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SIDNEY DONATO DA SILVA

Low Min-Max Widths of the Unit Disk

Maceió - AL
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Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática da Universidade Federal de Alagoas em Associação com a Universidade Federal da Bahia, como requisito parcial para obtenção do título de Doutor em Matemática.

Orientador: Prof. Dr. Feliciano Marcílio Aguiar Vitório.

Coorientador: Prof. Dr. Fernando Codá Marques.

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To my family.

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RESUMO

Para 2-variedades com bordo convexo e não vazio, provamos um resultado de regularidade para varifolds estacioários V de dimensão um, os quais são $\mathbb{Z}_2$-quase minimizantes em aneis. Essa regularidade diz que V é uma rede finita de geodésicas com fronteira livre. Usando essa regularidade, podemos deduzir algumas propriedades de V, como uma estimativa para a densidade em dimensão um. Juntamente com o Teorema Min-Max e as p-varreduras dadas por variedades algébricas reais, fomos capazes de calcular as primeiras p-larguras da bola unitária B^2 e de regiões fechadas planas cujo bordo é uma elipse com excentricidade suficientemente baixa. Essas p-varreduras nem sempre são ótimas. Contudo, nos casos que consideramos elas são quase ótimas.

Keywords: Rede de geodésicas com fronteira livre. P-varreduras. P-larguras.

ABSTRACT

For 2-manifolds with non-empty convex boundary, we prove a regularity result for one dimensional stationary varifolds V with free boundary, such that are $\mathbb{Z}_2$-almost minimizing in annuli. That regularity says that V is a free boundary finite geodesic network. Using that regularity we can deduce some properties of V, as an estimate for the one-dimensional density. Together with the Min-Max Theorem and the p-sweepouts given by real algebraic varieties, we were able to calculate the first p-widths of the unit ball B^2 and of planar full ellipses close to B^2. Those p-sweepouts are not always optimal. However, in our situations they are almost optimal.

Keywords: Free boundary geodesic networks. P-sweepouts. P-widths.

CONTENTS

1 INTRODUCTION

Almgren-Pitts min-max theory

Minimal surfaces have been an object of study for centuries in the fields of geometry, analysis and partial differential equations. The solution of the Plateau problem in 1930 stimulated several research activities in related problems, as the free boundary problems for minimal surfaces. One of the mathematicians that studied related problems was Almgren [6] that introduced the concepts of varifolds, rectifiable varifolds and integral varifolds. Essentially, the set of integral varifolds of dimension k, for $0 < k \leqslant n+1$, on a Riemannian manifold M^{n+1}, $n > 0$, is the set $I\mathcal{V}_k(M)$ such that each element $V \in I\mathcal{V}_k(M)$ is a finite union of k-dimensional submanifolds with multiplicity, where we consider a measure defined by integration of the Hausdorff measure $\mathcal{H}^k$ over each submanifold, endowed with the weak convergence topology of measures. We can define a first variation of varifolds and we say that it is stationary if its first variation vanishes. When the varifold is a surface, its first variation coincides with the usual first variation of area formula. In this case, in particular, its support is a minimal surface possibly with multiplicity and self intersections. The notion of area for a varifold V over a measurable set A is given by the weight $\|V\|(A)$.

In the work of Almgren, the study of variational calculus for minimal surfaces is generalized in the sense that it considers any dimension and codimension, no curvature hypothesis, and includes the cases with fixed or free boundary. In that work he proposed a free boundary problem for manifolds possibly with non-empty boundary: to find a smooth, embedded k-dimensional minimal submanifold $\Sigma \subset M^{n+1}$ solution for the free boundary problem, and such that $\partial\Sigma \subset \partial M$. In the case with empty boundary, he found a non-trivial and general weak solution that is an integral stationary varifold V. The regularity for the codimension-one case was proved by Pitts [31] for $2 \leqslant n \leqslant 5$. Pitts introduced the concept of almost minimizing, and he shown that V can be chosen to be almost minimizing and then the support of V is given by a smooth embedded minimal hypersurface. Essentially, the regularity comes from the fact that almost minimizing varifolds are locally stable almost everywhere. Later Schoen and Simon [34] extended that result for $n \geqslant 6$.

The Almgren-Pitts min-max theory is a Morse-type theory which may explain better a manifold and its hypersurfaces. For example, some important and surprising applications are the solution of the famous Willmore conjecture by Marques and Neves [23], the Yau's conjecture by Marques and Neves [25] and Song [36], the Weyl law for the Volume Spectrum by Liokumovich, Marques and Neves [22], and the Freedman, He and Wang conjecture by Agol, Marques and Neves [1]. The particularities, comparing with the Morse theory, is the use of integral n-cycles with coefficients in $\mathbb{Z}_2$, and the functional mass $\mathbf{M}$, which is a notion of volume. Roughly speaking, the space of integral n-cycles with coefficients in $\mathbb{Z}_2$ is the space of compact n-dimensional hypersurfaces in M without boundary and without orientation. Another differences is that the Almgren-Pitts min-max theory works with varifolds, which allows degenerations. Also, it works with homotopy instead of homology, thus it is necessary to consider different variations to obtain the critical points (varifolds).

Essentially, the Almgren-Pitts min-max theory [31] says that: if $[\Phi]$ denotes a certain homotopy class of a map Φ from a cubical subcomplex $X \subset I^m$, for some $m \in \mathbb{N}$, into the space of integral n-cycles, and defining the width of $[\Phi]$ by

$$L([\Phi]) = \inf_{\Psi \in [\Phi]} \sup_{x \in X} \mathbf{M}(\Psi(x)),$$

then for $2 \leqslant n \leqslant 5$ and $L([\Phi]) > 0$, there exists a closed embedded minimal hypersurface Σ (possibly with multiplicity) satisfying:

$$L([\Phi]) = \mathrm{Vol}(\Sigma).$$

A similar result was extended by Schoen and Simon [34] for all $n \geqslant 2$. Also, recently Marques and Neves [24] and [26] obtained from this theory a characterization of the Morse index for minimal hypersurfaces obtained from a homotopy class of p-parameters.

In the case $\partial M \neq \varnothing$, the deformations that we consider are deformations given by vector fields on M which are tangential to ∂M. This is because the free boundary minimal hypersurfaces are critical points to the area functional with respect to deformations in M that preserves ∂M. In this set, we call the stationary varifolds of stationary varifolds with free boundary, but this notation does not mean that the varifold has the same properties as a free boundary minimal surface Σ. In fact, in the regular case we have that Σ meets the boundary ∂M orthogonally along its boundary $\partial\Sigma$. On the other hand, any constant multiply of a connected component of ∂M is a stationary varifold with free boundary, even though it can be nothing like a free boundary minimal hypersurface in M.

We say that a varifold V is an almost minimizing varifold with free boundary on a relatively open set $U \subset M$ if we can approximate it by a varifold induced from a cycle T such that for any deformation of T by a discrete family supported in U, and with the mass not increasing too much, then at the end of the deformation the mass can not be deformed down too much. The main difference here, comparing with the definition for empty boundary, is that we are working with relative cycles, so in that approximation the induced varifolds have no mass on ∂M.

In contrast with higher dimensions, where to show the regularity of a stationary varifold V is used the almost minimizing property, in the one-dimensional case the interior regularity does not depend on this property. In fact, as showed by Allard and Almgren [4], one dimensional integral stationary varifolds are given by a geodesic network in any compact set contained in the interior of M, that is, the varifold is a finite union of geodesic segments such that the singularities are given by their possible stationary junctions. However, to obtain the free boundary property, we assume the almost minimizing property in annuli (Main Theorem A).

Sweepouts and width

In the space of flat chains we can define three different topologies: the mass norm, the flat norm and the $\mathbf{F}$-distance. In fact, since the space of k-currents is the dual space of the space of differential k-forms, it is naturally endowed with the weak topology and has a boundary operator. A similar notion to area in the space of currents is given by the mass norm $\mathbf{M}(T)$ of a current T. When T is induced by a manifold, the mass norm is exactly the definition of area. Also, this norm induces a topology that is stronger than the weak topology. The flat distance (or area-distance) between two k-currents T_1, T_2 is the minimal area of any $(k+1)$-current whose boundary is $T_1 - T_2$. The space of flat k-chains is given by the closure under the flat distance over the set of k-rectifiable currents, where the latter are essentially currents whose support is given by k-rectifiable sets.

The flat chains with no boundary are called of flat cycles, and we denote by $\mathcal{Z}_k(M; \mathbb{Z}_2)$ the space of integral k-cycles with coefficients in $\mathbb{Z}_2$. Roughly speaking, this space is the space of compact k-dimensional submanifolds in M without boundary and without orientation.

We are interested in calculating the volume of cycles for certain families of planar curves with k-parameters in a planar full ellipse $M^2 = E^2 \subset \mathbb{R}^2$. In the simplest case where E^2 is the unit ball B^2, and we take an 1-parameter family of smooth curves that sweep B^2 out, we have that one of that curves have to pass through the origin, in particular the length of that curve is at least 2. On the other hand, if our family is given by vertical lines in the unit ball, we have that the maximum length will be 2. Denoting by $\mathbb{S}$ the set of all 1-parameter families of curves in the unit ball which sweep B^2 out, and by C a curve in a family $S \in \mathbb{S}$, we obtain

$$W(B^2) = \inf_{S \in \mathbb{S}} \sup_{C \in S} L(C) = 2,$$

where $L(C)$ is the length of C in B^2. We call the min-max estimate above of a width for B^2. An analogous can be made for higher dimensions, taking families of surfaces Σ and the area instead of curves and length, respectively. Also, for the precise definition of the problem above it is necessary that $\partial\Sigma \subset \partial M$.

For M^{n+1}, $n > 0$, such that $\partial M = \varnothing$, the problem that we will consider is the min-max problem for continuous families of mod 2 n-cycles such that each family is a p-sweepout of M, for some $p \in \mathbb{N}$. By a continuous family we mean that this family is given by a map $\Phi : X \to \mathbb{Z}_n(M; \mathbb{Z}_2)$ which is continuous in the flat topology, where $X \subset I^m$ is a cubical subcomplex for some $m \in \mathbb{N}$. And by a p-sweepout we mean that the p-th cup power of $\Phi^*(\overline{\lambda})$ is nonzero, where $\overline{\lambda}$ is the generator of $H^1(\mathcal{Z}_n(M; \mathbb{Z}_2))$, given by the Almgren isomorphism [5] and the Universal Coefficient Theorem [16, Section 3.1]. Roughly speaking, the geometric meaning of a family of cycles be a p-sweepout is that for any choice of p points in M, there exists a cycle $\Phi(x)$ that passes through these points. We will see a more precisely definition in the next section, also we will consider these families without concentration of mass to avoid currents (cycles) such that the mass is accumulated in points. Similar considerations apply when M has non-empty boundary, but in this case we use the space of the relative mod 2 n-cycles $\mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2)$, which is the space given by a quotient in the space of integral flat n-chains with coefficients in $\mathbb{Z}_2$ and boundary lying on ∂M. Essentially, when we take relative cycles we forget the part of the cycle lying on ∂M. The relative cycles was used by Almgren [5] and its use is motivated by the fact that, in the min-max problem, the cycles need to have its boundary lying on ∂M.

Our definition of relative cycles on M is identical to the definition of cycles when M has empty boundary. So, we will consider the space of relative cycles for the general case.

We denote by $\mathcal{P}_p(M)$ the set of all p-sweepouts with no concentration of mass, and we define the p-width of M as

$$\omega_p(M) = \inf_{\Phi \in \mathcal{P}_p(M)} \sup\{\mathbf{M}(\Phi(x)) : x \in \operatorname{dmn}(\Phi)\},$$

where $\operatorname{dmn}(\Phi)$ is the domain of each Φ (the domain is not fixed).

Clearly, $\omega_p(M) \leqslant \omega_{p+1}(M)$ for all p.

