stekloff eigenvalue problem* (5.2)*. Then,*

$$q_1 \leqslant \frac{A}{V}.$$

Moreover, if in addition that the Ric_f^k *of M is non-negative and that there is a point $x_0 \in \partial M$ such that $H_f(x_0) \geqslant \frac{(k-1)A}{knV}$, and $q_1 = \frac{A}{V}$ implies that M is isometric to an $(n+1)$-dimensional Euclidean ball, f is constant and $k = n + 1$.*

and

Theorem 5.4 *Let M^{n+1} be a compact connected weighted Riemannian manifold with $\mathrm{Ric}_f^k \geqslant 0$ and boundary ∂M nonempty. Assume that $H_f \geqslant \frac{(k-1)c}{n}$, for some positive constant c. Let q_1 be the first eigenvalue of the problem (5.3). Then*

$$q_1 \geqslant c.$$

Moreover, equality occurs if and only if M is isometric to a ball of radius $\frac{1}{c}$ in $\mathbb{R}^{n+1}$, f is constant and $k = n+1$.

Lastly, we announce a sharp estimate of the first non-zero Stekloff eigenvalue of surfaces on suitable hypotheses.

Theorem 5.5 *Let M^2 be a compact weighted Riemannian manifold with boundary. Assume that M has non-negative Ric_f, and that the geodesic curvature of ∂M, k_g satisfies $k_g - f_\nu \geqslant c > 0$. Let p_1 the first non-zero eigenvalue of the Stekloff problem (5.1). Assume that f is constant on the boundary ∂M and its derivative in the direction normal exterior is nonnegative, then $p_1 \geqslant c$. Moreover, the equality occur if and only if M is the Euclidean ball of radius c^{-1} and f is constant.*

5.3 Preliminaries

In this section we will present some results necessary to prove the theorems enunciated in the previous section. We will present some proofs for the sake of completeness.

In [5] the authors proved the following useful inequality.

Proposition 5.6 *Let u be a smooth function on M^{n+1}. then we have*

$$|\mathrm{Hess}u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u) \geqslant \frac{(\Delta_f u)^2}{k} + \mathrm{Ric}_f^k(\nabla u, \nabla u),$$

for every $k > n+1$ or $k = n+1$ and f is a constant. Moreover, equality holds if and only if $\mathrm{Hess}u = \frac{\Delta u}{n+1}\langle\, , \rangle$ and $\langle \nabla u, \nabla f\rangle = -\frac{k-n-1}{k}\Delta_f u$[1]*.*

Proof. Let $\{e_1, \ldots, e_{n+1}\}$ be a orthonormal basis of T_pM, then by Cauchy-Schwarz inequality we have that

$$(\Delta u)^2 \leqslant (n+1)|\mathrm{Hess}u|^2. \tag{5.7}$$

Using that $\frac{1}{n+1}a^2 + \frac{1}{k-n-1}b^2 \geqslant \frac{1}{k}(a-b)^2$ with equality if and only if

$$a = -\frac{(n+1)b}{k-n-1}, \tag{5.8}$$

[1]This term only appear in the case of a non constant function.

we obtain

$$\begin{aligned}
|\mathrm{Hess}u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u) &\geqslant \frac{1}{n+1}(\Delta u)^2 + \mathrm{Ric}_f^k(\nabla u, \nabla u) + \frac{\langle \nabla f, \nabla u\rangle^2}{k-n-1} \\
&\geqslant \frac{1}{k}(\Delta u - \langle \nabla f, \nabla u\rangle)^2 + \mathrm{Ric}_f^k(\nabla u, \nabla u) \\
&= \frac{1}{k}(\Delta_f u)^2 + \mathrm{Ric}_f^k(\nabla u, \nabla u).
\end{aligned} \tag{5.9}$$

If the equality holds, then since we use the Cauchy-Schwarz's inequality in (5.7) we obtain that $\mathrm{Hess}u = \lambda\langle\, ,\rangle$, and by (5.8)

$$\Delta u = -\frac{(n+1)\langle \nabla f, \nabla u\rangle}{k-n-1},$$

Consequently

$$\Delta_f u = -\frac{(n+1)\langle \nabla f, \nabla u\rangle}{k-n-1} - \langle \nabla f, \nabla u\rangle = -\frac{k}{k-n-1}\langle \nabla f, \nabla u\rangle.$$

The converse is immediate. ■

In [34] the authors showed that, for a smooth function u defined on an n-dimensional compact weighted Riemannian manifold M with boundary ∂M, the following identity holds if $h = \frac{\partial u}{\partial \nu}$, $z = u|_{\partial M}$ and Ric_f denote the generalized Ricci curvature of M:

$$\begin{aligned}
\int_M [(\Delta_f u)^2 - |\mathrm{Hess}u|^2 - \mathrm{Ric}_f(\nabla u, \nabla u)] = \\
= \int_{\partial M} \left[nH_f h^2 + 2h\overline{\Delta}_f z + II(\overline{\nabla} z, \overline{\nabla} z)\right]
\end{aligned} \tag{5.10}$$

that is a generalization of the Reilly's formula. Here, $\overline{\Delta}$ and $\overline{\nabla}$ represent the Laplacian and the gradient on ∂M with respect to the induced metric on ∂M, respectively.

Using the Proposition 5.6 we have that

$$\begin{aligned}
\int_M \frac{k-1}{k}[(\Delta_f u)^2 - \mathrm{Ric}_f^k(\nabla u, \nabla u)] \geqslant \\
\geqslant \int_{\partial M} [nH_f h^2 + 2h\overline{\Delta}_f z + II(\overline{\nabla} z, \overline{\nabla} z)].
\end{aligned} \tag{5.11}$$

The next result is an estimate for the first non-zero eigenvalue of the f-Laplacian on closed submanifolds.

Proposition 5.7 *Let M^{n+1} be a compact weighted Riemannian manifold with nonempty boundary ∂M and $\mathrm{Ric}_f^k \geqslant 0$. If the second fundamental form of ∂M satisfies $II \geqslant cI$, in the quadratic form sense, and $H_f \geqslant \frac{k-1}{n}c$, then*

$$\lambda_1(\partial M) \geqslant (k-1)c^2,$$

where λ_1 is the first non-zero eigenvalue of the f-Laplacian acting on functions on ∂M. The equality holds if and only if M is isometric to an Euclidean ball of radius $\frac{1}{c}$, f is constant and $k = n+1$.

Proof. Let z be an eigenfunction corresponding to the first non-zero eigenvalue λ_1 of the f-Laplacian of ∂M, that is,

$$\overline{\Delta}_f z + \lambda_1 z = 0. \tag{5.12}$$

Let $u \in C^\infty(M)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ u = z & \text{on } \partial M. \end{cases}$$

It then follows from (5.11) and the non-negativity of Ric_f^k of M that

$$0 \geqslant \int_{\partial M} [nH_f h^2 + 2h\overline{\Delta}_f z + II(\overline{\nabla} z, \overline{\nabla} z)]. \tag{5.13}$$

Since $II \geqslant cI$, we have

$$II(\overline{\nabla} z, \overline{\nabla} z) \geqslant c|\overline{\nabla z}|^2,$$

and noticing that

$$\int_{\partial M} |\overline{\nabla z}|^2 = -\int_{\partial M} z\overline{\Delta} z = \lambda_1 \int_{\partial M} z^2,$$

we obtain

$$\begin{aligned} 0 &\geqslant \int_{\partial M} [nH_f h^2 + 2h\overline{\Delta}_f z + II(\overline{\nabla} z, \overline{\nabla} z)] \\ &\geqslant \int_{\partial M} [(k-1)ch^2 - 2\lambda_1 zh + c\lambda_1 z^2] \\ &= \int_{\partial M} \left[(k-1)c\left(h - \frac{\lambda_1 z}{(k-1)c}\right)^2 + \lambda_1\left(c - \frac{\lambda_1}{(k-1)c}\right) z^2\right] \\ &\geqslant \lambda_1\left(c - \frac{\lambda_1}{(k-1)c}\right) \int_{\partial M} z^2. \end{aligned}$$

Consequently,

$$\lambda_1 \geqslant (k-1)c^2,$$

which proof the first part of theorem. The equality case follows by Proposition 5.6 and a careful analysis in the equalities that occur. The converse is immediate. ■

Recall the following version of Hopf boundary point lemma, which will be important in our proofs. See the proof in [26, Lemma 3.4].

Proposition 5.8 (Hopf boundary point lemma) *Let $(M^n, \langle\,,\rangle)$ be a complete Riemannian manifold and let $\Omega \subset M$ be a closed domain. If $u : \Omega \to \mathbb{R}$ is a function with $u \in C^2(int(\Omega))$ satisfying*

$$\Delta u + \langle X, \nabla u\rangle \geqslant 0,$$

where X is a bounded vector field, $x_0 \in \partial\Omega$ is a point where

$$u(x) < u(x_0) \;\; \forall x \in \Omega,$$

u is continuous at x_0, and Ω satisfies the interior sphere condition at x_0, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

if this outward normal derivative exists.

5.4 Proof of Eigenvalue Estimates and Rigidity

In this section we will give the proof of the four first results announced in the introduction and for this we will use all tools presented in the preliminaries.

Proof of Theorem 5.1. Let u be the solution of the following problem

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ u|_{\partial M} = z, \end{cases}$$

where z is a first eigenfunction corresponding to λ_1, that is, z satisfies $\overline{\Delta}_f z + \lambda_1 z = 0$. Set $h = \frac{\partial u}{\partial \nu}\big|_{\partial M}$, then we have from the Rayleigh inequality that (cf.[30])

$$p_1 \leqslant \frac{\int_{\partial M} h^2}{\int_M |\nabla u|^2} \tag{5.14}$$

and

$$p_1 \leqslant \frac{\int_M |\nabla u|^2}{\int_{\partial M} z^2} \tag{5.15}$$

Notice that (5.15) it is the variational principle, and (5.14) it is obtained as follows,

$$\begin{aligned} p_1 \leqslant \frac{\int_M |\nabla u|^2}{\int_{\partial M} z^2} &= \frac{-\int_M u\Delta_f u + \int_{\partial M} u\langle \nabla u, \nu\rangle}{\int_{\partial M} z^2} \\ &= \frac{\int_M |\nabla u|^2}{\int_{\partial M} z^2} \cdot \frac{\int_{\partial M} u\langle \nabla u, \nu\rangle}{\int_M |\nabla u|^2} \\ &= \frac{1}{\int_{\partial M} z^2} \cdot \frac{\left(\int_{\partial M} u\langle \nabla u, \nu\rangle\right)^2}{\int_M |\nabla u|^2} \\ &\leqslant \frac{\int_{\partial M} z^2}{\int_{\partial M} z^2} \cdot \frac{\int_{\partial M}\langle \nabla u, \nu\rangle^2}{\int_M |\nabla u|^2} \\ &= \frac{\int_{\partial M} h^2}{\int_M |\nabla u|^2}, \end{aligned}$$

which gives

$$p_1^2 \leqslant \frac{\int_{\partial M} h^2}{\int_{\partial M} z^2}. \tag{5.16}$$

It then follows by substituting u into the equation (5.11), we obtain

$$\begin{aligned} 0 \geqslant \int_M \frac{k-1}{k}[(\Delta_f u)^2 - \mathrm{Ric}_f^k(\nabla u, \nabla u)] &\geqslant \\ &\geqslant \int_{\partial M} [nH_f h^2 + 2h\overline{\Delta}_f z + II(\overline{\nabla} z, \overline{\nabla} z)] \\ &\geqslant \int_{\partial M} [(k-1)ch^2 - 2h\lambda_1 z + c|\overline{\nabla} z|^2]. \end{aligned} \tag{5.17}$$

Note that, by Green's formula,

$$\int_{\partial M} |\overline{\nabla} z|^2 = \int_{\partial M} \langle \overline{\nabla} z, \overline{\nabla} z \rangle = -\int_{\partial M} z \overline{\Delta}_f z = \lambda_1 \int_{\partial M} z^2.$$

Putting this expression in (5.17) we have that

$$\begin{aligned}
0 &\geqslant (k-1)c \int_{\partial M} h^2 - 2\lambda_1 \int_{\partial M} hz + c\lambda_1 \int_{\partial M} z^2 \\
&\geqslant (k-1)c \int_{\partial M} h^2 - 2\lambda_1 \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}} + c\lambda_1 \int_{\partial M} z^2 \\
&= \frac{(k-1)c^2 - \lambda_1}{c} \int_{\partial M} h^2 + \left[\sqrt{\frac{\lambda_1}{c}} \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} - \sqrt{c\lambda_1} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}} \right]^2,
\end{aligned}$$

from where

$$\frac{\sqrt{\lambda_1 - (k-1)c^2}}{\sqrt{c}} \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} \geqslant \sqrt{\frac{\lambda_1}{c}} \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} - \sqrt{c\lambda_1} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}}$$

and

$$\frac{\sqrt{\lambda_1} - \sqrt{\lambda_1 - (k-1)c^2}}{\sqrt{c}} \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} \leqslant \sqrt{c\lambda_1} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}},$$

that is,

$$\begin{aligned}
\left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} &\leqslant \frac{c\sqrt{\lambda_1}}{\sqrt{\lambda_1} - \sqrt{\lambda_1 - (k-1)c^2}} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}} \\
&= \frac{\sqrt{\lambda_1}}{(k-1)c} (\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}) \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Using (5.16), we obtain

$$p_1 \leqslant \frac{\sqrt{\lambda_1}}{(k-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2} \right).$$

Now, assume that

$$p_1 = \frac{\sqrt{\lambda_1}}{(k-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2} \right).$$

So, we also have that

$$\left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} = \frac{\sqrt{\lambda_1}}{(k-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2} \right) \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}}$$

and all inequalities above become equality. Thus $h = \alpha z$ and

$$\alpha = \frac{\left(\alpha^2 \int_{\partial M} z^2\right)^{\frac{1}{2}}}{\left(\int_{\partial M} z^2\right)^{\frac{1}{2}}} = \frac{\sqrt{\lambda_1}}{(k-1)c}\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}\right),$$

that is,

$$h = \frac{\sqrt{\lambda_1}}{(k-1)c}(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2})z.$$

Furthermore we infer, by Proposition 5.6, that Hess$u = 0$. Now, on the boundary ∂M, we can write

$$\begin{aligned}\nabla u &= (\nabla u)^{\top} + (\nabla u)^{\perp} \\ &= (\nabla u)^{\top} + \langle \nabla u, \nu \rangle \nu,\end{aligned}$$

where $(\nabla u)^{\top}$ is tangent to ∂M and $(\nabla u)^{\perp}$ is normal to ∂M. Then, take a local orthonormal fields $\{e_i\}_{i=1}^n$ tangent to ∂M. We obtain

$$\begin{aligned}0 &= \sum_{i=1}^n \mathrm{Hess}u(e_i, e_i) = \sum_{i=1}^n \langle \nabla_{e_i} \nabla u, e_i \rangle \\ &= \sum_{i=1}^n \langle \nabla_{e_i} [(\nabla u)^{\top} + \langle \nabla u, \nu \rangle \nu], e_i \rangle \\ &= \sum_{i=1}^n \langle \nabla_{e_i} (\nabla u)^{\top} + \langle \nabla u, \nu \rangle \nabla_{e_i} \nu + e_i(\langle \nabla u, \nu \rangle) \nu, e_i \rangle \\ &= \overline{\Delta} z + \sum_{i=1}^n \langle \nabla u, \nu \rangle II(e_i, e_i) \\ &= \overline{\Delta} z + nHh \\ &= \overline{\Delta}_f z - f_\nu h + nHh \\ &= \overline{\Delta}_f z + nH_f h \\ &= -\lambda_1 z + c(k-1)h \\ &= -\lambda_1 z + c(k-1)\frac{\sqrt{\lambda_1}}{(k-1)c}(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2})z,\end{aligned}$$

from where

$$\lambda_1 = (k-1)c^2.$$

Therefore, follow by Proposition 5.7, that M is isometric to an $(n+1)$-dimensional Euclidean ball of radius $\frac{1}{c}$, f is constant and so $k = n+1$. The converse follows the ideas of the Riemannian case.

Proof of Theorem 5.2. Let w be an eigenfunction corresponding to the first eigenvalue q_1 of problem (5.2), that is

$$\begin{cases} \Delta_f^2 w = 0 & \text{in } M, \\ w = \Delta_f w - q_1 \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{5.18}$$

Set $\eta = \frac{\partial w}{\partial \nu}|_{\partial M}$; then by divergence theorem we obtain

$$\begin{aligned}\int_M (\Delta_f w)^2 &= -\int_M \langle \nabla(\Delta_f w), \nabla w\rangle + \int_{\partial M} \Delta_f w \, \langle \nabla w, \nu\rangle \\ &= \int_M w \, \Delta_f(\Delta_f w) - \int_{\partial M} w \, \langle \nabla(\Delta_f w), \nu\rangle + \int_{\partial M} \Delta_f w \, \langle \nabla w, \nu\rangle \\ &= q_1 \int_{\partial M} \eta^2,\end{aligned}$$

that is,

$$q_1 = \frac{\int_M (\Delta_f w)^2}{\int_{\partial M} \eta^2}.$$

Substituting w in (5.11), and noting that $w|_{\partial M} = z$, we have

$$\begin{aligned}\frac{k-1}{k} \int_M (\Delta_f w)^2 &\geqslant \int_M \mathrm{Ric}_f^k(\nabla w, \nabla w) + \int_{\partial M} n H_f \eta^2 \\ &\geqslant \frac{(k-1)nc}{k} \int_{\partial M} \eta^2,\end{aligned}$$

from where $q_1 \geqslant nc$, as we desired.

Assume now that $q_1 = nc$, then the inequalities above become equalities and consequently $H_f = \frac{k-1}{k} c$. Furthermore, we have equality in the Proposition 5.6, thus $\mathrm{Hess} w = \frac{\Delta w}{n+1}\langle \, , \rangle$ and $\Delta_f w = \frac{k}{n+1} \Delta w$.

Take an orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}\}$ on M such that when restricted to ∂M $e_{n+1} = \nu$. Since $w|_{\partial M} = 0$ we have

$$\begin{aligned}e_i(\eta) &= e_i \langle \nabla w, \nu\rangle \\ &= \langle \nabla_{e_i} \nabla w, \nu\rangle + \langle \nabla w, \nabla_{e_i} \nu\rangle \\ &= \mathrm{Hess} w(e_i, \nu) + II((\nabla w)^\top, e_i) = 0,\end{aligned}$$

that is, $\eta = \rho =$ constant, and so $(\Delta_f w)|_{\partial M} = q_1 \eta = nc\rho$ is also a constant. Using the fact that $\Delta_f w$ is a f-harmonic function on M, we conclude by maximum principle that $\Delta_f w$ is constant on M. Since $\Delta_f w = \frac{k}{n+1} \Delta w$, then w satisfies

$$\begin{cases} \mathrm{Hess} w = \frac{\Delta_f w}{k} \langle \, , \rangle & \text{in} \quad M, \\ w|_{\partial M} = 0. \end{cases}$$

Thus, by Lema 3 in [48], we conclude that M is isometric to a ball in $\mathbb{R}^{n+1}$ of radius c^{-1}. Now, using the hessian of w is possible see that $w = \frac{\lambda}{2} r^2 + C$, where $\lambda = \frac{\Delta_f w}{k}$ and r is the distance function from its minimal point, see [48] for more details for this technique.

Lastly, we will show that f is constant. In fact, if $k > n+1$, then $\langle \nabla f, \nabla w\rangle$ is constant and so $f = -(k-n-1)\ln r + C$. It is a contradiction, since f is a smooth function.

Proof of Theorem 5.3. Now, let w be the solution of the following Laplace equation

$$\begin{cases} \Delta_f w = 1 & \text{in } M, \\ w|_{\partial M} = 0. \end{cases} \tag{5.19}$$

Follows from Rayleigh characterization of q_1 that

$$q_1 \leqslant \frac{\int_M (\Delta_f w)^2}{\int_{\partial M} \eta^2} = \frac{V}{\int_{\partial M} \eta^2}, \tag{5.20}$$

where $\eta = \frac{\partial w}{\partial \nu}\big|_{\partial M}$. Integrating $\Delta_f w = 1$ on M and using the divergence theorem, it gives

$$V = \int_{\partial M} \eta.$$

Hence we infer from Schwarz inequality that

$$V^2 \leqslant A \int_{\partial M} \eta^2. \tag{5.21}$$

Consequently,

$$q_1 \leqslant \frac{V}{\int_{\partial M} \eta^2} \leqslant \frac{V}{V^2/A} = \frac{A}{V}.$$

Assume now that $\mathrm{Ric}_f^k \geqslant 0$, $H_f(x_0) \geqslant \frac{(k-1)A}{k\,n\,V}$ for some $x_0 \in \partial M$ and $q_1 = \frac{A}{V}$. In this case (5.21) become a equality and so $\eta = \frac{V}{A}$ is a constant. Consider the function ϕ on M given by

$$\phi = \frac{1}{2}|\nabla w|^2 - \frac{w}{k}.$$

Using the Bochner formula (5.32), $\Delta_f w = 1$, the Proposition 5.6 and that $\mathrm{Ric}_f^k \geqslant 0$, we have that

$$\begin{aligned} \frac{1}{2}\Delta_f \phi &= |\mathrm{Hess} w|^2 + \langle \nabla w, \nabla(\Delta_f w)\rangle + \mathrm{Ric}_f(\nabla w, \nabla w) - \frac{1}{n+1} \\ &\geqslant \frac{1}{k}(\Delta_f w)^2 - \frac{1}{k} = 0. \end{aligned} \tag{5.22}$$

Thus ϕ is f-subharmonic. Observe that $\phi = \frac{1}{2}\left(\frac{V}{A}\right)^2$ on the boundary. In fact, if we write $\nabla w = (\nabla w)^\top + (\nabla w)^\perp$, where $(\nabla w)^\top$ is tangent to ∂M and $(\nabla w)^\perp$ is normal to ∂M, and since $w|_{\partial M} = 0$, it follows that $\nabla w = (\nabla w)^\perp = C\nu$ on ∂M. On the other hand,

$$1 = \Delta_f w = q_1 \langle \nabla w, \nu\rangle = \frac{A}{V} C \;\text{ implies }\; C = \frac{V}{A} \quad \text{and} \quad |\nabla w| = \frac{V}{A}.$$