Gromov [14] and later Guth [15] estimated the growth of the p-widths. Precisely, there exist positive constants c and C depending only on (M^{n+1}, g) such that

$$cp^{\frac{1}{n+1}} \leqslant \omega_p(M) \leqslant Cp^{\frac{1}{n+1}}.$$

The upper bound arises from constructing a particular sweepout of small mass using the fact that the coefficients are in $\mathbb{Z}_2$. And the lower bound is due to the fact that any p-sweepout must divide any p disjoint balls in half, so applying the isoperimetric inequality for each ball is obtained this lower bound. The inequality above was an important tool used by Marques and Neves [25] to prove the existence of infinitely many embedded minimal hypersurfaces in certain compact Riemannian manifolds.

Main results and related results

The main goal of this thesis is to calculate the low p-widths for the unit ball $B^2 \subset \mathbb{R}^2$ and for perturbations of B^2, which are given by planar full ellipses $E^2 \subset \mathbb{R}^2$ that are sufficiently C^∞-close to B^2. Our results are inspired in the results of Aiex [2], which say that

Theorem 1. [2, Theorem 5.2] *If S^2 is the round 2-sphere of radius 1, then*

(i) $\omega_1(S^2) = \omega_2(S^2) = \omega_3(S^2) = 2\pi$;

(ii) $\omega_4(S^2) = \omega_5(S^2) = \omega_6(S^2) = \omega_7(S^2) = \omega_8(S^2) = 4\pi$.

Also, given an ellipsoid $\mathcal{E}^2 \subset \mathbb{R}^3$, take W_1, W_2, W_3 as the one-dimensional varifolds with multiplicity one and induced from the three principal closed geodesics in $\mathcal{E}^2$, respectively.

Consider the varifolds given by the combinations of those varifolds: $W_4 = 2W_1, W_5 = W_1 + W_2, W_6 = 2W_2, W_7 = W_1 + W_3, W_8 = 2W_3, W_9 = W_2 + W_3$. Then,

Theorem 2. [2, Theorem 5.6] *If $\mathcal{E}^2$ is an ellipsoid which is sufficiently C^∞-close to the round metric in S^2, then*

(i) $\omega_i(\mathcal{E}^2) = \|W_i\|(\mathcal{E}^2)$ for $i = 1, 2$ or 3;

(ii) $\omega_j(\mathcal{E}^2) = \|W_l\|(\mathcal{E}^2)$ for $j = 4, \cdots, 8$ and for some $l = 4, \cdots, 9$ without repetition.

As we see in item (ii) above, one of the j-widths for $j = 4, \cdots, 8$ has multiplicity two, in other words we have an example of a unstable min-max critical 1-varifold with multiplicity and smooth embedded support. So, working with min-max critical curves is a special case, since for min-max critical hypersurfaces this fact should not happen as claims the Multiplicity One Conjecture for closed manifolds by Marques and Neves [24], which was proved recently in dimension three by Chodosh and Mantoulidis [7], also other authors made partial solutions as Zhou [39]. Moreover, many of the recent results involving min-max techniques are not applicable for curves. So, to obtain the results above, we will need to adapt some tools from the min-max techniques for the set of curves. In fact, as we are working in the case with boundary, we need some results about regularity with free boundary for varifolds. One of our main results gives a regular-type result. Precisely:

Main Theorem A. *Let M^2 be a compact Riemannian manifold with non-empty strictly convex boundary. If $V \in I\mathcal{V}_1(M)$ is a stationary varifold such that it is $\mathbb{Z}_2$-almost minimizing in small anulli with free boundary, then V is a free boundary geodesic network.*

For us, free boundary have a specific meaning: $V \llcorner \partial M = 0$ as an 1-varifold and the varifold tangent $\mathrm{VarTan}(V, p) \notin T_p(\partial M)$ for all $p \in \mathrm{support}(V) \cap \partial M$.

This result will be useful, because the varifolds given by the Min-Max Theorem satisfy those hypotheses. So, the p-widths are reached by free boundary geodesic networks, and we will obtain a finite classification of free boundary geodesic networks, when M is a planar full ellipse E^2 close to B^2. Indeed, to classify the free boundary geodesic networks with low mass in E^2, we use an upper bound for the density as a function of the mass, so we forget certain junctions that can not happen. Finally, we use an interior result from Aiex [2, Theorem 4.13], which says that if a geodesic network is $\mathbb{Z}_2$-almost minimizing in annuli at interior of M, then the density in the interior points is an integer number.

The Theorem A is founded in the Section 3.4. Below we give an idea of the proof.

Idea of the proof of Theorem A: By the interior regularity result due to Allard and Almgren [4] we know that V is a geodesic network finite in every compact $K \subset \mathring{M}$. Near to ∂M, we will use Fermi coordinates, since the half ball and the half sphere have good properties in these coordinates. The convexity will be crucial to obtain a weak regularity result for replacements, which will be important to show that the interior of each geodesic segment of V will be in $\mathring{M}$. We consider replacements on overlapping annuli centered at boundary points, and by a maximum principle we can glue continuously those replacements. We do the following important observation: if a varifold is $\mathbb{Z}_2$-almost minimizing in an open ball in $\mathring{M}$, then the support of each respective varifold tangent will be a straight line. Using this observation, we can glue smoothly the replacements. It will not difficult to see that in a neighborhood of ∂M there will be a finite number of geodesic segments of $\text{spt}\|V\|$, which are in $\mathring{M}$ and each segment has endpoints touching ∂M. Finally, we apply the Constancy Theorem in $M\backslash\partial M$ to see that the support of V is a free boundary geodesic network.

Our main results for k-widths are very similar to the results above. The main difference is that we will work with manifolds with boundary. More precisely, we proved that

Main Theorem B. *The low p-widths of B^2 are given by*

(i) $\omega_1(B^2) = \omega_2(B^2) = 2$;

(ii) $\omega_3(B^2) = \omega_4(B^2) = 4$.

Also, if E^2 is a planar full ellipse C^∞-close to B^2 with small diameter d and large diameter D, then

(iii) $\omega_1(E^2) = d$ *and* $\omega_2(E^2) = D$;

(iv) $\omega_3(E^2), \omega_4(E^2) \in \{2d, d + D, 2D\}$ *such that* $\omega_3(E^2) \neq \omega_4(E^2)$. *In particular, one of those widths is reached by an one-varifold with multiplicity two.*

Again, from (iv) we obtained an example of min-max critical varifold with multiplicity. We do not know any development in a Multiplicity One Conjecture in the case with boundary. But, again the case for curves is a special case, which allows multiplicity. The Theorem B above is founded in the Section 4.3.

Idea of the proof of Theorem B: We use the p-sweepouts defined in Guth [15], whose images are given by real algebraic varieties. We be able to get a good upper estimate for the mass of those cycles. By the Min-Max Theorem there will be a varifold V that reaches the p-width, and such that it satisfies the hypotheses of Theorem A, so V will have regularity properties. Those regularity properties together with an upper bound for the density will be enough to obtain a finite number of possible values for the first p-widths. It will be crucial to use the fact that E^2 is close to B^2, then by continuity the respective p-widths will be close. Finally, for low mass, we will see that the widths of E^2 are all different from each other, then it will create 'gaps' which will enable us to calculate presicely the first widths of B^2 and E^2, even though some of those p-sweepouts are not optimal.

In the case of the sphere $M = S^2$ in Theorem 1 above, similar p-sweepouts given by algebraic varieties are optimal in the sense that

$$\omega_p(M) = \sup\{\mathbf{M}(\Phi(x)) : x \in \mathrm{dmn}(\Phi)\}.$$

In contrast, in our case the p-sweepouts for B^2 are not optimal if $p = 3, 4$ in Theorem B. Actually, as we will see in the Appendix, for $p = 3, 4$ the p-sweepout is almost optimal. We believe that this type of p-sweepout is not optimal for all $p \geqslant 3$, and the error increases. However, the regularity of Theorem A will be enough to classify the varifolds with low mass (Theorem 3.7 and Corollary 3.9). In particular, we will deduce gaps of values for the p-widths, so we will not need optimal p-sweepouts a priori.

Other important result is the thesis of Nurser [29], which calculates the first widths of S^3 :

$$\omega_1(S^3) = \omega_2(S_3) = \omega_3(S^3) = \omega_4(S^3) = 4\pi,$$

$$\omega_5(S^3) = \omega_6(S^3) = \omega_7(S^3) = 2\pi^2,$$

and gives estimates for $\omega_9(S^3)$ and $\omega_{13}(S^3)$. In general, for spheres S^n for all $n \geqslant 3$, we know that $\omega_p(S^n) = \mathrm{Vol}(S^{n-1})$ for $p = 1, \cdots, n+1$. This result was proved by Gaspar and Guaraco [12] using a notion of width in the context of the Allen-Cahn equation and the energy functional associated. A more recent result was done in the thesis of Lima [21], where he calculates that $\omega_1(\mathbb{RP}^3) = \omega_2(\mathbb{RP}^3) = \pi^2$ and gives estimates for $\omega_9(\mathbb{RP}^3)$, where $\mathbb{RP}^3$ is the three-dimensional real projective space.

A more general and open question is to find a general formula for the width of a certain manifold M^{n+1}. In fact, so far this problem seems to be very hard. A recent result in this direction was the proof by Liokumovich, Marques and Neves [22] of the Weyl law for the volume spectrum $\{\omega_p(M)\}$. Precisely, they proved:

Theorem 3. [22] *Given* (M^{n+1}, g) *a compact Riemannian manifold (with possible non-empty boundary). There exists a constant* $a(n) > 0$ *such that*

$$\lim_{p\to\infty} \omega_p(M)p^{-\frac{1}{n+1}} = a(n)vol(M)^{\frac{n}{n+1}}.$$

This formula is very similar to the Weyl law for eigenvalues. While for eigenvalues we know how to calculate the constant of the Weyl law in many cases, for the volume spectrum it is not know any example so far, even in our case of the unit disk.

Using the theorem above, Irie, Marques and Neves [17] proved the density of minimal surfaces for closed manifolds M^{n+1} with generic metric and $2 \leqslant n \leqslant 6$. Using that theorem again, Marques, Neves and Song [27] extended this result, showing that there exist an equidistribution of minimal surfaces in that case.

2 PRELIMINARIES

In this section, we summarize some definitions and results that we will use in the next sections.

Since we are assuming that the Riemannian manifold M has non-empty boundary, it will be useful consider Fermi coordinates at the points of ∂M. In fact, the geometric properties of the half-ball and half-sphere in those coordinates will be crucial to prove our regularity result.

From geometric measure theory we will talk about currents, flat chains, relative flat cycles and varifolds. Since we are supposing that M can have non-empty boundary, we will take more care in these definitions.

We will also talk about some notions of homotopy, definitions and results from the Almgren-Pitts Theory. In the case with boundary, we follow some modifications due to Liokumovich, Marques and Neves [22] and Marques and Neves [25]. We will do a version of the Min-Max Theorem similar to Pitts [31, Theorem 4.10] and Li and Zhou [19, Theorem 4.21], following the necessary modifications to our case. Roughly speaking, that theorem proves the existence of almost minimizing varifolds, moreover it says that there exists always such a varifold that reaches the width of a chosen homotopy class. The main application is to use this theorem for p-widths as done in the Corollary 2.23, which will be important to calculate the p-widths in the next sections, since our p-sweepout does not always reach the p-widths.

2.1 Manifolds with Boundary and Fermi Coordinates

Here we follow the notations of [19]. When M^{n+1}, $n > 0$, is a compact Riemannian Manifold with nonempty boundary ∂M, we can always extend M to a closed Riemannian manifold $\widetilde{M}$ with the same dimension such that $M \subset \widetilde{M}$ ([30]). Also, by the Nash's Theorem, we can fix an isometric embedding $\widetilde{M} \hookrightarrow \mathbb{R}^Q$ for some $Q \in \mathbb{N}$. We will denote by $B_r(p) \subset \mathbb{R}^Q$ the open Euclidean ball of radius r centered at $p \in \mathbb{R}^Q$, and by $\widetilde{\mathcal{B}}_r(p)$ the open geodesic ball in $\widetilde{M}$ of radius r centered at p. For $0 < s < r$ we define the following open annuli:

$$A_{s,r}(p) := B_r(p)\backslash\mathrm{Clos}(B_s(p)) \quad \text{and} \quad \widetilde{A}_{s,r}(p) := \widetilde{B}_r(p)\backslash\mathrm{Clos}(\widetilde{B}_s(p)),$$

where $\mathrm{Clos}(S)$ denotes the closure of the set S.

We consider the following spaces of vector fields:

$$\mathfrak{X}(M) := \{X \in \mathfrak{X}(\mathbb{R}^Q) : X(p) \in T_pM \text{ for all } p \in M\}$$

and

$$\mathfrak{X}_{tan}(M) := \{X \in \mathfrak{X}(M) : X(p) \in T_p(\partial M) \text{ for all } p \in \partial M\}.$$

Definition 2.1. (Relative Topology) Given any subset $A \subset M$, where M is equipped with the subspace topology, the *interior relative of* A, $\mathrm{int}_M(A)$, is defined as the set of all $p \in M$ such that there exists a relatively open neighborhood $U \subset A$ of p. The *exterior relative of* A will be the set $\mathrm{int}_M(M\backslash A)$. And the *relative boundary of* A, $\partial_{rel}A$, is the subset of M such that is neither in the relative interior nor exterior of A.

Definition 2.2. (Relative Convexity) A subset $\Omega \subset M$ is said to be a *relatively convex* (respect. *relatively strictly convex*) *domain in* M if it is a relatively open connected subset in M whose relative boundary $\partial_{rel}\Omega$ is smooth and convex (respect. strictly convex) in M.

Definition 2.3. (Fermi coordinates) Given $p \in \partial M$ and suppose that $(x_1, \cdots, x_n)$ is the geodesic normal coordinates of ∂M in a neighborhood of p. Take $t = \mathrm{dist}_M(\cdot, \partial M)$, which is a smooth map well-defined in a relatively open neighborhood of p in M. *The Fermi coordinates system of* $(M, \partial M)$ *centered at* p is given by the coordinates $(x_1, \cdots, x_n, t)$. Also, the *Fermi distance function from* p on a relatively open neighborhood of p in M is defined by

$$\widetilde{r} := \widetilde{r}_p(q) = |(x,t)| = \sqrt{x_1^2 + \cdots x_n^2 + t^2}.$$

Definition 2.4. Given $p \in \partial M$, we define the *Fermi half-ball and half-sphere of radius* r *centered at* p respectively by

$$\widetilde{\mathcal{B}}_r^+(p) := \{q \in M : \widetilde{r}_q(x) < r\}, \quad \widetilde{\mathcal{S}}_r^+(p) := \{q \in M : \widetilde{r}_q(p) = r\}.$$

The geometric properties of the Fermi half-ball and half-sphere can be summarized in the following proposition:

Proposition 2.5. ([19, Lemma A.5]) *There exists a small constant* $r_{Fermi} > 0$, *depending only on the isometric embedding* $M \subset \mathbb{R}^Q$, *such that for all* $0 < r < r_{Fermi}$

(i) $\widetilde{\mathcal{S}}_r^+(p)$ *is a smooth hypersurface meeting* ∂M *orthogonally;*

(ii) $\widetilde{\mathcal{B}}_r^+(p)$ *is a relatively strictly convex domain in* M*;*

(iii) $B_{r/2}(p) \cap M \subset \widetilde{\mathcal{B}}_r^+(p) \subset B_{2r}(p) \cap M$*.*

Remark 2.6. The convexity in [19] is assumed to be strictly convex. Also, the property (iii) above will imply that many properties that hold for small open sets in the Fermi coordinates also hold for small open sets in the Euclidean coordinates, and vice versa.

We want to define the following annuli neighborhood in the Fermi coordinates:

$$\mathcal{A}_{s,t}(p) := \widetilde{\mathcal{B}}_t^+(p) \backslash \widetilde{\mathcal{B}}_s^+(p)$$

for $p \in \partial M$, and $0 < s < t$. Also, when $p \in \mathring{M}$, we require that $t < \mathrm{dist}_M(p, \partial M)$.

2.2 Currents, Flat Chains and Relative Flat Cycles

In this section, we will define the main tools to deal with flat chains, whose classical reference is Federer [11, Section 4]. See also Morgan [28, Section 4]. As we are in the case with boundary, we also follow the treatment given in Liokumovich, Marques and Neves [22], Guth [15] and Li and Zhou [19].

In the ambient space $\mathbb{R}^Q$, let us define $\mathcal{D}^k = \{C^\infty$ differential k-forms with compact support$\}$, we define the *space of* k*-dimensional currents* $\mathcal{D}_k$ as being the dual space of $\mathcal{D}^k$. The space $\mathcal{D}_k$ is naturally endowed with the weak topology: $T_i \rightharpoonup T \Leftrightarrow T_i(\xi) \to T(\xi),\ \forall \xi \in \mathcal{D}^k$.

The *support* of a current T, denoted by $\mathrm{spt}(T)$, is the smallest closed set $C \subset \mathbb{R}^Q$ such that

$$(\mathrm{spt}(\xi)) \cap C = \varnothing \Rightarrow T(\xi) = 0.$$

When $k \geqslant 1$, the k-currents can be interpreted as a generalization of the k-dimensional oriented submanifolds N having locally finite $\mathcal{H}^k$-measure. Indeed, given such N with orientation given by a k-vector field η ($\eta_x = \pm\eta_1 \wedge \cdots \wedge \eta_k$ for all $x \in N$, where $\eta_1, \ldots, \eta_k$ is an orthonormal basis for $T_x N$), there exists a corresponding k-current $|N|$ defined by

$$|N|(\xi) = \int_N \xi_x(\eta_x)\ d\mathcal{H}^k, \quad \xi \in \mathcal{D}^k. \tag{1}$$

A subset $R \subset \mathbb{R}^Q$ is said to be a (countably) *k-rectifiable subset* if $R = R_0 \cup \big(\cup_{i=1}^{\infty} F_i(R_i)\big)$, where $\mathcal{H}^k(R_0) = 0$, $F_i : R_i \to \mathbb{R}^Q$ is Lipschitz, and each $R_i \subset \mathbb{R}^k$ is bounded.

As shown in Simon [35, Section 11.6], each $\mathcal{H}^k$-measurable k-rectifiable subset $R \subset \mathbb{R}^Q$ has approximate tangent space almost everywhere, so we can get a current as (1). Also, we can add a measurable function $\theta : R \to \mathbb{R}$ with $\int_R \|\theta\| d\mathcal{H}^k < \infty$ and define

$$[R, \eta, \theta](\xi) = \int_R \xi_x(\eta_x)\theta(x)\ d\mathcal{H}^k, \quad \xi \in \mathcal{D}^k. \tag{2}$$

These currents are actually equivalency classes $[\![R, \eta, \theta]\!]$ which identify triples that define the same current. Each class is said to be a *rectifiable k-current*. If $\theta : R \to \mathbb{Z}_+$, we said that each class is an *integer-multiplicity rectifiable current* and θ is the *multiplicity function*.

We define the *mass norm* $\mathbf{M}(T)$ of a $T \in \mathcal{D}_k$ by

$$\mathbf{M}(T) = \sup_{\|\xi\|^* \leqslant 1,\ \xi \in \mathcal{D}^k} T(\xi),$$

where $\|\xi\|^* = \sup_x \sup_{\xi \in \Lambda^k T_x M,\ \langle \xi,\xi\rangle_x = 1} \langle \xi, \xi\rangle_x$. For rectifiable currents,

$$\mathbf{M}([R, \eta, \theta]) = \int_R |\theta|\ d\mathcal{H}^k.$$

In particular, for submanifolds N as above we have that $\mathbf{M}(|N|) = \text{Area}(N)$.

Also, motivated by (1) and the Stoke's theorem $\big(\int_N d\xi = \int_{\partial N} \xi$, when N has smooth boundary$\big)$ the *boundary* $\partial T \in \mathcal{D}_{k-1}$ of a current $T \in \mathcal{D}_k$ is defined such that

$$T(d\xi) = \partial T(\xi), \quad \xi \in \mathcal{D}^{k-1}.$$

Note that when N has smooth boundary, we have that $|\partial N| = \partial |N|$.

The space of rectifiable (flat) chains with coefficients in a complete normed abelian group $(G, \|\ \|)$ is a more general space containing certain types of currents.

The group of *Lipschitz k-chains with coefficients in G* is defined by

$$\mathcal{L}_k(\mathbb{R}^Q; G) = \Big\{ \sum_{i=1}^{I} \theta_i [\![f_i(\Delta_i)]\!] : I < \infty, \theta_i \in G, \Delta_i \text{ is a } k\text{-simplex in } \mathbb{R}^k,$$
$$\text{and } f_i : \Delta_i \to \mathbb{R}^Q \text{ is Lipschitz}\Big\}.$$

Where $[\![f_i(\Delta_i)]\!]$ is the rectifiable current $[\![f_i(\Delta_i), \eta_i, 1]\!]$.

Given $T \in \mathcal{L}_k(\mathbb{R}^Q; G)$, we can always find a representation $\sum_i \theta_i [\![f_i(\Delta_i)]\!]$ of T such that $f_i(\Delta_i)$ are non-self-overlapping and we define the *mass norm of* T by $\mathbf{M}(T) = \sum_i \|\theta_i\| \mathbf{M}([f_i(\Delta_i, \eta_i, 1)])$. In other words:

$$\mathbf{M}(T) = \inf\Big\{ \sum_i \|\theta_i\| \mathbf{M}([f_i(\Delta_i, \eta_i, 1)]) : T = \sum_i \theta_i [\![f_i(\Delta_i)]\!] \Big\}.$$

We define the group of the *rectifiable flat k-chains*, $\mathcal{R}_k(\mathbb{R}^Q; G)$, as the M-completion of $\mathcal{L}_k(\mathbb{R}^Q; G)$. In the space $\mathcal{L}_k(\mathbb{R}^Q; G)$ we can define a boundary map $\partial : \mathcal{L}_k(\mathbb{R}^Q; G) \to \mathcal{L}_{k-1}(\mathbb{R}^Q; G)$ as in the singular homology theory. The *flat norm* $\mathcal{F}$ of $T \in \mathcal{L}_k(\mathbb{R}^Q; G)$ is defined by

$$\mathcal{F}(T) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) : T = A + \partial B, A \in \mathcal{L}_k(\mathbb{R}^Q; G), B \in \mathcal{L}_{k+1}(\mathbb{R}^Q, G)\}.$$

The $\mathcal{F}$-completion of $\mathcal{L}_k(\mathbb{R}^Q; G)$ is the group of *flat k-chains with coefficients in G*, denoted by $\mathcal{F}_k(\mathbb{R}^Q; G)$. The boundary map $\partial : \mathcal{L}_k(\mathbb{R}^Q; G) \to \mathcal{L}_{k-1}(\mathbb{R}^Q; G)$ admits a unique continuous extension $\partial : (\mathcal{F}_k(\mathbb{R}^Q; G), \mathcal{F}) \to (\mathcal{F}_{k-1}(\mathbb{R}^Q; G), \mathcal{F})$.

The group of *integral flat k-chains in* $\mathbb{R}^Q$ *with coefficients in* G is given by

$$\mathcal{I}_k(\mathbb{R}^Q; G) = \{T \in \mathcal{R}_k(\mathbb{R}^Q; G) : \partial T \in \mathcal{R}_{k-1}(\mathbb{R}^Q; G)\}.$$

The groups above can be defined on M taking the groups on $\mathbb{R}^Q$ restricted to the elements with support on M.

The min-max constructions considered by Almgren [5] for equivalents classes are made for relative cycles that are integral cycles. We follow Liokumovich, Marques and Neves [22] approach, although we could work more generally with rectifiable cycles not necessary integrals (see Li and Zhou [19, Section 3.1]). The *space of flat cycles* restricted to integral chains is given by

$$\mathcal{Z}_k(M; G) = \{T \in \mathcal{I}_k(M; G) : \partial T = 0\}.$$

We also consider the space

$$\mathcal{Z}_k(M, \partial M; G) = \{T \in \mathcal{I}_k(M; G) : \mathrm{spt}(\partial T) \subset \partial M\}$$

and the s*pace of equivalent classes of relative cycles* by the quotient

$$\mathcal{Z}_{k,rel}(M, \partial M; G) = \mathcal{Z}_k(M, \partial M; G)/\mathcal{I}_k(\partial M; G).$$

Then the support of a *class* $[T] \in \mathcal{Z}_{k,rel,}(M, \partial M; G)$ is given by $\mathrm{spt}([T]) = \bigcap_{T \in [T]} \mathrm{spt}(T)$.