Therefore $\phi = \frac{1}{2}\left(\frac{V}{A}\right)^2$ on the boundary, and so we conclude by Proposition 5.8 that either

$$\phi = \frac{1}{2}\left(\frac{V}{A}\right)^2 \quad \text{in } M \tag{5.23}$$

or

$$\frac{\partial \phi}{\partial \nu}(y) > 0, \quad \forall\, y \in \partial M. \tag{5.24}$$

From $w|_{\partial M} = 0$, we have

$$
\begin{aligned}
1 = (\Delta_f w)|_{\partial M} &= nH\eta + \mathrm{Hess}w(\nu,\nu) - \frac{V}{A}\langle \nabla f, \nu\rangle \\
&= \frac{nV}{A}\left(H_f + \frac{\langle \nabla f, \nu\rangle}{n}\right) + \mathrm{Hess}w(\nu,\nu) - \frac{V}{A}\langle \nabla f, \nu\rangle \\
&= \frac{nV}{A}H_f + \mathrm{Hess}w(\nu,\nu).
\end{aligned}
$$

Hence it holds on ∂M that

$$
\begin{aligned}
\frac{\partial \phi}{\partial \nu} &= \frac{V}{A}\mathrm{Hess}w(\nu,\nu) - \frac{V}{k\,A} \\
&= \frac{V}{A}\left(1 - \frac{nV}{A}H_f\right) - \frac{V}{k\,A} \\
&= n\frac{V}{A}\left(\frac{k-1}{k\,n} - H_f\frac{V}{A}\right),
\end{aligned}
$$

which shows that (5.24) is not true since $H_f(x_0) \geqslant \frac{(k-1)A}{k\,n\,V}$. Therefore ϕ is constant on M. Since the f-Laplacian of ϕ vanishes, we infer that equality must hold in (5.22) and that give us equality in the Proposition 5.6, and consequently $1 = \Delta_f w = \frac{k}{n+1}\Delta w$ and $\mathrm{Hess}w = \frac{\Delta w}{n+1}\langle\, , \rangle$. The remainder of the proof follows a similar arguments as in proof of Theorem 5.2.

Proof of Theorem 5.4. Let w be an eigenfunction corresponding to the first eigenvalue q_1 of the problem (5.3):

$$
\begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \frac{\partial^2 u}{\partial \nu^2} - q\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{5.25}
$$

Observe that w is not a constant. Otherwise, we would conclude from $w|_{\partial M} = 0$ that $w \equiv 0$. Set $\eta = \frac{\partial w}{\partial \nu}|_{\partial M}$; then $\eta \neq 0$. In fact, if $\eta = 0$ then

$$
w|_{\partial M} = (\nabla w)|_{\partial M} = \frac{\partial^2 w}{\partial \nu^2} = 0
$$

this implies, by (5.4), that $(\Delta_f w)|_{\partial M} = 0$ and so $\Delta_f w = 0$ on M by the maximum principal, which in turn implies that $w = 0$. This is a contradiction.

Since $w|_{\partial M} = 0$, we have by the divergence theorem that

$$
\int_M \langle \nabla w, \nabla(\Delta_f w)\rangle = -\int_M w\Delta_f^2 w = 0, \tag{5.26}
$$

hence

$$
\int_{\partial M} \Delta_f w\,\frac{\partial w}{\partial \nu} = \int_M \langle \nabla(\Delta_f w), \nabla w\rangle + \int_M (\Delta_f w)^2 = \int_M (\Delta_f w)^2. \tag{5.27}
$$

Since $w|_{\partial M} = 0$, we have $\nabla w = \frac{\partial w}{\partial \nu}\nu$ and

$$
\begin{aligned}
(\Delta_f w)|_{\partial M} &= \frac{\partial^2 w}{\partial \nu^2} + nH\frac{\partial w}{\partial \nu} - \langle \nabla f, \nabla w\rangle \\
&= q_1\frac{\partial w}{\partial \nu} + nH_f\frac{\partial w}{\partial \nu} + \langle \nabla f, \nu\rangle\frac{\partial w}{\partial \nu} - \langle \nabla f, \nu\rangle\frac{\partial w}{\partial \nu} \\
&= q_1\frac{\partial w}{\partial \nu} + nH_f\frac{\partial w}{\partial \nu}
\end{aligned} \tag{5.28}
$$

using (5.27) and (5.28) we obtain that

$$q_1 = \frac{\int_M (\Delta_f w)^2 - n\int_{\partial M} H_f\eta^2}{\int_{\partial M}\eta^2}.$$

On the other hand, substituting w into (5.11), we obtain

$$\begin{aligned}\frac{k-1}{k}\int_M (\Delta_f w)^2 &= \int_M \mathrm{Ric}_f^k(\nabla w, \nabla w) + \int_{\partial M} nH_f\eta^2 \\ &\geqslant \int_{\partial M} nH_f\eta^2,\end{aligned} \tag{5.29}$$

that is,

$$\int_M (\Delta_f w)^2 - \int_{\partial M} nH_f\eta^2 \geqslant \frac{n}{k-1}\int_{\partial M} H_f\eta^2 \geqslant c\int_{\partial M}\eta^2.$$

By expression for q_1 and estimate above, we obtain the desired estimate

$$q_1 \geqslant c. \tag{5.30}$$

Assume now that $q_1 = c$. So all inequalities in (5.29) become equalities. Thus, by Proposition 5.6, we have that

$$\mathrm{Hess} w = \frac{\Delta w}{n+1}\langle\,,\rangle \qquad \text{and} \qquad \Delta_f w = -\frac{k}{k-n-1}\langle\nabla f, \nabla w\rangle. \tag{5.31}$$

Choice an orthonormal frame $\{e_1, \ldots, e_n\}$ on M so that restricted to ∂M, $e_n = \nu$. On the other side, to $i = 1, \ldots, n-1$, using that $w|_{\partial M} = 0$, we obtain

$$\begin{aligned} 0 = \mathrm{Hess} w(e_i, e_n) &= e_i e_n(w) - \nabla_{e_i} e_n(w) \\ &= e_i(\eta) - \langle\nabla_{e_i} e_n, e_n\rangle\eta = e_i(\eta),\end{aligned}$$

follow that $\eta = b_0 = const$. Since (5.30) takes equality and η is constant, we conclude that $H_f = \frac{k-1}{n}c$, which implies from (5.28) that $(\Delta_f w)|_{\partial M} = kcb_0$, therefore, by maximum principle $\Delta_f w$ is constant on M which implies from (5.31) that Δw is constant on M. The remainder of the proof follows a similar arguments as in proof of Theorem 5.2.

5.5 Sharp Estimate of the Stekloff's Eigenvalue for Surfaces

Recall the Bochner type formula for weighted Riemannian manifold, which says: Any smooth function u on M holds that

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess} u|^2 + \langle\nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \tag{5.32}$$

An immediate consequence of the Bochner type formula is the result below, however we believe that this is not a sharp estimate.

Theorem 5.9 *Let M^{n+1}, $n \geqslant 2$ be a compact weighted Riemannian manifold with boundary ∂M. Assume that $\mathrm{Ric}_f \geqslant 0, H_f \geqslant 0$ and that the second fundamental form satisfies $II \geqslant cI$ on ∂M, $c > 0$. Then*

$$p_1 > \frac{c}{2}.$$

Proof. Set $h = \frac{\partial u}{\partial \nu}$, and $z = u|_{\partial M}$ where u is solution of problem (5.1). We have $p_1 z = p_1 u = h$, thus $p_1 \overline{\nabla} z = \overline{\nabla} h$. By (5.10), we have

$$\begin{aligned}
0 > -\int_M |\mathrm{Hess} u|^2 &\geqslant \int_M [(\Delta_f u)^2 - |\mathrm{Hess} u|^2 - \mathrm{Ric}_f(\nabla u, \nabla u)] \\
&= \int_{\partial M} \left[n H_f h^2 + 2h\overline{\Delta}_f z + II(\overline{\nabla} z, \overline{\nabla} z) \right] \\
&\geqslant -2\int_{\partial M} \langle \overline{\nabla} h, \overline{\nabla} z \rangle + c\int_{\partial M} |\overline{\nabla} z|^2 \\
&\geqslant -2p_1 \int_{\partial M} |\overline{\nabla} z|^2 + c\int_{\partial M} |\overline{\nabla} z|^2
\end{aligned}$$

Note that

$$\int_{\partial M} |\overline{\nabla} z|^2 > 0.$$

Otherwise z is constant on the Boundary and hence f is constant on M which is a contradiction. Thus $p_1 > \frac{c}{2}$. ∎

Below we present the proof of the sharp estimate of the non-zero first Stekloff eigenvalue on surfaces. The technique was introduced by Escobar in [15], and just allows us to attack this problem in context of surfaces.

Proof of Theorem 5.5. Let ϕ be a non-constant eigenfunction for the Stekloff problem (5.1). Consider the function $v = \frac{1}{2}|\nabla \phi|^2$, then by (5.32)

$$\Delta_f v = |\mathrm{Hess}\phi|^2 + \langle \nabla \phi, \nabla(\Delta_f \phi) \rangle + \mathrm{Ric}_f(\nabla \phi, \nabla \phi).$$

Since ϕ is a f-harmonic function and $\mathrm{Ric}_f \geqslant 0$ we find that

$$\Delta_f v = |\mathrm{Hess}\phi|^2 + \mathrm{Ric}_f(\nabla \phi, \nabla \phi) \geqslant 0. \tag{5.33}$$

Therefore the maximum of v is achieved at some point $P \in \partial M$. The Proposition 5.8 implies that $(\partial v/\partial \eta)(P) > 0$ or v is identically constant.