Also the mass norm and flat norm in the space of relative cycles are defined, respectively, by

$$\mathbf{M}([T]) = \inf_{T \in [T]} \mathbf{M}(T), \quad \mathcal{F}([T]) = \inf_{T \in [T]} \mathcal{F}([T]),$$

for $[T] \in \mathcal{Z}_{k,rel,}(M, \partial M; G)$.

We will consider the space of relative flat cycles endowed with the flat norm $\mathcal{F}$. When it is endowed with the topology of the mass norm, we denote it by $\mathcal{Z}_k(M, \partial M; \mathbf{M}; G)$.

Note that each $[T] \in \mathcal{Z}_{k,rel}(M, \partial M; G)$ has a unique *canonical representative* k-chain $T^0 \in [T]$ such that $T^0 \llcorner \partial M = 0$, in particular, $\mathbf{M}([T]) = \mathbf{M}(T^0)$ and $\mathrm{spt}([T]) = \mathrm{spt}(T^0)$, see [19, Lemma 3.3]. Also, it follows that $\mathcal{F}([T]) \leqslant \mathbf{M}([T])$. This canonical representative is obtained take $T^0 = S \llcorner (M \backslash \partial M)$ for any $S \in [T]$. We will keep the notation simple and we denote $[T]$ by T and for us $G = \mathbb{Z}_2$.

When $\partial M = \varnothing$ we have that $\mathcal{Z}_{k,rel}(M, \partial M; G)$ is identical to $\mathcal{Z}_k(M; G)$.

The theorem below is an extension of the classical Federer-Fleming Compactness Theorem for integral flat cycles.

Theorem 2.7. (Compactness Theorem [22, Theorem 2.3]). *The set*

$$\{T \in \mathcal{Z}_{k,rel}(M, \partial M; G) : \boldsymbol{M}(T) \leqslant L\}$$

is compact in the flat topology for all $L > 0$.

Another extended theorem is the follow about the lower semicontinuity of the mass norm with respect to flat topology in $\mathcal{Z}_{k,rel}(M, \partial M; G)$.

Theorem 2.8. (Lower Semicontinuity, [22, Proposition 2.4]). *If $\{T_i\} \subset \mathcal{Z}_{k,rel}(M, \partial M; G)$ is a sequence converging to T in the flat topology then*

$$\boldsymbol{M}(T) \leqslant \liminf_{i \to \infty} \boldsymbol{M}(T_i).$$

2.3 Varifolds

Here we follow Pitts [31] and Simon [35].

Let $U \subset \mathbb{R}^Q$ be an open set. The *Grassmannian bundle of k-planes over U* is defined by $\mathcal{G}_k(U) = \{(x,S) : x \in U \text{ and } S \text{ is an } k\text{-dimensional subspace of } \mathbb{R}^Q\}$.

The space of *k-dimensional varifolds* in U, $\mathcal{V}_k(U)$, is such that $V \in \mathcal{V}_k(U)$ if V is a Radon measure on $\mathcal{G}_k(U)$ and is equipped with the weak topology. The *weight* $\|V\|$ of a varifold $V \in \mathcal{V}_k(U)$ is a Radon measure given by

$$\|V\|(A) = (V \llcorner A)(\mathcal{G}_k(U)) = V\big(\mathcal{G}_k(U) \cap \{(x,S) : x \in A\}\big) \quad \text{whenever } A \subset U.$$

The *mass* of $V \in \mathcal{V}_k(U)$ is defined by $\mathbf{M}(V) = \|V\|(U)$, and the *support* of V, $\operatorname{spt}\|V\|$, is the smallest closed subset $C \subset \mathbb{R}^Q$ such that $V \llcorner (\mathbb{R}^Q \backslash C) = 0$.

Given a varifold $V \in \mathcal{V}_k(U)$ and $x \in \operatorname{spt}\|V\|$, we denote by $\operatorname{VarTan}(V,x) \subset \mathcal{V}(\mathbb{R}^Q)$ as the set of *the varifold tangents of V at x* [35, Definition 42.3], which is a natural generalization of tangent planes for smooth surfaces. When the density is positive, this set is not empty [3, 3.4].

The *Pitts' $\mathbf{F}$-metric* on $\mathcal{V}_k(U)$ is such that $\mathbf{F} : \mathcal{V}_k(U) \times \mathcal{V}_k(U) \to \mathbb{R}^+$ and

$$\mathbf{F}(V,W) = \sup\{V(f) - W(f) : f \in C^0_c(\mathcal{G}_k(U)), |f| \leqslant 1, \operatorname{Lip}(f) \leqslant 1\}.$$

This metric induces precisely the usual weak topology on the set $\{V \in \mathcal{V}_k(U) : \|V\|(U) \leqslant L\}$, for each constant $L > 0$.

If $R \subset U$ is k-rectifiable and θ is a $\mathcal{H}^k$-integrable non-negative function on R, we denote by $V = \upsilon(R,\theta) \in \mathcal{V}_k(U)$ to be the *rectifiable k-varifold associated to R with multiplicity function θ* defined by

$$\int_{\mathcal{G}_k(U)} \varphi(x,S)\, dV(x,S) = \int_R \theta(x)\varphi(x,T_xR)\, d\mathcal{H}^k \quad \forall \varphi \in C^0_c(\mathcal{G}_k(U)),$$

where T_xR is the approximate tangent space of R in x. If θ assumes only positive integers values, we say that $\upsilon(R,\theta)$ is an *integral varifold*.

We denote by $R\mathcal{V}_k(U)$ and $I\mathcal{V}_k(U)$ the spaces of *k-dimensional rectifiable varifolds in U* and *k-dimensional integral varifolds in U*, respectively.

Given an integer-multiplicity rectifiable k-current $T = [R,\theta,\eta]$, we can define an associated k-varifold $|T| = \upsilon\big(R,\theta\big) \in I\mathcal{V}_k$. If T is a k-chain with coefficients in $\mathbb{Z}_2$, $|T| \in I\mathcal{V}_k$ denotes the varifold induced by the support of T.

Conversely, given a k-varifold $V \in \mathcal{V}_k(U)$ we can define the natural k-current

$$|V|(\omega) = \int_{\mathcal{G}_k(U)} \omega_x(S)\ dV(x,S) \quad \omega \in \mathcal{D}^k.$$

Let $U, U' \subset \mathbb{R}^Q$ be open sets. Given $V \in \mathcal{V}_k(U)$ and a C^1 map $f : U \to U'$ with $f|(\text{support}(V))$ proper. The *pushfoward varifold of V by f*, denoted by $f_\sharp V$, is defined by: for any Borel set $A \subset \mathbb{R}^Q$,

$$f_\sharp V = \int_{F^{-1}(A)} J_S f(x)\ dV(x,S),$$

where $F(x,S) = (f(x), df_x(S))$ and $J_S f(x) = \sqrt{\det[(df_x|_S)^* \circ (df_x|_S)]}$ (see [35, §39]).

2.3.1 Varifolds in manifolds with boundary

We denote by $R\mathcal{V}_k(M)$ and $I\mathcal{V}_k(M)$ the sets of k-dimensional rectifiable varifolds and integral varifolds in $\mathbb{R}^Q$ with support contained in M and equipped with the weak topology, respectively. Also $\mathcal{V}_k(M)$ will be the closure of $R\mathcal{V}_k(M)$ in the weak topology.

Given $V \in \mathcal{V}_k(M)$, let $X \in \mathfrak{X}_{tan}(M)$ be a generator of a one-parameter family of diffeomorphisms ϕ_t of $\mathbb{R}^Q$ with $\phi_0(M) = M$, we have that the *first variation of V along the vector field X* is given by ([35, 39.2])

$$\delta V(X) := \frac{d}{dt}\bigg|_{t=0} \mathbf{M}((\phi_t)_\sharp V).$$

Definition 2.9. Let $U \subset M$ be a relatively open subset. A varifold $V \in \mathcal{V}_k(M)$ is said to be *stationary in U with free boundary* if $\delta V(X) = 0$ for any $X \in \mathfrak{X}_{tan}(M)$ compactly supported in U.

Note that a free boundary minimal submanifold is also stationary with free boundary. However, the reverse may not be true. Indeed, as we commented at the introduction, any constant multiple of a connected component of ∂M is a stationary varifold with free boundary, even though it can be nothing like a minimal hypersurface in M.

By the relative topology we will consider the *k-dimensional density*, $\Theta^k(V,x)$, of a stationary varifold $V \in \mathcal{V}_k(M)$ as the density restricted to M, that is, given $x \in M$, we take

$$\Theta^k(V,x) := \lim_{\rho \to 0} \frac{\|V\|(B_\rho(x) \cap M)}{\rho^k |B^k|},$$

where $|B^k|$ is the volume of the k-dimensional unit Euclidean ball B^k. For a fixed x, define the function

$$\Theta^k_x(V,\rho) := \frac{\|V\|(B_\rho(x)\cap M)}{\rho^k|B^k|}.$$

In the case $\partial M = \varnothing$, we have $B_\rho(x) \subset M$ and it is known that the function above for stationary varifolds satisfies the monotonicity formula [35, Sections 17 and 40]: $\Theta^k_x(V,\rho)$ is non-decreasing in ρ. A similar monotonicity formula for the case $\partial M \neq \varnothing$ can be found in [19, Theorem 2.3]. Also, it is well known that any tangent varifold of a stationary varifold is a stationary Euclidean cone and $\Theta^k_x(C,\rho) = \Theta^k(V,x)$ for any $C \in \mathrm{VarTan}(V,x)$ and for all $\rho > 0$. We will write that as $\Theta^k_x(C,\infty) = \Theta^k(V,x)$.

2.4 Cell Complex

Here we follow the notations of Marques and Neves [23, Section 7].

For each $m \in \mathbb{N}$, we denote by I^m the *m-dimensional cube* $I^m = [0,1]^m$.

For each $j \in \mathbb{N}$, $I(1,j)$ denotes the cube complex on I^1 whose 1-cells and 0-cells (vertices) are, respectively,

$$[0,3^{-j}],[3^{-j},2\cdot 3^{-j}],\cdots,[1-3^{-j},1] \quad \text{and} \quad [0],[3^{-j}],\cdots,[1-3^{-j}],[1].$$

We denote by $I(m,j)$ the cell complex on I^m:

$$I(m,j) = I(1,j)\otimes\cdots\otimes I(1,j) \quad (m \text{ times}).$$

Then a cell $\alpha = \alpha_1\otimes\cdots\otimes\alpha_m$ of $I(m,j)$ is a q-cell if and only if α_i is a cell of $I(1,j)$ for each i, and $\sum_{i=1}^m \dim(\alpha_i) = q$. By abuse of notation a q-cell α will be identified with its support: $\alpha_1\times\cdots\times\alpha_m$.

We denote by $I(n,j)_p$ the set of all p-cells in $I(m,j)$.

Let $X \subset I(m,j)$ be a subcomplex (cubical subcomplex), the cube complex $X(j)$ is the union of all cells of $I(m,j)$ whose support is contained in some cell of X. The set of all q-cells in $X(j)$ is denoted by $X(j)_q$ and we say that two vertices $x,y \in X(j)_0$ are adjacent if there is a 1-cell $\alpha \in X(j)_1$ containing both x and y.

Given $i,j \in \mathbb{N}$, the map $\mathbf{n}(i,j) : X(i)_0 \to X(j)_0$ is defined as follows: $\mathbf{n}(i,j)(x)$ is the element in $X(j)_0$ that is closest to x.

Definition 2.10. The *fineness* of a map $\phi : X(j)_0 \to \mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2)$ is defined by

$$\mathbf{f}(\phi) = \sup\{\mathbf{M}(\phi(x) - \phi(y)); x, y \text{ are adjacent vertices in } X(j)_0\}.$$

2.5 Homotopy

In the following we define the notions of homotopy with fineness follows Marques and Neves [25].

Definition 2.11. Let $\phi_i : X(k_i)_0 \to \mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2)$, $i = 1, 2$. We say that ϕ_1 is *X-homotopic to ϕ_2* in $\mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)$ *with fineness δ* if we can find $k_3 \in \mathbb{N}$ and a map

$$\psi : I(1, k_3)_0 \times X(k_3)_0 \to \mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2),$$

such that

(i) $\mathbf{f}(\psi) < \delta$;

(ii) $\psi([i-1], x) = \phi_i(\mathbf{n}(k_3, k_i)(x))$, $i = 1, 2$.

Definition 2.12. We say that a sequence of mappings

$$\phi_i : X(k_i)_0 \to \mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2)$$

is an *$(X, \mathbf{M})$-homotopy sequence of mappings into $\mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)$*, if each ϕ_i is X-homotopy to ϕ_{i+1} in $\mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)$ with fineness δ_i and

(i) $\lim_{i\to\infty} \delta_i = 0$;

(ii) $\sup\{\mathbf{M}(\phi_i(x)); x \in X(k_i)_0, i \in \mathbb{N}\} < \infty$.

Definition 2.13. Let $S^1 = \{\phi_i^1\}_{i\in\mathbb{N}}$ and $S^2 = \{\phi_i^2\}_{i\in\mathbb{N}}$ be $(X, \mathbf{M})$-homotopy sequences of mappings into $\mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)$. We say that S^1 is *homotopic with* S^2 if there exists a sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that

(i) ϕ_i^1 is X-homotopic to ϕ_i^2 in $\mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)$ with fineness δ_i;

(ii) $\lim_{i\to\infty} \delta_i = 0$.

The relation "homotopy with" is an equivalence relation on $(X, \mathbf{M})$-homotopy sequences of mappings into $\mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)$. So we can define the $(X, \mathbf{M})$-*homotopy classes of mappings into* $\mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)$. We denote the set of all equivalence classes by $[X, \mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)]^\sharp$.

2.6 Min-Max Definitions

Let $\Pi \in [X, \mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)]^\sharp$. For each $S = \{\phi_i\}_{i \in \mathbb{N}} \in \Pi$, we define

$$\mathbf{L}(S) = \limsup_{i \to \infty} \max\{\mathbf{M}(\phi_i(x)); x \in \operatorname{dmn}(\phi_i)\}.$$

Definition 2.14. The *width* of Π is defined by

$$\mathbf{L}(\Pi) = \inf\{\mathbf{L}(S) : S \in \Pi\}.$$

We say that $S \in \Pi$ is a *critical sequence* for Π if $\mathbf{L}(S) = \mathbf{L}(\Pi)$, and the *critical set* $\mathbf{C}(S)$ of a critical sequence S is given by

$$\mathbf{C}(S) = \mathbf{K}(S) \cap \{V \in \mathcal{V}_n(M) : \|V\|(M) = \mathbf{L}(S)\},$$

where

$$\mathbf{K}(S) = \Big\{V \in \mathcal{V}_n(M) : V = \lim_{j \to \infty} |\phi_{i_j}(x_j)| \text{ as varifolds, for some subsequence}$$

$$\{\phi_{i_j}\} \subset S \text{ and } x_j \in \operatorname{dmn}(\phi_{i_j})\Big\}.$$

From Marques and Neves [23, Theorem 15.1] (See also Pitts [31, 4.1 (4)]) we know that there exist critical sequences for each class Π, and from [31, 4.2 (2)], $\mathbf{C}(S)$ is compact and non-empty.

Definition 2.15. Let $X \subset I^m$ be a cubical subcomplex. For $p \in \mathbb{N}$ and $0 \leqslant k < n+1$, we say that a continuous map in the flat topology $\Phi : X \to \mathcal{Z}_{k,rel}(M, \partial M; \mathbb{Z}_2)$ is a *p-sweepout* if the p-th cup power of $\Phi^*(\overline{\lambda}_k)$ is nonzero in $H^{p(n+1-k)}(X; \mathbb{Z}_2)$, where $\overline{\lambda}_k$ is the generator of $H^{n+1-k}(\mathcal{Z}_{k,rel}(M, \partial M; \mathbb{Z}_2); \mathbb{Z}_2)$.

Obviously, a $(p+1)$-sweepout is a p-sweepout.

Remark 2.16. In the above definition we used the fact that $H^{n+1-k}(\mathcal{Z}_{k,rel}(M,\partial M;\mathbb{Z}_2);\mathbb{Z}_2)$ has a generator $\overline{\lambda}_k$. This is consequence of Almgren Isomorphism Theorem [5], Hurewicz Theorem and Universal Coefficients Theorem (see Liokumovich, Marques and Neves [22, Section 2.5]). More precisely,

$$H^{n+1-k}(\mathcal{Z}_{k,rel}(M,\partial M;\mathbb{Z}_2);\mathbb{Z}_2) = \mathbb{Z}_2 = \{0,\overline{\lambda}_k\}.$$

Also, the pullback map Φ^* is the map

$$\Phi^* : H^{n+1-k}(\mathcal{Z}_{k,rel}(M,\partial M;\mathbb{Z}_2);\mathbb{Z}_2) \to H^{n+1-k}(X;\mathbb{Z}_2),$$

defined by $\Phi^*(\sigma)(\cdot) = \sigma(\Phi(\cdot))$ for $\sigma \in H^{n+1-k}(\mathcal{Z}_{k,rel}(M,\partial M;\mathbb{Z}_2);\mathbb{Z}_2)$.

Remark 2.17. Geometrically, if a map $\Phi : X \to \mathcal{Z}_{k,rel}(M,\partial M;\mathbb{Z}_2)$ is a p-sweepout, then for each point $x \in M$ there exists a cycle $\Phi(t)$, for some t, such that it passes through the point x ([22], Section 2.5). For the case $k = n$, another consequence noted by Gromov [13] and a consequence of Lusternik-Schnirelmann Theory [9, p. 2-3] is that if $U_1,\ldots,U_p$ are disjoint open sets in M, then there exists a cycle $\Phi(s)$, for some s, that separates each U_i into two sets of equal area. In particular and roughly speaking, for every set of points $\{x_1,\ldots,x_p\} \subset M$, we can find a cycle $\Phi(r)$, for some r, so that $\{x_1,\ldots,x_p\} \subset \Phi(r)$.

Definition 2.18. A flat continuous map $\Phi : X \to \mathcal{Z}_{k,rel}(M,\partial M;\mathbb{Z}_2)$ has *no concentration of mass* if

$$\lim_{r\to 0}\sup\{\|\Phi(x)\|(B_r(p)\backslash\partial M) : x \in \mathrm{dmn}(\Phi), p \in M\} = 0.$$

The set of all p-sweepouts with no concentration of mass is denoted by $\mathcal{P}_p^k(M)$.

Definition 2.19. The *p-width of M* (of dimension k) is given by

$$\omega_p^k(M) = \inf_{\Phi\in\mathcal{P}_p^k(M)} \sup\{\mathbf{M}(\Phi(x)) : x \in \mathrm{dmn}(\Phi)\}.$$

For the case $k = n$, we just write $\mathcal{P}_p(M)$ and $\omega_p(M)$.

As showed by Gromov [14] and later by Guth [15], there exist positive constants c and C depending only on (M^{n+1}, g) such that

$$cp^{\frac{1}{n+1}} \leqslant \omega_p(M) \leqslant Cp^{\frac{1}{n+1}} \tag{3}$$

for all $p \in \mathbb{N}$.

2.7 Min-Max Theorem

In this section we will talk about the existence of $\mathbb{Z}_2$-almost minimizing varifolds with free boundary. The main theorem will be consequence of Pitts [31], Marques and Neves [23] and Liokumovich, Marques and Neves [22].

Definition 2.20. Let $U \subset M$ be a relatively open subset, we say that a varifold $V \in \mathcal{V}_k$ is $\mathbb{Z}_2$-*almost minimizing in* U *with free boundary* if for every $\epsilon > 0$ we can find $\delta > 0$ and $T \in \mathcal{Z}_{k,rel}(M, \partial M; \mathbb{Z}_2)$ with $\mathbf{F}(V, |T|) < \epsilon$ and such that the following property holds true: if $T = T_0, T_1, \ldots, T_m \in \mathcal{Z}_{k,rel}(M, \partial M; \mathbb{Z}_2)$ with

- support$(T - T_i) \subset U$ for $i = 1, \ldots, m$;
- $\mathcal{F}(T_i - T_{i-1}) \leqslant \delta$ for $i = 1, \ldots, m$ and
- $\mathbf{M}(T_i) \leqslant \mathbf{M}(T) + \delta$ for $i = 1, \ldots, m$

then $\mathbf{M}(T_m) \geqslant \mathbf{M}(T) - \epsilon$.

Roughly speaking, it means that we can approximate V by a varifold induced from a current T such that for any deformation of T by a discrete family supported in U, and with the mass not increasing too much (parameter δ), then at the end of deformation the mass cannot be deformed down too much (parameter ϵ).

A varifold $V \in \mathcal{V}_k(M)$ is said to be $\mathbb{Z}_2$-*almost minimizing in annuli with free boundary* if for each $p \in$ support(V) there exists $r > 0$ such that V is $\mathbb{Z}_2$-almost minimizing in the annuli $M \cap A_{s,r}(p) = M \cap B_r(p) \backslash \overline{B}_s(p)$ for all $0 < s < r$. If $p \notin \partial M$, we require that $r < \text{dist}(p, \partial M)$. By Proposition 2.5 (iii), this definition with respect to $A_{s,r}(p)$ or $\mathcal{A}_{s,r}(p)$ is equivalent.

As shown in Pitts ([31], Theorem 3.3), if $V \in \mathcal{V}_k(M)$ is $\mathbb{Z}_2$-almost minimizing in a relatively open set $U \subset M$ with free boundary, then V is stationary in U with free boundary.

When $\partial M = \varnothing$, we do not need use the expression 'with free boundary'.

In the following, we talk about the existence of such varifolds. First of all, we can do a *tightening process* to a critical sequence $S \in \Pi$ so that every $V \in \mathbf{C}(S)$ becomes a stationary varifold with free boundary. For the case without boundary this is proved in [31] and [23]. For the case with boundary, we discuss the proof below.

Theorem 2.21. *Let $\Pi \in [X, \mathcal{Z}_{n,rel}(M, \partial M; \boldsymbol{M}; \mathbb{Z}_2)]^\sharp$. For each critical sequence $S^* \in \Pi$, there exists another critical sequence $S \in \Pi$ such that $\boldsymbol{C}(S) \subset \boldsymbol{C}(S^*)$ and each $V \in \boldsymbol{C}(S)$ is stationary in M with free boundary.*

Proof. The proof of this result is essentially the same as [23, Theorem 8.5]. The only modifications are the use of Theorems 13.1 and 14.1 of [23], as noted in [19, Theorem 4.17]. In place of [23, Theorem 14.1] we use [22, Theorem 2.11]; and a compatible version of [23, Theorem 13.1] follows from [22, Lemma A. 1] in the same way that the [23, Theorem 13.1] follows from [23, Lemma 13.4].

□

With the tightening process above we can prove the existence of a $\mathbb{Z}_2$-almost minimizing varifolds with free boundary such that it reaches the width of a chosen $(X; \mathbf{M})$-homotopy class $\Pi \in [X, \mathcal{Z}_{n,rel}(M, \partial M; \mathbf{M}; \mathbb{Z}_2)]^\sharp$. For $\partial M = \varnothing$, it was first proved by Pitts [31, Theorem 4.10] with maps in cubical domains for $1 \leqslant k \leqslant n$ and later by Marques and Neves [25, Theorem 2.9] for cubical subcomplex domains when $k = n$. For the case with boundary, a version for cubical domains was proved by Li and Zhou [19, Theorem 4.21]. We present below a version for the case $\partial M \neq \varnothing$ and take maps in cubical subcomplex domains when $k = n$.

Theorem 2.22. *For any $\Pi \in [X, \mathcal{Z}_{n,rel}(M, \partial M; \boldsymbol{M}; \mathbb{Z}_2)]^\sharp$, there exists $V \in I\mathcal{V}_n(M)$ such that*

(i) $\|V\|(M) = \boldsymbol{L}(\Pi)$;

(ii) V is stationary in M with free boundary;

(iii) V is $\mathbb{Z}_2$-almost minimizing in small annuli with free boundary.