Let's assume $(\partial v/\partial \eta)(P) > 0$ and let (t, x) be Fermi coordinates around the point P, that is, x represents a point on the curve ∂M and t represents the distance to the boundary point x. The metric has the form

$$ds^2 = dt^2 + h^2(t, x)dx^2, \tag{5.34}$$

where $h(P) = 1$, $(\partial h/\partial x)(P) = 0$. Thus

$$|\nabla \phi|^2 = \left(\frac{\partial \phi}{\partial t}\right)^2 + h^{-2}\left(\frac{\partial \phi}{\partial x}\right)^2,$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial x \partial t} + h^{-2}\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} - h^{-3}\frac{\partial h}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)^2.$$

Evaluating at the point P we obtain

$$\frac{\partial v}{\partial x}(P) = \frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial x \partial t} + \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} = 0. \tag{5.35}$$

The f-Laplacian with respect to the metric given by (5.34) in Fermi coordinates (t, x) is

$$\Delta_f = \frac{\partial^2}{\partial t^2} + h^{-1}\frac{\partial h}{\partial t}\frac{\partial}{\partial t} + h^{-1}\frac{\partial}{\partial x}\left(h^{-1}\frac{\partial}{\partial x}\right) - \frac{\partial f}{\partial t}\frac{\partial}{\partial t} - h^{-2}\frac{\partial f}{\partial x}\frac{\partial}{\partial x}.$$

The geodesic curvature of ∂M can be calculated in terms of the function f and its first derivative as follows:

$$\begin{aligned} k_g &= -\left\langle \nabla_{\partial/\partial x}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right\rangle = -\left\langle \nabla_{\partial/\partial t}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle \\ &= -\frac{1}{2}\frac{\partial}{\partial t}\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle = -\frac{1}{2}\frac{\partial}{\partial t}(h^2) = -hh'. \end{aligned} \tag{5.36}$$

Hence at P we find that

$$0 = \Delta_f \phi = \frac{\partial^2 \phi}{\partial t^2} - k_g\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x} - \frac{\partial f}{\partial t}\frac{\partial \phi}{\partial t} - \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial x}. \tag{5.37}$$

Using the equality (5.36) we get that

$$\frac{\partial v}{\partial t}(P) = \frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial t \partial x} + k_g\left(\frac{\partial \phi}{\partial x}\right)^2. \tag{5.38}$$

Multiplying the equation (5.37) by $-\frac{\partial \phi}{\partial t}$ and adding with the equation (5.38) we obtain

$$\frac{\partial v}{\partial t}(P) = k_g|\nabla \phi|^2 - \frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial t \partial x} + \frac{\partial f}{\partial t}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial t}. \tag{5.39}$$

If $\frac{\partial \phi}{\partial x}(P) \neq 0$, the equation (5.35) and the boundary condition yields

$$\frac{\partial^2 \phi}{\partial x^2}(P) = p_1\frac{\partial \phi}{\partial t}(P). \tag{5.40}$$

Therefore the equation (5.39) can be re-written using the boundary condition as

$$\frac{\partial v}{\partial t}(P) = (k_g - p_1)|\nabla \phi|^2 + p_1\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial t \partial x} + \frac{\partial f}{\partial t}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial t}. \tag{5.41}$$

Notice that by (5.35) we obtain, using (5.40),

$$0 = \frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial x \partial t} + \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial x \partial t} + p_1\frac{\partial \phi}{\partial x}\right), \tag{5.42}$$

that is,

$$p_1 \frac{\partial \phi}{\partial x} = -\frac{\partial^2 \phi}{\partial x \partial t}.$$

Thus (5.41) becomes

$$\frac{\partial v}{\partial t}(P) = (k_g - p_1)|\nabla \phi|^2 + \frac{\partial f}{\partial t}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial t}$$

and we write

$$\frac{\partial v}{\partial t}(P) = (k_g - p_1)|\nabla \phi|^2 + \frac{\partial \phi}{\partial t}\langle \nabla \phi, \nabla f \rangle.$$

Since $f|_{\partial M}$ is constant, so $\frac{\partial f}{\partial x}(P) = 0$, and using that $\frac{\partial f}{\partial t} \leqslant 0$

$$\begin{aligned}\frac{\partial v}{\partial t}(P) &= (k_g - p_1)|\nabla \phi|^2 + \frac{\partial f}{\partial t}\left(\frac{\partial \phi}{\partial t}\right)^2 \\ &\geqslant \left(k_g + \frac{\partial f}{\partial t} - p_1\right)|\nabla \phi|^2,\end{aligned}$$

hence

$$(k_g + \frac{\partial f}{\partial t} - p_1)|\nabla \phi|^2 < 0, \tag{5.43}$$

and $p_1 > k_g + \frac{\partial f}{\partial t} = k_g - f_\nu \geqslant c$.

Now we assume that $\frac{\partial \phi}{\partial x}(P) = 0$. A straighforward calculation yields

$$\frac{\partial^2 v}{\partial x^2}(P) = \left(\frac{\partial^2 \phi}{\partial x \partial t}\right)^2 + \frac{\partial \phi}{\partial t}\frac{\partial^3 \phi}{\partial x^2 \partial t} + \left(\frac{\partial^2 \phi}{\partial x^2}\right)^2.$$

Using the boundary condition we get that

$$\frac{\partial^2 v}{\partial x^2}(P) = p_1^2 \phi \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 \leqslant 0. \tag{5.44}$$

Since $\frac{\partial \phi}{\partial x}(P) = 0$, the equation (5.39) implies that

$$\frac{\partial v}{\partial t}(P) = k_g \left(\frac{\partial \phi}{\partial t}\right)^2 + p_1 \phi \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial f}{\partial t}\left(\frac{\partial \phi}{\partial t}\right)^2 = \left(k_g + \frac{\partial f}{\partial t}\right) p_1^2 \phi^2 + p_1 \phi \frac{\partial^2 \phi}{\partial x^2}.$$

Thus

$$\left(k_g + \frac{\partial f}{\partial t}\right) p_1^3 \phi^2 + p_1^2 \phi \frac{\partial^2 \phi}{\partial x^2} < 0. \tag{5.45}$$

Adding inequality (5.44) with (5.45) we obtain

$$\left(\frac{\partial^2 \phi}{\partial x^2} + p_1^2 \phi\right)^2 + p_1^3 \left(k_g + \frac{\partial f}{\partial t} - p_1\right)\phi^2 < 0.$$

Hence

$$p_1 > k_g + \frac{\partial f}{\partial t} = k_g - f_\nu \geqslant c.$$

Let's assume that v is the constant function. Observe that $v \not\equiv 0$ because ϕ is non-constant. Since v is f-harmonic, inequality (5.33) implies that

$$\mathrm{Hess}\phi = 0 \quad \text{and} \quad \mathrm{Ric}_f(\nabla\phi, \nabla\phi) = 0, \quad \text{on} \quad M.$$

Now, using that $\Delta_f \phi = 0$, we obtain that $\langle \nabla\phi, \nabla f\rangle = 0$ and hereby $\mathrm{Hess}f(\nabla\phi, \nabla\phi) = 0$. Thus, the Gaussian curvature K of M vanishes. Moreover, using the structure of surfaces,

$$\nabla f = \lambda\, J(\nabla\phi), \tag{5.46}$$

where J is the anti-clockwise rotation of $\pi/2$ in the tangent plane.

Let $\{e_1, e_2\}$ be a local orthonormal frame field such that e_1 is tangent to ∂M and $e_2 = \eta$. So,

$$\begin{aligned} 0 = \mathrm{Hess}\phi(e_1, e_2) &= e_1 e_2(\phi) - \nabla_{e_1} e_2(\phi) \\ &= e_1(p_1\phi) - \langle \nabla_{e_1} e_2, e_1\rangle \phi_1 \\ &= (p_1 - k_g)\phi_1. \end{aligned}$$

Observe that if $\phi_1 = 0$ on ∂M, then ϕ =constant on ∂M and hence ϕ is a constant function on M which is a contradiction. Thus $p_1 = k_g$ except maybe when in the points where $\phi_1 = 0$. Since $\mathrm{Hess}\phi(e_1, e_1) = 0$ we have

$$\begin{aligned} 0 = \mathrm{Hess}\phi(e_1, e_1) &= e_1 e_1(\phi) - \nabla_{e_1} e_1(\phi) \\ &= e_1(e_1\phi) - \langle \nabla_{e_1} e_1, e_2\rangle e_2(\phi) \\ &= e_1(e_1\phi) + k_g p_1 \phi. \end{aligned}$$

Hence ϕ satisfies on the boundary a second order differential equation

$$\begin{gathered} \frac{d^2\phi}{dx^2} + k_g p_1 \phi = 0 \\ \phi(0) = \phi(\ell) \end{gathered} \tag{5.47}$$

where ℓ represents the length of ∂M. The function ϕ does not vanishes identically, thus $\phi_1 = 0$ except for a finite number of points. Therefore $p_1 = k_g$ except for a finite number of points and using the continuity of k_g, we conclude that $p_1 = k_g$ everywhere. Therefore,

$$p_1 = k_g - f_\nu + f_\nu \geqslant c,$$

and the equality between p_1 and c occurs if $k_g = k_0$ and $f_\nu = 0$. Using $K = 0$ and k_g is a positive constant, we conclude that M is an Euclidean ball.

Furthermore, by the identity (5.46), and using that $\mathrm{Hess}\varphi = 0$, we obtain

$$\begin{aligned} \nabla_X \nabla f &= X(\lambda) J(\nabla\varphi) + \lambda J(\nabla_X \nabla\varphi) \\ &= X(\lambda) J(\nabla\varphi), \end{aligned}$$

and note that

$$|\nabla f|^2 = \lambda^2 |\nabla\varphi|^2 = 2\lambda^2 v^2 \quad \Rightarrow \quad \lambda^2 = \frac{|\nabla f|^2}{2v^2}.$$

By other hand,

$$\langle J(\nabla_X \nabla\varphi), \nabla\varphi)\rangle = \langle \nabla_X J(\nabla\varphi), \nabla\varphi)\rangle = -\langle J(\nabla\varphi), \nabla_X \nabla\varphi\rangle.$$

Let $\{e_1, e_2\}$ be a orthonormal basis of the T_pM, then

$$\begin{aligned}\Delta f &= \sum_{i=1}^{2} \langle \nabla_{e_i} \nabla f, e_i\rangle \\ &= \sum_{i=1}^{2} \langle \nabla\lambda, e_i\rangle\langle J(\nabla\varphi), e_i\rangle \\ &= \langle \nabla\lambda, J(\nabla\varphi)\rangle,\end{aligned}$$

and, using the symmetry of the Hess$f$, we obtain

$$\langle \nabla\lambda, \nabla\varphi\rangle |J(\nabla\varphi)|^2 = \langle \nabla_{J(\nabla\varphi)} \nabla f, \nabla\varphi\rangle = \langle \nabla\lambda, \nabla\varphi\rangle\langle J(\nabla\varphi), \nabla\varphi\rangle = 0.$$

Therefore, $\nabla\lambda = \xi J(\nabla\varphi)$ and $\nabla_X \nabla f = \xi\langle J(\nabla\varphi), X\rangle J(\nabla\varphi)$. Consequently,

$$\mathrm{Hess} f = \xi J(\nabla\varphi) \otimes J(\nabla\varphi)$$

and

$$\Delta f = \xi |J(\nabla\varphi)|^2 = 2\xi v \quad \Rightarrow \quad \xi = \frac{\Delta f}{2v},$$

from where

$$\mathrm{Hess} f = \frac{\Delta f}{2v} \left(J(\nabla\phi) \otimes J(\nabla\phi) \right).$$

It easy to see, using that M is an Euclidian ball, that $\phi = x_i$, that is, ϕ is a coordinate function. Thus, using the expression of ϕ, f satisfies Hess$f = 0$ and as f is constant on the boundary, we have f constant.

CHAPTER 6

A WEIGHTED SPLITTING THEOREM

6.1 Introduction

Given $g \in C^{\infty}(M)$ we consider the closed Dirichlet problem

$$\Delta_f u + g(u) = 0. \tag{6.1}$$

A solution of that problem is a critical point of an energy functional, which we will denote by E_f. We say that a solution u is *stable* if the second variation of E_f is non-negative on $W_c^{1,2}(M)$, where

$$W_c^{1,2}(M) = \{u \in L^2(M); \frac{\partial u}{\partial x_i} \in L^2(M),\, i = 1, 2, \ldots, m = \dim(M)\}$$

with compact support in M, see [13] for a good overview about Sobolev's spaces.

We say that a weighted Riemannian manifold is *f-parabolic* if there exists no non-constant and bounded below function which is f-superharmonic.