Proof. Using the previous theorem, we can follow the same procedure in the proof of [31, Theorem 4.10] (see also [19, Theorem 4.21]). To prove that V is $\mathbb{Z}_2$-almost minimizing in small annuli with free boundary on ∂M, just do as in the proof of [19, Theorem 4.21]. □

When $\partial M = \varnothing$, as noted by Aiex [2, Theorem 4.4], if we take $\Phi : X \to \mathcal{Z}_k(M; \mathbb{Z}_2)$ a p-sweepout with no concentration of mass and define Π_Φ the class of all flat continuous maps $\Psi : X \to \mathcal{Z}_k(M; \mathbb{Z}_2)$ with no concentration of mass that are flat homotopic to Φ and define

$$L(\Pi_\Phi) = \inf_{\Psi \in \Pi_\Phi} \sup_{x \in X} \mathbf{M}(\Psi(x)),$$

then the same conclusion of the theorem above is still true.

We present the most useful result for us of this section.

Corollary 2.23. *For $p \in \mathbb{N}$ and each $\epsilon > 0$, we can find $V \in I\mathcal{V}_n(M)$ such that*

(i) $\omega_p(M) \leqslant \|V\|(M) \leqslant \omega_p(M) + \epsilon$;

(ii) V is stationary in M with free boundary;

(iii) V is $\mathbb{Z}_2$-almost minimizing in small annuli with free boundary.

Proof. Note that the results in Section 3.3 of Marques and Neves [25] can be extended for compact manifolds (with or without boundary) from the results in Section 2 of Liokumovich, Marques and Neves [22]. So we can use the results from Section 3.3 of [25].

By definition we can find $\Phi : X \to \mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2)$ a p-sweepout with no concentration of mass such that $\sup\{\mathbf{M}(\Phi(x)) : x \in \mathrm{dmn}(\Phi)\} \leqslant \omega_p(M) + \epsilon$. From 3.6 of [25] there exists an $(X, \mathbf{M})$-homotopy sequence of mappings $S = \{\phi_i\}_{i\in\mathbb{N}} \in \Pi$ associated. By 3.7 and 3.9 (ii) of [25] we can extended this sequence to a sequence $\{\Phi_i\}_{i\in\mathbb{N}}$ of maps continuous in the mass norm and homotopics to Φ in the flat topology for large i. Moreover

$$L(\Pi) \leqslant L(S) = \lim_{i\to\infty} \sup \sup\{M(\Phi_i(x)) : x \in X\} \leqslant \sup_{x\in X} M(\Phi(x)).$$

As Φ is a p-sweepout and Φ_i is flat continuous and homotopic to Φ for large i, then Φ_i is also a p-sweepout for large i with no concentration of mass by 3.5 of [25]. Also from 3.9 (i) of [25] we have that $\{\widetilde{\Phi}_i\}_{i\in\mathbb{N}} \in \mathcal{P}_p(M)$ for each $\widetilde{S} = \{\widetilde{\phi}_i\}_{i\in\mathbb{N}} \in \Pi$ and for large i. Together with the above inequality we conclude that

$$\omega_p(M) \leqslant L(\Pi) \leqslant \sup_{x\in X} M(\Phi(x)) \leqslant \omega_p(M) + \epsilon.$$

The remaining items are deduced from the above theorem.

∎

3 ONE DIMENSIONAL STATIONARY VARIFOLDS

In this section we will talk about results related to one dimensional stationary varifolds. In that case, we have a classical result about interior regularity due to Allard-Almgren [4] which says that those varifolds are geodesic networks. Our main result about free boundary geodesic networks is used to give an upper bound for the density. When M is the unit ball $B^2 \subset \mathbb{R}^2$, or a planar full ellipse $E^2 \subset \mathbb{R}^2$ sufficiently close to B^2, we will be able to classify the free boundary geodesic networks, provided they are $\mathbb{Z}_2$-almost minimizing in annuli and has mass bounded by 6. Also we will prove our main theorem about regularity (Main Theorem A), which it will be important to show that the varifold obtained in the Min-Max Theorem is a free boundary geodesic network.

3.1 Free Boundary Geodesic Networks

We now define in this section certain 1-dimensional stationary varifolds whose support is given by geodesic segments. We follow the notations of Aiex [2].

Definition 3.1. Let $U \subset M$ be a relatively open set. A varifold $V \in I\mathcal{V}(M)$ is called a *free boundary geodesic network* in U if there exist geodesic segments $\{\alpha_1, \ldots, \alpha_l\} \subset \mathring{M}$ and $\{\theta_1, \ldots, \theta_l\} \subset \mathbb{Z}_+$ such that

(i) $V \llcorner U = \sum_{i=1}^{l} v(\alpha_i \cap U, \theta_i)$;

(ii) The *set of junctions* is the set $\Sigma_V = \cup_{i=1}^{l}(\partial\alpha_i) \cap U$. If $p \in \Sigma_V$, then there exist geodesic segments $\{\alpha_{i_1}, \ldots, \alpha_{i_m}\} \subset \mathring{M}$ for some $m = m(p)$ and we require $m \geqslant 3$ for $p \in \mathring{M}$. Each geodesic segment is parameterized by arc-length with initial point p,

$$\sum_{k=1}^{m} \theta_{i_k}\dot{\alpha}_{i_k}(0) \;=\; 0, \quad \text{if} \quad p \in \Sigma_V \cap \mathring{M}, \quad \text{and} \tag{4}$$

$$\sum_{k=1}^{m} \theta_{i_k}\dot{\alpha}_{i_k}(0) \;\perp\; \partial M, \quad \text{if} \quad p \in \Sigma_V \cap \partial M. \tag{5}$$

A junction $p \in \Sigma_V \cap \mathring{M}$ is said to be *singular in* $\mathring{M}$ if there exist at least two geodesic segments with $\theta_{i_k}\dot{\alpha}_{i_k}(0) \neq -\theta_{i_{k'}}\dot{\alpha}_{i_{k'}}(0)$, and *regular in* $\mathring{M}$ otherwise. In other words, an interior regular junction belong to the intersection of longer geodesic segments. When $p \in \Sigma_V \cap \partial M$, we said that it is regular if $\dot{\alpha}_{i_k}(0) \perp \partial M$ for every α_i such that $p \in \alpha_i$. A *triple junction* is a point $p \in \Sigma_V$ such that it belongs to exactly three geodesic segments with multiplicity 1 each. Obviously a triple junctions is not regular in $\mathring{M}$.

We can deduce the following properties as did in [2]:

Proposition 3.2. (Proposition 3.2 and Corollaries 3.3 and 3.4 of [2]). *Let V be as above.*

(i) *V is stationary in U;*

(ii) $\Theta^1(V,x) = \sum_{k=1}^{s} \frac{\theta_{i_k}}{2}$ *for* $x \in \bigcap_{k=1}^{s} v(\alpha_{i_k} \cap U, \theta_{i_k})$;

(iii) *If $\Theta^1(V,x) < 2$ for all $x \in spt\|V\| \cap \mathring{M}$, then every $p \in \Sigma_V \cap \mathring{M}$ is a triple junction;*

(iv) *If $\Theta^1(V,x) \leqslant 2$ for all $x \in spt\|V\| \cap \mathring{M}$, then either $\Sigma_V \cap \mathring{M}$ contains a triple junction or all junctions are regular in $\mathring{M}$ and the geodesic segments of each junction have multiplicity one;*

(v) *If $\Theta^1(V,x) \leqslant 1$ for $x \in spt\|V\| \cap \partial M$, then a junction on x is given by a geodesic segment with multiplicity one or two and orthogonal to ∂M, or by two geodesic segments with multiplicity one each and with the same angles with respect to ∂M.*

3.2 Upper Bound for the Density

Now we prove the main property for us about free boundary geodesic networks. We do similar results to Proposition 3.6 and Theorem 3.7 from Aiex [2], but in a different way.

Lemma 3.3. *Consider M^2 a compact region in $\mathbb{R}^2$ with non-empty boundary and $V \in I\mathcal{V}_1(M)$ a free boundary geodesic network. For each $p_i \in \Sigma_V \cap \partial M$, let*

$$F_i = \sum_{k=1}^{m} \theta_{i_k} \dot{\alpha}_{i_k}(0),$$

as in the Definition 3.1 (ii).

(ii) *For $M^2 = B^2$ we have that $\sum_i |F_i| = \|V\|(M)$.*

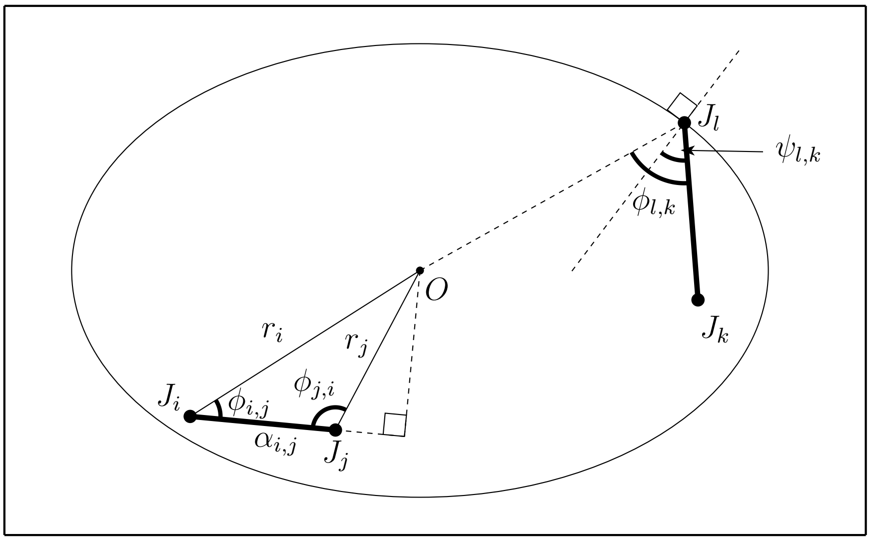
(ii) *If $M^2 \to B^2$ in the manifold sense, then $\sum_i |F_i| \to \|V\|(M)$. More precisely, let $\varepsilon > 0$. For M^2 sufficiently close to B^2, depending only on a parameter $C > 0$, we have*

$$\left| \|V\|(M) - \sum_i |F_i| \right| < \varepsilon \tag{6}$$

for every free boundary geodesic network $V \in I\mathcal{V}_1(M)$ with $\|V\|(M) < C$.

Proof. Denote by J_i the i-th junction of V. Each segment of V is determined by two junctions J_i and J_j. Denote that segment by $\alpha_{i,j}$. Also we have two angles $\phi_{i,j}$ and $\phi_{j,i}$ associated, which are the intern angles of the respective junctions in the triangle given by the origin O and the junctions J_i and J_j (see Figure 1). Note that, in those notations, we have $\alpha_{i,j} = \alpha_{j,i}$.

Figure 1



Source: Own construction

Suppose that each segment $\alpha_{i,j}$ has multiplicity $\theta_{i,j}$, for some $\theta_{i,j} \in \mathbb{N}$. Note that

$$\sum_{i,j} \theta_{i,j} \cos(\phi_{i,j}) = 0, \tag{7}$$

for all i or j fixed such that $J_i \in \mathring{M}$, or $J_j \in \mathring{M}$, respectively. Indeed, $\cos(\phi_{i,j})$ is the projection of $\dot{\alpha}_{i,j}(0)$ (recall that $|\dot{\alpha}_{i,j}(0)| = 1$) on the straight line that passes through O and J_i. So, the condition (4) concludes.

Let r_i be the distance from the origin O to the the junction J_i. The length $|\alpha_{i,j}|$ of each $\alpha_{i,j}$ is given by

$$|\alpha_{i,j}| = (r_i \cos(\phi_{i,j}) + r_j \cos(\phi_{j,i})).$$

Using this and (7),

$$\|V\|(M) = \sum_{i,j} \theta_{i,j}|\alpha_{i,j}| = \sum_{i,j} \theta_{i,j} r_i \cos(\phi_{i,j}) = \sum_{k,l} \theta_{k,l} r_l \cos(\phi_{l,k}) \text{ for } J_l \in \partial M. \tag{8}$$

For $J_l \in \partial M$, let $\psi_{l,k}$ be the angle between the segment $\alpha_{l,k}$ and the normal to ∂M at J_l (see Figure 1). Thus,

$$|F_l| = \sum_k \theta_{l,k} \cos(\psi_{l,k}).$$

If $M = B^2$, then $r_l = 1$ and $\phi_{l,k} = \psi_{l,k}$ for all l, k such that $J_l \in \partial M$. So,

$$\|V\|(B^2) = \sum_l |F_l|.$$

For M close to B^2, we have $r_l \approx 1$ and $\phi_{l,k} \approx \psi_{l,k}$ for all l, k such that $J_l \in \partial M$. Then,

$$\|V\|(M) = (1 \pm \varepsilon_1) \sum_l |F_l|$$

for some $\varepsilon_1 > 0$, which depends only on the approximation $M \approx B^2$. Note that, as $\|V\|(M) < C$, we see by the above expression that $\sum_l |F_l| < C_1 = C_1(C)$ for some constant $C_1 > 0$. Therefore, taking $\varepsilon_1 = \varepsilon/C_1$, we obtain

$$\left| \|V\|(M) - \sum_l |F_l| \right| \leqslant \varepsilon_1 \sum_l |F_l| < \varepsilon_1 C_1 = \varepsilon.$$

$\square$

Note that, when V is a smooth free boundary submanifold of M, the above result is an immediate consequence of the Divergence Theorem. Furthermore, we can find the same formula (8) applying the Divergence Theorem at each segment of V along of the position vector.

From the above theorem we have the following upper bound for the density. Compare with Aiex [2, Theorem 3.7].

Theorem 3.4. *Let $V \in I\mathcal{V}_1(B^2)$ be a free boundary geodesic network. If $\|V\|(B^2) < m+1$ for some integer positive m, then*

(i) $\Theta^1(V, x) \leqslant \dfrac{m}{2}$ *for all $x \in int(B^2)$.*

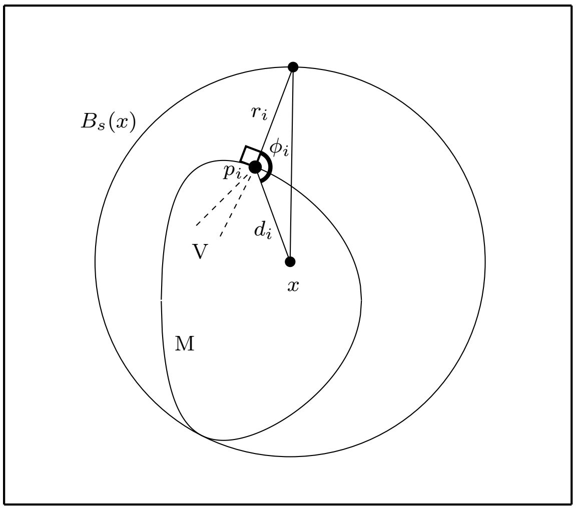
(ii) $\Theta^1(V, x) \leqslant \dfrac{m}{4}$ *for all $x \in \partial B^2$.*

Futhermore, let $V \in I\mathcal{V}_1(M^2)$ *be a free boundary geodesic network and* $0 < \varepsilon < m+1$ *such that* $\|V\|(M^2) < m+1-\varepsilon$*, where* M^2 *is a compact region of* $\mathbb{R}^2$ *with convex boundary, sufficiently close to* B^2 *and satisfying (6) for* $C = m+1-\varepsilon$*. Then, the conclusions (i) and (ii) above are still true for* M^2 *in place of* B^2*.*

Proof. We can extend the geodesic network $V \in I\mathcal{V}_1(M^2)$ for a varifold $\widetilde{V} \in \mathcal{V}_1(\mathbb{R}^2)$ (not necessary a geodesic network) in the followings way: for each $p_i \in \Sigma_V \cap \partial M$ we take the semi-straight line r_i starting in p_i with direction $-F_i = -\sum_{k=1}^{m} \theta_{i_k}\dot{\alpha}_{i_k}(0)$ and multiplicity $|F_i|$. Then $\widetilde{V}$ is a stationary varifold on $\mathbb{R}^2$.

Let $x \in \mathrm{spt}\|V\|$, $d_i = \mathrm{dist}(p_i, x)$, and ϕ_i be the angle between $\overrightarrow{p_i - x}$ and r_i at p_i. Also, consider d_0 as the minimal value of s such that $B_s(x)$ contains M. See Figure 2.

Figure 2



Source: Own construction

As M has convex boundary, we have that each r_i does not intersect M in $\mathbb{R}^2 \backslash M$. So, for $s \geqslant d_0$ we obtain the following expression for the monotonicity formula:

$$\begin{aligned}\Theta^1_x(s) &= \frac{\|\widetilde{V}\|(B_s(x))}{2s} \\ &= \frac{\|V\|(M) + \sum_i |F_i| \left(d_i\cos(\phi_i) + \sqrt{d_i^2\cos^2(\phi_i) - (d_i^2 - s^2)}\right)}{2s}\end{aligned}$$

Consider first the case M^2 close to B^2 and $\|V\|(M) < m+1-\varepsilon$. By the expression above $\Theta^1_x(s) \to \frac{1}{2}\sum_i |F_i|$ for $s \to \infty$. And by the above theorem we know that $\sum_i |F_i|$ is close to $\|V\|(M)$, so for s large

$$\Theta^1(\tilde{V}, x) \leqslant \Theta^1_x(s) < \frac{m+1-\varepsilon}{2} < \frac{m}{2} + \frac{1}{2}.$$

Where we used the fact that the function $\Theta^1_x(s)$ is non-decreasing for each x fixed, so $\Theta^1(x, \tilde{V}) \leqslant \Theta^1_x(s)$ for all $s > 0$.

(i) If $x \in \operatorname{spt}\|V\| \cap \mathring{M}$, then $\Theta^1(x, V) = \Theta^1(x, \tilde{V}) < m/2 + 1/2$. By the property (ii) of 3.2, we actually have that $\Theta^1(x, V) \leqslant m/2$.

(ii) If $x \in \operatorname{spt}\|V\| \cap \partial M$, then $\Theta^1(x, V) = \Theta^1(x, \tilde{V})/2 < m/4 + 1/4$. Again, by the property (ii) of 3.2, we actually have that $\Theta^1(x, V) \leqslant m/4$.

For the case $M^2 = B^2$ and $\|V\|(B^2) < m + 1$, just take $\varepsilon = 0$ in the above expressions.

□

3.3 Free Boundary Geodesic Networks with Low Mass

In this section we describe the free boundary geodesic networks with low mass when M^2 is the unit ball B^2 or a full ellipse E^2 sufficiently close to B^2.

We will need the following important theorem from Aiex [2]:

Theorem 3.5. ([2], Theorem 4.13). *Given $V \in I\mathcal{V}_1(M)$ a geodesic network with free boundary and $p \in \Sigma_V \cap \mathring{M}$. If V is $\mathbb{Z}_2$-almost minimizing in annuli with free boundary at p, then*

$$\Theta^1(V, p) \in \mathbb{N}.$$

Remark 3.6. This theorem is proved in [2] for closed manifolds but it still holds for manifolds with boundary, provided that p is a junction of V in the interior of M, as established above. Indeed, the only care are the use of the Constancy Theorem, the Compactness Theorem for relative flat cycles and the lower semi-continuity of $\mathbf{M}$. The last two are proved for the context with boundary in Liokumovich, Marques and Neves [22]: Theorem 2.3 and Proposition 2.4, respectively. Also, if we consider the open sets in Section 4 from [2] as open sets in the interior of M, then the Constancy Theorem is still applicable in the situations that appear in [2, Section 4]. Finally, to prove the Theorem 4.13, this author uses the theorems of Section 4 for any sufficiently small open ball containing p.

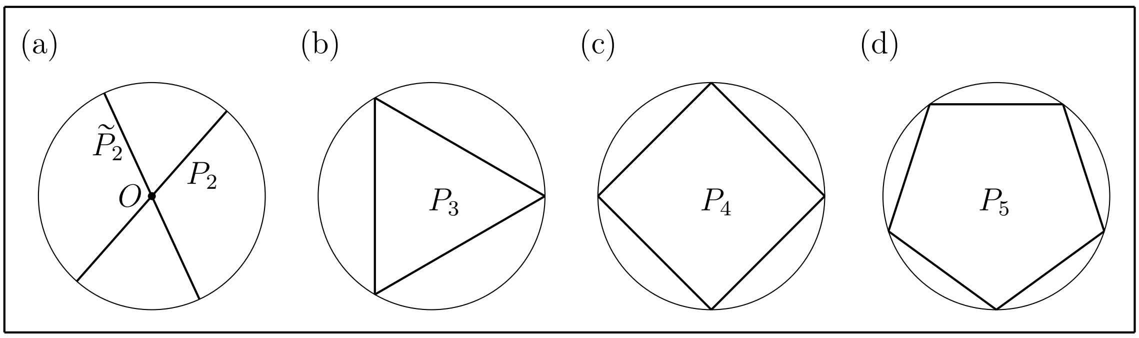
For $k \geqslant 3$, let P_k be a regular k-sided polygon inscribed in the unit circle. Here P_2 will be a diameter of the unit ball B^2. Clearly, for $k \geqslant 2$ we have that P_k and $\widetilde{P}_k$ are distinguished by a rotation. Also, note that $P_2, P_2 + \widetilde{P}_2, P_3, P_4$ and P_5 are the only ones such that the perimeter $|P_k|$ is smaller than 6, whose values are $2, 4, 3\sqrt{3}, 4\sqrt{2}$ and $10\sin(\sqrt{\pi/5})$, respectively.

More generally, we will use the term *closed k-polygon*, $k \geqslant 3$, to denote a periodic billiard trajectory inside of a domain Ω with boundary $\partial\Omega$, which is obtained by k reflexions on Ω.

Theorem 3.7. *Let $V \in I\mathcal{V}(B^2)$ be a free boundary geodesic network and $\mathbb{Z}_2$-almost minimizing in annuli with free boundary in B^2. If $0 < \|V\|(B^2) < 6$, then $V = P_k$ for some $k = 2, \ldots, 5$, or $V = P_2 + \widetilde{P}_2$.*

Proof. From Theorem 3.4 we know that $\Theta^1(V, x) \leqslant 5/2$ for $x \in \mathring{M}$, and $\Theta^1(V, x) \leqslant 5/4$ for $x \in \partial M$. Now using Proposition 3.2 (ii) and Theorem 3.5, we deduce that $\Theta^1(V, x) = 1$ or 2 for $x \in \mathring{M}$, and $\Theta^1(V, x) = 0.5$, or 1 for $x \in \partial M$. Therefore, Proposition 3.2 (iv) says that all junctions of V in $\mathring{M}$ are regular and the geodesic segments of each junction have multiplicity one. Also, Proposition 3.2 (iv) and (v) say that each segment of V has multiplicity one or two and touches ∂M orthogonally, or has multiplicity one and touches ∂M making a reflexion generating another segment with multiplicity one also. As $\|V\|(B^2) < 6$, we claim that: if V touches ∂M orthogonally in some point, we have that V will be a diameter ($V = P_2$) or two diameters ($V = P_2 + \widetilde{P}_2$) of B^2 (see Figure 3 (a)); and if V does a reflexion in some point of ∂M, then V will be a regular polygon P_k for some $k = 3, 4$ ou 5 (see Figure 3 (b), (c) and (d)).