In this chapter our aim is to prove the following weighted splitting theorem. It is read as follow:

Theorem 6.1 *Let M be a complete and non-compact weighted Riemannian manifold without boundary and satisfying $Ric_f \geqslant 0$. Assume that $u \in C^{\infty}(M)$ is a non-constant and stable solution of* (6.1)*.*

If either

(i) M is $f-$parabolic and $\nabla u \in L^{\infty}(M)$, or

(ii) the function $|\nabla u|$ satisfies

$$\int_{B_R} |\nabla u|^2 d\nu_f = o(R^2 \log R) \quad \text{as } R \to +\infty. \tag{6.2}$$

Then, $M = N \times \mathbb{R}$ with the product metric $g_M = g_N + dt^2$, for some complete, totally geodesic, f-parabolic hypersurface N. In particular, $Ric_f^N \geqslant 0$ if $m \geqslant 3$, and $M = \mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{R}$, with their flat metric, if $m = 2$. Moreover, u depends only on t, has no critical points, and writing $u = y(t)$ it holds $-y'' + ky' = g(y)$ where k is a constant.

Moreover, if (ii) is met,

$$\mathrm{Vol}_f(B_R^N) = o(R^2 \log R) \qquad \text{as } R \to +\infty. \tag{6.3}$$

$$\int_{-R}^{R} |y'(t)|^2 dt = o\left(\frac{R^2 \log R}{\mathrm{Vol}_f(B_R^N)}\right) \qquad \text{as } R \to +\infty. \tag{6.4}$$

This chapter is organized as follow; in section 2 we recall all concepts and equivalences that we use in the chapter; in section 3 we present some technical propositions that will auxiliary in the proofs of the principal results; in section 4 we dedicated it to proof of the Theorem 6.1.

6.2 Preliminaries

Throughout the chapter M will denote a connect weighted Riemannian manifold of dimension $m \geqslant 2$, without boundary. We briefly fix some notation. Having fixed an origin p_0, we set $r(x) = dist(x, p_o)$, and we write B_R for geodesic ball centered at p_o. If we need to emphasize the set under consideration, we will add a superscript symbol, so that, for instance, we will also write Ric_f^M and B_R^M. The Riemannian m-dimensional volume will be indicated with Vol, and the measure with density by $d\nu_f = e^{-f} d\mathrm{Vol}$. While will write $\mathcal{H}^{m-1}$ for the induced $(m-1)$-dimensional Hausdorff measure and $d\mathcal{H}_f^{m-1} = e^{-f} d\mathcal{H}^{m-1}$. We will use the symbol $\{\Omega_j\} \uparrow M$ for indicate a family $\{\Omega_j\}_{j \in \mathbb{N}}$ of relativity compact, open sets with smooth boundary and satisfying

$$\Omega_j \Subset \Omega_{j+1} \Subset M, \qquad M = \bigcup_{j=0}^{+\infty} \Omega_j,$$

where $A \Subset B$ means $\overline{A} \subseteq B$. Such a family will be called an exhaustion of M. Hereafter, we consider

$$g \in C^\infty(\mathbb{R}),$$

and a solution u on M of

$$\Delta_f u + g(u) = 0 \quad \text{ on } M. \tag{6.5}$$

We recall that u is characterized, on each open subset $U \Subset M$, as a critical point of the energy functional $E_f : W_c^{1,2}(M) \to \mathbb{R}$ given by

$$E_f(w) = \frac{1}{2} \int_M |\nabla w|^2 d\nu_f - \int_M G(w) d\nu_f, \quad \text{where } G(t) = \int_0^t g(s) ds, \tag{6.6}$$

with respect to compactly variation in U. Let J_f the Jacob operator of E_f at u, that is,

$$J_f \phi = -\Delta_f \phi - g'(u)\phi, \qquad \forall\, \phi \in C_c^\infty(M), \tag{6.7}$$

where $C_c^\infty(M)$ is the space of the smooth functions compactly supported in M.

Definition 3 *The function u solving* (6.5) *is said to be a stable solution if J_f is non-negative on $C_c^\infty(M)$, that is, if $(\phi, J_f\phi)_{L^2} \geqslant 0$, for all $\phi \in C_c^\infty(M)$. In other words,*

$$\int_M g'(u)\phi^2 d\nu_f \leqslant \int_M |\nabla\phi|^2 d\nu_f, \qquad \text{for all } \phi \in C_c^\infty(M). \tag{6.8}$$

By density, we can replace $C_c^\infty(M)$ in (6.8) with $Lip_c(M)$. By a simple adaptation of the [21, Theorem 1], the stability of u turns out to be equivalent to the existence of a positive $w \in C^\infty(M)$ solving $\Delta_f w + g'(u)w = 0$ on M.

Let Ω be an open set on M and K be a compact set in Ω. We call the pair (K, Ω) of a f-capacitor and define the f-capacity $\mathrm{cap}_f(K, \Omega)$ by

$$\mathrm{cap}_f(K, \Omega) = \inf_{\phi \in \mathcal{L}(K,\Omega)} \int_\Omega |\nabla\phi|^2 d\nu_f, \tag{6.9}$$

where $\mathcal{L}(K, \Omega)$ is a set of Lipschitz functions ϕ on M with a compact support in $\overline{\Omega}$ such that $0 \leqslant \phi \leqslant 1$ and $\phi|_K = 1$.

For an open precompact set $K \subset \Omega$, we define its f-capacity by

$$\mathrm{cap}_f(K, \Omega) := \mathrm{cap}_f(\overline{K}, \Omega).$$

In case that $\Omega = M$, we write $\mathrm{cap}_f(K)$ for $\mathrm{cap}_f(K, \Omega)$. It is obvious from the definition that the set $\mathcal{L}(K, \Omega)$ increases on expansion of Ω (and on shrinking of K). Therefore, the capacity $\mathrm{cap}_f(K, \Omega)$ decreases on expanding of Ω (and on shrinking of K). In particular, one can prove that, for any exhaustion sequence $\{\mathcal{E}_k\}$

$$\mathrm{cap}_f(K) := \lim_{k\to\infty} \mathrm{cap}_f(K, \mathcal{E}_k).$$

Definition 4 *A weighted Riemnnian manifold is f-parabolic if there exists no non-constant bounded below f-superharmonic function u, that is, if $\Delta_f u \leqslant 0$ and $u \geqslant k$, for some $k \in \mathbb{R}$, then u is constant.*

Hence, we have the following characterization of f-parabolicity. For the proof see [25].

Proposition 6.2 *Let M be a complete weighted Riemannian manifold. Then, the following are equivalent:*

1. *M is f-parabolic.*

2. *$cap_f(K) = 0$ for some (then any) compact set $K \subset M$.*

The following criterion of f-parabolicity is well known, for more details see for instance [22, Proposition 3.4].

Proposition 6.3 *Let p_o be a fixed point in a weighted Riemannian manifold M and let*

$$L(r) = \int_{\partial B(p_o,r)} d\mathcal{H}_f^{m-1} \qquad \text{and} \qquad V(r) = \int_{B(p_o,r)} d\nu_f.$$

If

$$\int_1^\infty \frac{dr}{L(r)} = +\infty \qquad \text{or} \qquad \int_1^\infty \frac{rdr}{V(r)} = +\infty,$$

then M is f-parabolic.

6.3 Technical Computations

We start this section with a Picone type identity in a weighted Riemannian manifold.

Lemma 2 *Let $\Omega \subseteq M$ be a domain with C^3 boundary (possibly empty) and let $u \in C^3(\overline{\Omega})$ be a solution of $-\Delta_f u = g(u)$ on Ω. Let $w \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of $\Delta_f w + g'(u)w \leqslant 0$ such that $w > 0$ on Ω. Then the following inequality holds true: for every $\varepsilon > 0$ and for every $\phi \in Lip_c(M)$,*

$$\begin{aligned}\int_{\partial\Omega} \frac{\phi^2}{w+\varepsilon}(\partial_\nu w) d\mathcal{H}_f^{m-1} \leqslant \int_\Omega |\nabla\phi|^2 d\nu_f - \int_\Omega g'(u)\frac{w}{w+\varepsilon}\phi^2 d\nu_f \\ - \int_\Omega (w+\varepsilon)^2 \left|\nabla\left(\frac{\phi}{w+\varepsilon}\right)\right|^2 d\nu_f.\end{aligned} \tag{6.10}$$

Furthermore, if either $\Omega = M$ or $w > 0$ on $\overline{\Omega}$, one can also take $\varepsilon = 0$ inside the above inequality. The inequality is indeed an equality if w solves $\Delta_f w + g'(u)w = 0$ on Ω.

Proof. We integrate $\Delta_f w + g'(u)w \leqslant 0$ against the test function $\phi^2/(w+\varepsilon)$ to deduce

$$\begin{aligned}0 \leqslant -\int_\Omega (\Delta_f w + g'(u)w)\frac{\phi^2}{w+\varepsilon} d\nu_f = -\int_{\partial\Omega} \frac{\phi^2}{w+\varepsilon}(\partial_\nu w) d\mathcal{H}_f^{m-1} \\ + \int_\Omega \left\langle \nabla\left(\frac{\phi^2}{w+\varepsilon}\right), \nabla w\right\rangle d\nu_f - \int_\Omega \frac{g'(u)w\phi^2}{w+\varepsilon} d\nu_f\end{aligned} \tag{6.11}$$

Since

$$\left\langle \nabla\left(\frac{\phi^2}{w+\varepsilon}\right), \nabla w\right\rangle = 2\frac{\phi}{w+\varepsilon}\langle\nabla\phi, \nabla w\rangle - \frac{\phi^2}{(w+\varepsilon)^2}|\nabla w|^2,$$

using the identity

$$(w+\varepsilon)^2\left|\nabla\left(\frac{\phi}{w+\varepsilon}\right)\right|^2 = |\nabla\phi|^2 + \frac{\phi^2}{(w+\varepsilon)^2}|\nabla w|^2 - 2\frac{\phi}{w+\varepsilon}\langle\nabla w, \nabla\phi\rangle,$$

we infer that

$$\left\langle \nabla\left(\frac{\phi^2}{w+\varepsilon}\right), \nabla w\right\rangle = |\nabla\phi|^2 - (w+\varepsilon)^2\left|\nabla\left(\frac{\phi}{w+\varepsilon}\right)\right|^2. \tag{6.12}$$

Inserting (6.12) into (6.11) we get the desired (6.10). ■

Proposition 6.4 *In the above assumptions, for every $\varepsilon > 0$ the following integral inequality holds true:*

$$\begin{aligned}
&\int_\Omega [|\mathrm{Hess} u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u)]\frac{\phi^2 w}{w+\varepsilon} d\nu_f - \int_\Omega \phi^2 |\nabla|\nabla u||^2 d\nu_f \leqslant \qquad (6.13)\\
&\leqslant \int_{\partial\Omega} \frac{\phi^2}{w+\varepsilon}\left[w\partial_\nu\left(\frac{|\nabla u|^2}{2}\right) - |\nabla u|^2 \partial_\nu w\right] d\mathcal{H}_f^{m-1} +\\
&\quad + \varepsilon \int_\Omega \frac{\phi}{w+\varepsilon}\langle \nabla\phi, \nabla|\nabla u|^2\rangle d\nu_f - \frac{1}{2}\int_\Omega \phi^2 \left\langle \nabla|\nabla u|^2, \nabla\left(\frac{w}{w+\varepsilon}\right)\right\rangle d\nu_f +\\
&\quad + \int_\Omega |\nabla\phi|^2 |\nabla u|^2 d\nu_f - \int_\Omega (w+\varepsilon)^2 \left|\nabla\left(\frac{\phi|\nabla u|}{w+\varepsilon}\right)\right|^2 d\nu_f.
\end{aligned}$$

Furthermore, if either $\Omega = M$ or $w > 0$ on $\overline{\Omega}$, one can also take $\varepsilon = 0$. The inequality is indeed an equality if $\Delta_f w + g'(u)w = 0$ on Ω.

Proof. We start with the Böchner formula

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess} u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u),$$

valid for each $u \in C^3(\overline{\Omega})$. Since u solves $-\Delta_f u = g(u)$, we get

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess} u|^2 - g'(u)|\nabla u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u). \qquad (6.14)$$

Integrating (6.14) on Ω against the test function $\psi = \phi^2 w/(w+\varepsilon)$ we deduce

$$\begin{aligned}
&\int_\Omega [|\mathrm{Hess} u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u)]\psi \, d\nu_f = \qquad (6.15)\\
&= \int_\Omega g'(u)|\nabla u|^2 \frac{\phi^2 w}{(w+\varepsilon)} d\nu_f + \frac{1}{2}\int_\Omega \frac{\phi^2 w}{(w+\varepsilon)}\Delta_f |\nabla u|^2 d\nu_f\\
&= \int_\Omega g'(u)|\nabla u|^2 \frac{\phi^2 w}{(w+\varepsilon)} d\nu_f + \frac{1}{2}\int_{\partial\Omega} \frac{w\phi^2}{w+\varepsilon}\partial_\nu |\nabla u|^2 d\mathcal{H}_f^{m-1}\\
&\quad - \frac{1}{2}\int_\Omega \left\langle \nabla\left(\frac{w\phi^2}{w+\varepsilon}\right), \nabla|\nabla u|^2\right\rangle d\nu_f\\
&= \int_\Omega g'(u)|\nabla u|^2 \frac{\phi^2 w}{(w+\varepsilon)} d\nu_f + \frac{1}{2}\int_{\partial\Omega} \frac{w\phi^2}{w+\varepsilon}\partial_\nu |\nabla u|^2 d\mathcal{H}_f^{m-1}\\
&\quad - \int_\Omega \frac{w\phi}{w+\varepsilon}\langle \nabla\phi, \nabla|\nabla u|^2\rangle d\nu_f\\
&\quad - \frac{1}{2}\int_\Omega \phi^2 \left\langle \nabla|\nabla u|^2, \nabla\left(\frac{w}{w+\varepsilon}\right)\right\rangle d\nu_f.
\end{aligned}$$

Next, we consider the inequality (6.10) with the test function $\phi|\nabla u| \in Lip_c(M)$:

$$\begin{aligned}\int_{\partial\Omega} |\nabla u|^2 \frac{\phi^2}{w+\varepsilon}(\partial_\nu w) d\mathcal{H}_f^{m-1} &\leqslant \int_\Omega |\nabla(\phi|\nabla u|)|^2 \, d\nu_f - \int_\Omega g'(u)\frac{w}{w+\varepsilon}\phi^2|\nabla u|^2 d\nu_f \\ &\quad - \int_\Omega (w+\varepsilon)^2 \left|\nabla\left(\frac{\phi|\nabla u|}{w+\varepsilon}\right)\right|^2 d\nu_f \\ &= \int_\Omega |\nabla\phi|^2|\nabla u|^2 d\nu_f + \int_\Omega \phi^2|\nabla|\nabla u||^2 d\nu_f \\ &\quad + 2\int_\Omega \phi|\nabla u|\langle\nabla\phi, \nabla|\nabla u|\rangle d\nu_f \\ &\quad - \int_\Omega g'(u)\frac{w}{w+\varepsilon}|\nabla\phi|^2\phi^2 d\nu_f \\ &\quad - \int_\Omega (w+\varepsilon)^2 \left|\nabla\left(\frac{\phi|\nabla u|}{w+\varepsilon}\right)\right|^2 d\nu_f.\end{aligned} \tag{6.16}$$

Recalling that $\nabla|\nabla u|^2 = 2|\nabla u|\nabla|\nabla u|$ weakly on M, summing up 6.15 and (6.16), putting together the terms of the same kind and rearranging we deduce (6.13) as desired. ∎

Corollary 6.1 *In the above assumptions, if it holds*

$$\liminf_{\varepsilon\to 0^+} \int_\Omega \phi^2 \langle \nabla|\nabla u|^2, \nabla\left(\frac{w}{w+\varepsilon}\right)\rangle d\nu_f \geqslant 0. \tag{6.17}$$

Then

$$\begin{aligned}&\int_\Omega \left[|\mathrm{Hess}u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u) - |\nabla|\nabla u||^2\right]\phi^2 d\nu_f + \\ &+ \liminf_{\varepsilon\to 0^+} \int_\Omega (w+\varepsilon)^2 \left|\nabla\left(\frac{\phi|\nabla u|}{w+\varepsilon}\right)\right|^2 d\nu_f \leqslant \int_\Omega |\nabla\phi|^2|\nabla u|^2 d\nu_f + \\ &+ \liminf_{\varepsilon\to 0^+} \int_{\partial\Omega} \frac{\phi^2}{w+\varepsilon}\left[w\partial_\nu\left(\frac{|\nabla u|^2}{2}\right) - |\nabla u|^2\partial_\nu w\right] d\mathcal{H}_f^{m-1}.\end{aligned} \tag{6.18}$$

Proof. It is an immediate consequence of the Proposition 6.4. ∎

The next result is known in the literature, we present it here for sake of completeness.

Proposition 6.5 *Let $u \in C^2(M)$, and let $p \in M$ be a point such that $\nabla u(p) \neq 0$. Then, denoting with $|A|^2$ the square norm of second fundamental form of the level set $\Sigma = \{u = u(p)\}$ in a neighborhood of p, it holds*

$$|\mathrm{Hess}u|^2 - |\nabla|\nabla u||^2 = |\nabla u|^2|A|^2 + |\nabla^T|\nabla u||^2,$$

where ∇^T is the tangential gradient on the level set Σ.

Proof. Fix a local orthonormal frame $\{e_i\}$ on Σ, and let $\nu = \nabla u/|\nabla u|$ be the normal vector. For every vector field $X \in \mathfrak{X}(M)$,

$$\mathrm{Hess}u(\nu, X) = \frac{1}{|\nabla u|}\mathrm{Hess}u(\nabla u, X) = \frac{1}{2|\nabla u|}\langle\nabla|\nabla u|^2, X\rangle = \langle\nabla|\nabla u|, X\rangle.$$

Moreover, for a level set

$$A = -\frac{\mathrm{Hess}u|_{T\Sigma\times T\Sigma}}{|\nabla u|},$$

we have

$$\begin{aligned}
|\mathrm{Hess}u|^2 &= \sum_{i,j}(\mathrm{Hess}u(e_i,e_j))^2 + 2\sum_j(\mathrm{Hess}u(\nu,e_j))^2 + (\mathrm{Hess}u(\nu,\nu))^2\\
&= |\nabla u|^2|A|^2 + 2\sum_j\langle\nabla|\nabla u|, e_j\rangle^2 + \langle\nabla|\nabla u|,\nu\rangle^2\\
&= |\nabla u|^2|A|^2 + |\nabla^T|\nabla u||^2 + |\nabla|\nabla u||^2,
\end{aligned}$$

proving the proposition. ■

6.4 Proof of the Theorem 6.1

Now we are ready to prove our main theorem.

Proof of Theorem 6.1. In our assumption, we consider the integral formula (6.13) with $\Omega = M$ and $\varepsilon = 0$. Since $\mathrm{Ric}_f \geqslant 0$ we deduce

$$\int_M \left[|\mathrm{Hess}u|^2 - |\nabla|\nabla u||^2\right]\phi^2 d\nu_f \leqslant \int_M |\nabla\phi|^2|\nabla u|^2 d\nu_f - \int_M w^2\left|\nabla\left(\nabla\frac{\phi|\nabla u|}{w}\right)\right|^2 d\nu_f. \tag{6.19}$$

Next, we rearrange the right hand side as follows: using the inequality

$$|X+Y|^2 \geqslant |X|^2 + |Y|^2 - 2|X||Y| \geqslant (1-\delta)|X|^2 + (1-\delta^{-1})|Y|^2,$$

valid for each $\delta > 0$, we obtain

$$w^2\left|\nabla\left(\frac{\phi|\nabla u|}{w}\right)\right|^2 = w^2\left|\frac{|\nabla u|\nabla\phi}{w} + \phi\nabla\left(\frac{|\nabla u|}{w}\right)\right|^2 \tag{6.20}$$

$$\geqslant (1-\delta^{-1})|\nabla u|^2|\nabla\phi|^2 + (1-\delta)\phi^2 w^2\left|\nabla\left(\frac{|\nabla u|}{w}\right)\right|^2. \tag{6.21}$$

Substituting in (6.19) yields

$$\begin{aligned}
\int_M \left[|\mathrm{Hess}u|^2 - |\nabla|\nabla u||^2\right]\phi^2 d\nu_f + (1-\delta)\int_M \phi^2 w^2\left|\nabla\left(\frac{|\nabla u|}{w}\right)\right|^2 d\nu_f&\\
\leqslant \frac{1}{\delta}\int_M |\nabla\phi|^2|\nabla u|^2 d\nu_f.&
\end{aligned}$$

Choose $\delta < 1$. We claim that, for suitable families $\{\phi_\alpha\}_{\alpha\in I\subseteq\mathbb{R}^+}$, it holds

$$\{\phi_\alpha\} \text{ is monotone increasing to } 1, \quad \lim_{\alpha\to+\infty}\int_M |\nabla\phi_\alpha|^2|\nabla u|^2 d\nu_f = 0. \tag{6.22}$$

Choose ϕ as follows, according to the case.

In the case **(i)**, fix $\Omega \Subset M$ with smooth boundary and let $\{\Omega_j\} \uparrow M$ be a smooth exhaustion with $\Omega \subset \Omega_1$. Choose $\phi = \phi_j \in Lip_0(M)$ to be identity 1 on Ω, 0 on $M \backslash \Omega_j$ and the f-harmonic capacitor on $\Omega_j \backslash \Omega$, that is, the solution of

$$\begin{cases} \Delta_f \phi_j = 0 & \text{on } \Omega_j \backslash \Omega \\ \phi_j = 1 & \text{on } \partial\Omega, \\ \phi_j = 0 & \text{on } \partial\Omega_j. \end{cases}$$

By comparison and since M is f-parabolic, $\{\phi_j\}$ is monotonically increasing and pointwise convergent to 1, and moreover

$$\int_{\Omega_j} |\nabla \phi_j|^2 |\nabla u|^2 d\nu_f \leqslant |\nabla u|^2_{L^\infty} \mathrm{cap}_f(\Omega, \Omega_j) \to |\nabla u|^2_{L^\infty} \mathrm{cap}_f(\Omega) = 0,$$

the last equality follows of Proposition 6.2 since M is f-parabolic. This proves (6.22).

In the case (ii), we apply a logarithmic cut-off argument. For fixed $R > 0$, choose the following radial function $\phi(x) = \phi_R(r(x))$:

$$\phi_R(r) = \begin{cases} 1 & \text{if } r \leqslant \sqrt{R}, \\ 2 - 2\frac{\log r}{\log R} & \text{if } r \in [\sqrt{R}, R], \\ 0 & \text{if } r \geqslant R. \end{cases} \tag{6.23}$$

Note that

$$|\nabla \phi(x)|^2 = \frac{4}{r(x)^2 \log^2 R} \chi_{B_R \backslash B_{\sqrt{R}}}(x),$$

where χ_A is the characteristic function of a subset $A \subseteq M$. Choose R in such a ways that $\log R/2$ is an integer. Then

$$\begin{aligned} \int_M |\nabla \phi|^2 |\nabla u|^2 d\nu_f &= \int_{B_R \backslash B_{\sqrt{R}}} |\nabla \phi|^2 |\nabla u|^2 d\nu_f \\ &= \frac{4}{\log^2 R} \sum_{k=\log R/2}^{\log R - 1} \int_{B_{e^{k+1}} \backslash B_{e^k}} \frac{|\nabla u|^2}{r(x)^2} d\nu_f \\ &\leqslant \frac{4}{\log^2 R} \sum_{k=\log R/2}^{\log R} \frac{1}{e^{2k}} \int_{B_{e^{k+1}}} |\nabla u|^2 d\nu_f. \end{aligned} \tag{6.24}$$

By assumption

$$\int_{B_{e^{k+1}}} |\nabla u|^2 d\nu_f \leqslant (k+1) e^{2(k+1)} \delta(k)$$

for some $\delta(k)$ satisfying $\delta(k) \to 0$ as $k \to +\infty$. Without loss of generality, we can assume $\delta(k)$ to be decreasing as a function of k. Whence,

$$\begin{aligned}\frac{4}{\log^2 R}\sum_{k=\log R/2}^{\log R}\frac{1}{e^{2k}}\int_{B_{e^{k+1}}}|\nabla u|^2 d\rho &\leqslant \frac{8}{\log^2 R}\sum_{k=\log R/2}^{\log R}\frac{e^{2(k+1)}}{e^{2k}}(k+1)\delta(k) \\ &\leqslant \frac{8e^2}{\log^2 R}\delta(\log R/2)\sum_{k=0}^{\log R}(k+1) \\ &\leqslant \frac{C}{\log^2 R}\delta(\log R/2)\log^2 R \\ &= C\delta(\log R/2),\end{aligned} \tag{6.25}$$

for some constant $C > 0$. Combining (6.24) and 6.25 and letting $R \to +\infty$ we deduce (6.22). Therefore, in both the cases, we can infer from the integral formula (6.19) that

$$|\nabla u| = cw, \quad \text{for some } c \geqslant 0, \quad |\mathrm{Hess} u|^2 = |\nabla|\nabla u||^2, \quad \mathrm{Ric}_f(\nabla u, \nabla u) = 0. \tag{6.26}$$

Since u is non-constant by assumption, we have $c > 0$ and thus $|\nabla u| > 0$ on M. From Böchner formula, it holds

$$|\nabla u|\Delta_f|\nabla u| + |\nabla|\nabla u||^2 = \frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess} u|^2 - g'(u)|\nabla u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u)$$

on M. Using (6.26), we obtain that $\Delta_f|\nabla u| + g'(u)|\nabla u| = 0$ on M, hence $|\nabla u|$ (and so w) both solve the linearized equation $J_f v = 0$.

Now, the flow Φ of $\nu = \nabla u/|\nabla u|$ is well defined on M. Since M is complete and $|\nu| = 1$, Φ is defined on $M \times \mathbb{R}$. By (6.26) and Proposition 6.5, $|\nabla u|$ is constant on each connected component of a level set N, and N is totally geodesic. Therefore, in a local Darboux frame $\{e_j, \nu\}$ for the level surface N, we have that

$$\begin{cases} 0 = |II|^2 \text{ implies } \mathrm{Hess} u(e_i, e_j) = 0, \\ 0 = \langle \nabla|\nabla u|, e_j\rangle = \mathrm{Hess} u(\nu, e_j), \end{cases} \tag{6.27}$$

so the unique component of $\mathrm{Hess} u$ is that corresponding to the pair (ν, ν). Now we will prove that γ is a geodesic. Indeed, let $X \in \mathfrak{X}(M)$ be a vector field, we have that

$$\begin{aligned}\langle \nabla_{\gamma'}\gamma', X\rangle &= \frac{1}{|\nabla u|}\langle \nabla_{\nabla u}\left(\frac{\nabla u}{|\nabla u|}\right), X\rangle \\ &= \frac{1}{|\nabla u|^2}\langle \nabla_{\nabla u}\nabla u, X\rangle - \frac{1}{|\nabla u|^3}\langle \nabla u(|\nabla u|)\nabla u, X\rangle \\ &= \frac{1}{|\nabla u|^2}\mathrm{Hess} u(\nabla u, X) - \frac{1}{|\nabla u|^3}\langle \nabla|\nabla u|, \nabla u\rangle\langle \nabla u, X\rangle \\ &= \frac{1}{|\nabla u|}\mathrm{Hess} u(\nu, X) - \frac{1}{|\nabla u|}\langle \nabla|\nabla u|, \nu\rangle\langle \nu, X\rangle \\ &= \frac{1}{|\nabla u|}\mathrm{Hess} u(\nu, X) - \frac{1}{|\nabla u|}\mathrm{Hess} u(\nu, \nu)\langle \nu, X\rangle = 0,\end{aligned}$$

where the last line follows from (6.27). So, $\nabla_{\gamma'}\gamma' = 0$ and γ is a geodesic as desired.

Following the arguments in the proof of [40, Theorem 9.3], we will prove the topological splitting. Since $|\nabla u|$ is constant on level sets of u, $|\nabla u| = \beta(u)$ for some function β. Evaluating along curves $\Phi_t(x)$, since $u \circ \Phi_t$ is a local bijection we deduce that β is continuous.

Claim 6.6 *Φ_t moves level sets of u to level sets of u.*

Indeed, integrating $\frac{d}{ds}(u \circ \Phi_s) = |\nabla u| \circ \Phi_s = \beta(u \circ \Phi_s)$ we get

$$t = \int_{u(x)}^{u(\Phi_t(x))} \frac{d\xi}{\beta(\xi)},$$

thus $u(\Phi_t(x))$ is independent of x varying in a level set. As $\beta(\xi) > 0$, this also show that flow lines starting from a level set of u do not touch the same level set and we conclude the Claim.

Let N be a connected component of a level set of u.

Claim 6.7 *$\Phi|_{N\times\mathbb{R}}$ is surjective.*

In fact, since the flow of ν is through geodesics, for each $x \in N$, Φ_t coincides with the normal exponential map $\exp^{\perp}(t\nu(x))$. Moreover, since N is closed in M and M is complete, the normal exponential map is surjective because each geodesic from $x \in M$ to N minimizing $\mathrm{dist}(x, N)$ is perpendicular to N (by variational arguments).

Claim 6.8 *$\Phi|_{N\times\mathbb{R}}$ is injective.*

Suppose that $\Phi(x_1, t_1) = \Phi(x_2, t_2)$. Then, since Φ moves level sets to level sets, necessarily $t_1 = t_2 = t$. If by contradiction $x_1 \neq x_2$, two distinct flow lines of Φ_t would intersect at the point $\Phi_t(x_1) = \Phi_t(x_2)$, contradicting the fact that Φ_t is a diffeomorphism on M for every t, as desired.

Thus, we conclude that $\Phi : N \times \mathbb{R} \to M$ is a diffeomorphism. In particular, each level set $\Phi_t(N)$ is connected. This proves the topological part of the splitting.

To conclude the splitting, we will prove that Φ_t is an isometry for all t, that is, we will prove that ν is a Killing vector field. Indeed, we consider the Lie derivative of the metric in the direction of ν:

$$\begin{aligned}(L_\nu g_M)(X, Y) &= \langle \nabla_X \nu, Y\rangle + \langle X, \nabla_Y \nu\rangle \\ &= \frac{2}{|\nabla u|}\mathrm{Hess}u(X, Y) + X\left(\frac{1}{|\nabla u|}\right)\langle \nabla u, Y\rangle + Y\left(\frac{1}{|\nabla u|}\right)\langle \nabla u, X\rangle.\end{aligned}$$

From the expression, using that $|\nabla u|$ is constant on N and the properties of $\mathrm{Hess}u$ we deduce that

$$(L_\nu g_M)(X, Y) = \frac{2}{|\nabla u|}\mathrm{Hess}u(X, Y) = 0,$$

if at least one between X and Y is in the tangent space of N. If, however, X and Y are normal (w.l.o.g. $X = Y = \nabla u$), we have

$$
\begin{aligned}
(L_\nu g_M)(X,Y) &= \frac{2}{|\nabla u|}\mathrm{Hess}u(\nabla u, \nabla u) + 2\nabla u\left(\frac{1}{|\nabla u|}\right)|\nabla u|^2 \\
&= \frac{2}{|\nabla u|}\mathrm{Hess}u(\nabla u, \nabla u) - 2\nabla u(|\nabla u|) \\
&= 2\mathrm{Hess}u(\nu, \nabla u) - 2\langle \nabla|\nabla u|, \nabla u\rangle = 0.
\end{aligned}
$$

Thus, we conclude that ν is a Killing field and so Φ_t is a flow of isometries. Since $\nabla u \perp TN$, M splits as a Riemannian product, as desired. In particular, $\mathrm{Ric}_f^N \geqslant 0$ if $m \geqslant 3$, while, if $m = 2$, $M = \mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{R}$ with the flat metric.

Lastly, we will verify the properties of the function u. Let γ be any integral curve of ν. Then

$$
\frac{d}{dt}(u \circ \gamma) = \langle \nabla u, \nu\rangle = |\nabla u| \circ \gamma > 0,
$$

since $|\nabla u| > 0$. Now, as M splits isometrically in the direction of ∇u we obtain that $\mathrm{Ric}(\nu, \nu) = 0$ and this implies that $\mathrm{Hess}f(\nu, \nu) = 0$. Consequently $\langle \nabla f, \nu\rangle = k$ is constant in the splitting direction.

By the other hand,

$$
\begin{aligned}
-g(u \circ \gamma) = \Delta_f u(\gamma) &= \mathrm{Hess}u(\nu, \nu)(\gamma) - \langle \nabla f, \nabla u\rangle(\gamma) \\
&= \langle \nabla|\nabla u|, \nu\rangle(\gamma) - \langle \nabla f, \nu\rangle|\nabla u|(\gamma) \\
&= \frac{d}{dt}(|\nabla u| \circ \gamma) - k|\nabla u| \circ (\gamma) \\
&= \frac{d^2}{dt^2}(u \circ \gamma) - k\frac{d}{dt}(u \circ \gamma),
\end{aligned}
$$

and thus $y = u \circ \gamma$ solves the ODE $-y'' + ky' = g(y)$ with $y' > 0$.

We next address the parabolicity. Under assumption (i), M is f-parabolic and so N is necessarily f-parabolic too. We are going to deduce the same under assumption (ii). Note that the chain of inequalities

$$
\begin{aligned}
\left(\int_{-R}^{R} |y'(t)|^2 dt\right)\mathrm{Vol}_f(B_R^N) &\leqslant \int_{[-R,R]\times B_R^N} |y'(t)|^2 dt\, d\nu_f^N \\
&\leqslant \int_{B_{R\sqrt{2}}} |\nabla u|^2 d\nu_f = o(R^2 \log R)
\end{aligned}
$$

gives immediately (6.3) and (6.4), since $|y'| > 0$ everywhere. Thus, since $\mathrm{Vol}_f(B_R^N) = o(R^2 \log R)$, we know that there is a constant A such that $\mathrm{Vol}_f(B_R^N) \leqslant AR^2 \log R$, that is,

$$
\frac{R}{\mathrm{Vol}_f(B_R^N)} \geqslant \frac{1}{AR\log R},
$$

hence

$$
\begin{aligned}
\lim_{t\to\infty}\int_1^t \frac{R\,dR}{\mathrm{Vol}_f(B_R^N)} &\geqslant \lim_{t\to\infty}\int_1^t \frac{dR}{AR\log R} \\
&= A^{-1}\lim_{t\to\infty}\log(\log t) = \infty.
\end{aligned}
$$

Thus, by proposition 6.3, N is f-parabolic. So we conclude the proof.

BIBLIOGRAPHY

[1] Alencar, H.; do Carmo, M. *hypersurfaces with constant mean curvature in spheres*, Proc. Math. Soc. 120, 1223-1229, 1994.

[2] Alías, L. J.; Meroño, M. A.; Ortiz, I. *On the first stability eigenvalue of constant mean curvature surfaces into homogeneous 3-manifolds*, Mediterr. J. Math. 12, no. 1, 147 - 158, 2015.

[3] Alías, L. J.; Brasil, A. Jr.; Sousa, L. A. M. *A characterization of Clifford tori with constant scalar curvature one by the first stability eigenvalue*, Bull. Braz. Math. Soc. (N.S.) 35, no. 2, 165 - 175, 2004.

[4] Alías, L. J.; Barros, A.; Brasil, A. Jr. *A spectral characterization of the H(r)-torus by the first stability eigenvalue*, Proc. Amer. Math. Soc. 133, no. 3, 875 - 884, 2005.

[5] Batista, M.; Cavalcante, M.P.; Pyo, J. *Some Isoperimetric Inequalities and Eigenvalue Estimates in Weighted Manifolds*, J. Math. Anal. Appl. v. 419, p. 617-626, 2014.

[6] Binoy, R.; Santhanam, G. *Sharp upperbound and a comparison theorem for the first nonzero Steklov eigenvalue*, J. Ramanujan Math. Soc. 29, no. 2, 133 - 154, 2014.

[7] Bakry, D.; Émery, M. *Diffusions hypercontractives. Séminaire de probabilities*, XIX, 1983/84, Lecture Notes in Math, vol. 1123, Springer, Berlin, pp 177 - 206, 1985.

[8] Bayle, V. *Propriétées de concavité du profil isopérimétrique et applications*, Thése de Doctorat, 2003.

[9] Chang, S.Y. A.; Gursky, M. J.; Yang, P. *Conformal invariants associated to a measure*, Proc. Natl. Acad. Sci. USA 103, no. 8, 2535 - 2540, 2006.

[10] Cheng, X.; Mejia, T.; Zhou, D. *Stability and Compactness for Complete f-minimal Surfaces*, Trans. Amer. Math. Soc. 367, no. 6, 4041–4059, 2015.

[11] Cheeger, J.; Gromoll, D. *The Splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom, v. 6, 119-128, 1971.

[12] Cheng, S. Y.; Yau, S.T. *Hypersurfaces with constant scalar curvature*, Math. Ann. 225, 195–204, 1977.

[13] Ebey, E. *Sobolev Spaces on Riemannian Manifolds*, Lecture Notes in Mathematics 1635, Springer, Paris 1996.

[14] Escobar, J. F. *The geometry of the first non-zero Stekloff eigenvalue*, J. Funct. Anal. 150, no. 2, 544 - 556, 1997.

[15] Escobar, J.F. *An isoperimetric inequality and the first Steklov eigenvalue*, J. Funct. Anal. 165 101 - 116, 1999.

[16] Escobar, J. F. *A comparison theorem for the first non-zero Steklov eigenvalue*, J. Funct. Anal. 178, no. 1, 143 - 155, 2000.

[17] Espinar, J.M. *Manifolds with Density, applications and Gradient Schrödinger Operators*, arXiv:1209.6162v6 [math.DG], 2012.

[18] Farina, A.; Mari, L.; Valdinoci, E. *Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds*, Comm. Partial Differential Equations 38, no. 10, 1818–1862, 2013.

[19] Fan, E. M. *Topology of three-manifolds with positive P-scalar curvature*, Proc. Amer. Math. Soc. 136, no. 9, 3255 - 3261, 2008.

[20] Fischer, M. *Intermediate Real Analysis*, Springer-Verlag, New York, 1983.

[21] Fischer, C.; Schoen, R. *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. 33, no. 2, 199 - 211, 1980.

[22] G. Friedman, E.; Hunsicker, A.; Libgober, L. *Maxim. Topology of stratified spaces*, MSRI Publications Vol.58, USA 2011.

[23] Gromov, M. *Isoperimetric of waists and concentration of maps*, Geom. Funct. Anal., 13 no. 1, 178-215, 2003.

[24] Girouard, A.; Polterovich, I. *Spectral Geometry of the Steklov problem*, arXiv:1411.6567v1 [math.SP], (2014).

[25] Grigoryan, A. *Analytic and geometric background of recurrence and non-explosion of Brownian motion on Riemannian manifolds*, Bull, of A.M.S, 36 no. 2,135-249, 1999.

[26] Gilbarg, D.; Trudinger, N. S. *Elliptic partial differential equations of second order*, Reprint of the 1998 ed. Cassics in Mathematics. Springer-Verlag, Berlin, 2001.

[27] Hua, B.; Liu, S.; Xia, C. *Liouville theorems for f-harmonic maps into Hadamard Spaces*, arxiv.org/pdf/1305.0485.pdf, 2013.

[28] Impera, D.; Rimoldi, M. *Stability properties and topology at infinity of f-minimal hypersurfaces*, Geom. Dedicata 178, 21-47, 2015.

[29] Hoffman; K., Kunze, R. *Linear Algbra.* Prentice Hall, Inc., Englewood Cliffs, 1961.

[30] Kuttler, J.R.; Sigillito, V.G. *Inequalities for membrane and Stekloff eigenvalues*, J. Math. Anal. Appl. 23, 148–160, 1968.

[31] Lichnerowich, A. *Variétés riemanniennes à tenseur C non négatif*, C. R. Acad. Sci. Paris Sér. A-B 271, A650–A653, 1970.

[32] Lichnerowich, A. *Variétés kählériennes à première classe de Chern non negative et variétés riemanniennes à courbure de Ricci généralisée non negative.* J. Diff. Geom. 6, 47–94, 1971.

[33] Morgan, F. *Manifolds with density*, Notices Amer. Math. Soc. 52, no. 8, 853 – 858, 2005.

[34] Ma, L.; Du, S. H. *Extension of Reilly formula with applications to eigenvalue estimates for drifting Laplacians*, C. R. Math. Acad. Sci. Paris 348, no. 21-22, 1203–1206, 2010.

[35] Mendonça, S. *Splitting, Parallel Gradient and Bakry-Émery Ricci Curvature*, arXiv:150200185v1, 2015.

[36] Monteanu, O.; Wang, J. *Analysis of weighted Laplacian and applications to Ricci solitons*, Comm. Anal. Geom. 20, no. 1, 55 - 94, 2012.

[37] Monteanu, O.; Wang, J. *Geometry of Manifolds with Densities*, Adv. Math. 259, 269 - 305, 2014.

[38] Nomizu, K.; Smyth, B. *A formula of Simon's type and hypersurfaces with constant mean curvature*, J. Differential Geom. 3, 367-377, 1969.

[39] Perelman, G. *The entropy formula for the Ricci flow and its geometric applications*, arXiv: .0211159 [math.DG], 2002.

[40] Pigola, S., Rigoli, M., Setti, A. G.*Vanish and Finiteness Results in Geometric Analysis: A Generalizations of the Bochner Techinique*, Progress in Mathematics V.26, Birkhauser Verlag AG. 2008.

[41] Perdomo, O. *First stability eigenvalue characterization of Clifford hypersurfaces*, Proc. Amer. Math. Soc. 130, 3379 - 3384, 2002.

[42] Payne, L. E. *Some isoperimetric inequalities for harmonic functions*, SIAM J. Math. Anal. 1, 354 - 359, 1970.

[43] Pigola, S.; Rigoli, M.; Rimoldi, M.; Setti, A.G. *Ricci almost soliton*, Ann, Sc. norm. Super. Pisa Cl. Sci (5) 10, no. 4, 757-799, 2011.

[44] Petersen, P. and Wylie, W. *Rigidity of Gradient Ricci Solitons*, Pacific J. Math. 241, no. 2, 329 - 345, 2009.

[45] R. Schoen and S.T. Yau. *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) 110, no. 1, 127 - 142, 1979.

[46] Raulot, S. and Savo, A. *On the first eigenvalue of the Dirichlet-to-Neumann operator on forms*, J. Funct. Anal. 262, no. 3, 889 - 914, 2012.

[47] Reilly, R. *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J. 26, 459 - 472, 1977.

[48] Reilly, R. *Geometric applications of the solvability of Neumann problems on a Riemannian manifold*, Arch. Rational Mech. Anal. 75, no. 1, 23 - 29, 1980.

[49] Stekloff, M. W. *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. École Norm. Sup. 19, 455 - 490, 1902.

[50] Tashiro, Y. *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. 117, 251 - 275, 1965.

[51] Villani, C. *Optimal transport, old and new*. Volume 338 of Grundlehren der mathematischen Wissenschaften, Springer, 2009.

[52] Weinstock, R. *Inequalities for a classical eigenvalue problem*, J. Rational Mech. Anal. 3, 745 - 753, 1954.

[53] Wylie, W. *Sectional Curvature for Riemann Manifolds with Density*, Geom. Dedicata 178, 151–169, 2015.

[54] Walter, W. *A theorem on elliptic differential inequalities with an application to gradient bounds*, Math. Z., 200(2):293-299, 1989.

[55] Wei, G., Wylie, W. *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom., 83 , no. 2, 377 – 405, 2009.

[56] Xia, C. *Rigidity of compact manifolds with boundary and nonnegative Ricci curvature*, Proc. Amer. Math. Soc 125, no. 6, 1801–1806 1997.

[57] Xia, C. and Wang, Q. *Inequalities for the Steklov eigenvalues*, Chaos Solitons Fractals 48, 61 – 67, 2013.

[58] Xia, C. and Wang, Q. *Sharp bounds for the first non-zero Stekloff eigenvalues*, J. Funct. Anal. 257, no. 8, 2635 – 2644, 2009.