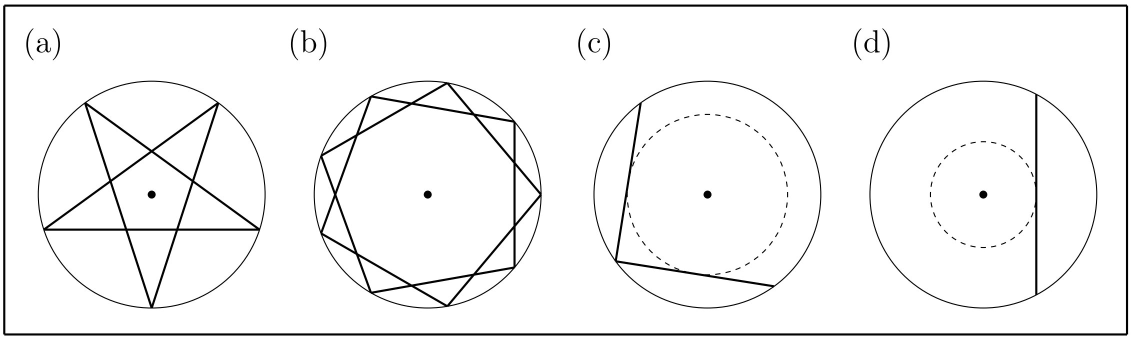
Figure 3



Source: Own construction

In fact, for $k \geqslant 6$ we have $|P_k| \geqslant |P_6| = 6$. From five reflexions, we can have non-convex closed polygons with self-intersection as in the Figure 4 (a) and (b). As a closed k-polygon in B^2 has all the sides with the same length, we have that each side is tangent to the same circle C_k concentric with ∂B^2 (see Figure 4 (c)), in particular the perimeter of that k-polygon is at least $|C_k|$. A regular convex k-polygon in B^2 gives a unique round around C_k, and a non-convex (closed) k-polygon in B^2 gives at least two rounds around C_k. So, if the radius of C_k is bigger than 0.5, then the perimeter of a non-convex k-polygon is bigger than $2 \cdot 2 \cdot 0.5\pi > 6$. Otherwise, if the radius of C_k is less or equal to 0.5 (see Figure 4 (d)), then each side of a non-convex k-polygon is bigger than 1.7, and so the perimeter is bigger than $5 \cdot 1.7 > 6$, since for non-convex k-polygons we have $k \geqslant 5$. Therefore, there is not candidates for V in the set of non-convex k-polygons.

Figure 4



Source: Own construction

$\square$

Remark 3.8. In the theorem above the case $V = P_2 + \widetilde{P}_2$ is the only possibility for V to have multiplicity, which occurs if $P_2 = \widetilde{P}_2$.

A similar result holds replacing B^2 by a planar full ellipse E^2 sufficiently close of B^2. We denote by P_k^E, for $k \geqslant 3$, the closed convex polygon (not necessary regular) inscribed in E^2 defined by k reflexions on k different points of ∂E^2. Here, P_2^E will be the smallest or the largest diameter of E^2. Cleary, as E^2 is close to B^2, we have that P_k^E is close to P_k. That polygons P_k^E are examples of closed billiard trajectories in ellipses (Poncelet polygons). We will see more properties of polygons in the proof below.

Corollary 3.9. *Let E^2 be a planar full ellipse and $0 < R < 6$ be a real number. For E^2 sufficiently close to B^2, depending only in the parameter R, the following is true: if $V \in I\mathcal{V}_1(E^2)$ is a free boundary geodesic network such that it is $\mathbb{Z}_2$-almost minimizing in annuli with free boundary in E^2 and $0 < \|V\|(E^2) < R$, then $V = P_k^E$ for some $k = 2, \ldots, 5$, or $V = P_2^E + \widetilde{P}_2^E$.*

Proof. Consider E^2 an planar full ellipse which boundary is given by an ellipse $x^2/a^2 + y^2/b^2 = 1$ for $a > b$ with focus $F_1, F_2 \in Ox$ (see Figure 5 (a)). Let d and D the values of the smallest and largest diameters of E^2, respectively. So, $d = 2b$ and $D = 2a$. Also, here we will always consider E^2 sufficiently close to B^2, so $d \approx D \approx 2$, for example.

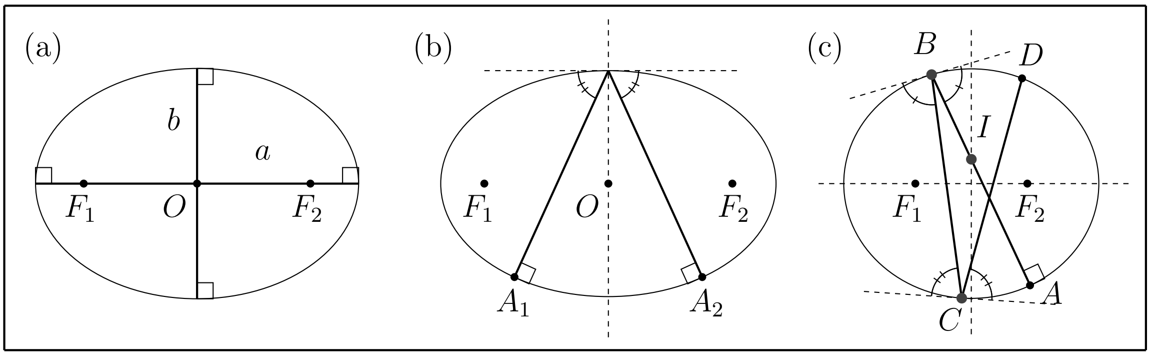
Take $E^2 \approx B^2$ such that it satisfies (6) from Lemma 3.3 for $C = 6 - R$. So we can use the Theorem 3.4 for $C = 6 - R = 5 + 1 - \varepsilon$ for some $\varepsilon > 0$ and, as in the proof of the theorem above, applying Proposition 3.2 and Theorem 3.5 we get: all junctions of V in $\mathring{M}$ are regular and the geodesic segments of each junction have multiplicity one; each segment of V has multiplicity one or two and touches ∂M orthogonally, or has multiplicity one and touches ∂M making a reflexion generating another segment with multiplicity one also. Therefore, V could be the smallest or the largest diameters of E^2, since they touch ∂E^2 orthogonally (see Figure 5 (a)). Also, V could be $P_2^E + \widetilde{P}_2^E$, and then $\|V\|(E^2) = 2d, d + D$ or $2D$, since $d \approx D \approx 2$ and $\|V\| < R < 6$. We could have V as in the Figure 5 (b): a segment touching ∂E^2 orthogonally at A_1, making a reflexion at $(0, b) \in \partial E^2$ with respect to ∂E^2, generating another segment which will be touch orthogonally ∂E^2 at $A_2 = (-x(A_1), y(A_1))$. This can always happen for $a >> b$. However, for E^2 close to B^2 we have $a, b \approx 1$, and the cases $V = P_2^E$ or $V = P_2^E + \widetilde{P}_2^E$ are the only possibility such that V touches ∂M orthogonally in some point and $\|V\|(E^2) < R$. Indeed, considering $(a\sin(t), b\sin(t))$ the polar coordinates on ∂E^2 for $t \in [0, 2\pi)$, and taking without loss of generality (by symmetry) $A \in \partial E^2$ such that $A = (a\sin(t_A), b\sin(t_A))$ for $t_A \in (3/4\pi, 2\pi)$, we claim that if a segment $\overline{AB} \subset E^2$ touches ∂M orthogonally at A, then $\overline{AB}$ is not orthogonal to ∂E^2 at $B \in \partial E^2$, and the segment $\overline{BC}$, reflexion of $\overline{AB}$ at B, is also not orthogonal to ∂E^2 at C (See Figure 5 (c)). In fact, the equation of the straight line which is perpendicular to ∂E^2 at A is given by

$$y = \frac{a\tan(t_A)}{b}x + \sin(t_A)\left(b - \frac{a^2}{b}\right).$$

We will require that $a^2 \leqslant 2b^2$, which is a condition satisfied for $E^2 \approx B^2$, since $a, b \approx 1$. Take $x = 0$ above, we see that this condition implies that $y(I) < b$, where I is the intersection of $\overline{AB}$ with Oy (Figure 5 (c)).

In an ellipse we have the following fact: if $\overline{AB}$ is orthogonal to ∂E^2 at A, then $\overline{AB}$ bisects the angle $\angle F_1AF_2$. In particular, $\overline{AB}$ passes through $\overline{F_1F_2}$ and, since $y(I) < b$, we have $t_B \in (\pi/2, \pi)$, where $B = (a\sin(t_B), b\cos(t_B))$. Also, if $\overline{AB}$ was orthogonal to

Figure 5



Source: Own construction

∂E^2 at B, the equation of the straight line through B will be the same as above, which implies that $\tan(t_A) = \tan(t_B)$ and $\sin(t_A) = \sin(t_B)$, which contradicts the fact that $t_A \in (3/4\pi, 2\pi)$ and $t_B \in (\pi/2, \pi)$. So, $\overline{AB}$ is not orthogonal to ∂E^2 at B and there exists $\overline{BC}$, reflexion of $\overline{AB}$ at B. Remember from billiard theory in ellipses that, if a segment in E^2 passes through $\overline{F_1F_2}$, then all the segments in that billiard trajectory (segments reflected at ∂E^2) pass through $\overline{F_1F_2}$ (see for example [18, Theorem 4]). So $\overline{BC}$ passes through $\overline{F_1F_2}$.

Supposing that $\overline{BC}$ is orthogonal to ∂E^2 at C, the same argument applied for $\overline{AB}$ could be apply to $\overline{BC}$ to get that $t_C \in (3/4\pi, 2\pi)$ and $t_C \neq t_A$, where $C = (a\sin(t_C), b\cos(t_C))$. Taking the equations of the straight lines that are perpendicular to A and C, respectively, we would have that they intersect at $B = (a\cos(t_B), b\sin(t_B))$, then

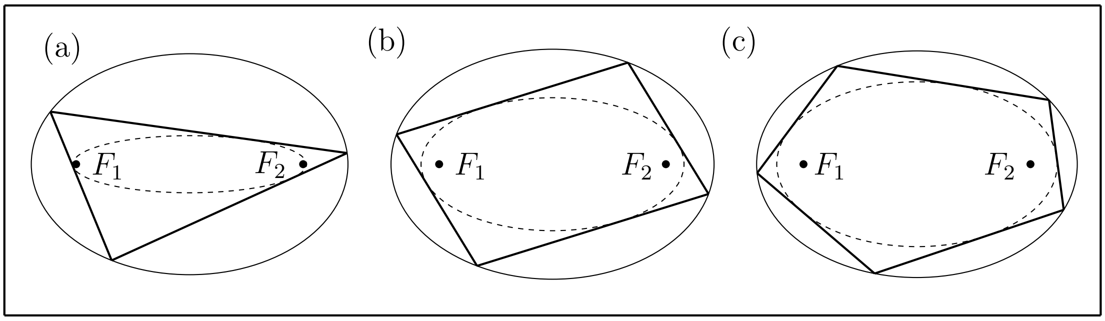
$$\frac{a\tan(t_A)}{b} a\cos(t_B) + \sin(t_A)\left(b - \frac{a^2}{b}\right) = \frac{a\tan(t_C)}{b} a\cos(t_B) + \sin(t_C)\left(b - \frac{a^2}{b}\right)$$

$$\therefore\ \frac{a^2}{b}\cos(t_B)(\tan(t_A) - \tan(t_C)) + \left(\frac{b^2 - a^2}{b}\right)(\sin(t_A) - \sin(t_C)) = 0.$$

As $t_A, t_C \in (3/4\pi, 2\pi)$, $t_A \neq t_B$ and $\cos(t_B), (b^2 - a^2) < 0$, we see that the left side of the last expression above is not equal to zero. Therefore, $\overline{BC}$ is not perpendicular to ∂E^2 at C and there will be another reflexion $\overline{CD}$ at C (see Figure 5 (c)).

Supposing that $V \neq P_2^E$ and $V \neq P_2^E + \widetilde{P}_2^E$, there exists a segment of V which is not orthogonal to ∂E^2. By the properties of billiard trajectories in ellipses ([18, Theorem 4]), we know that if some segment passes through $\overline{F_1F_2}$, then all the segments will be pass through $\overline{F_1F_2}$. Moreover, by the above arguments, there will be at least three segments that pass through $\overline{F_1F_2}$. We consider $E^2 \approx B^2$ such that the length of each of these segments is at least $R/3$, since the length of each of these segments tending to 2 as E^2 tends to B^2 and $R < 6$. So, there is not a segment of V that passes through $\overline{F_1F_2}$ for every V with $\|V\|(E^2) < R$. Since any segment that is orthogonal to ∂E^2 have to pass through F_1F_2, we see that V is a closed k-polygon. Also, as the segments of V do not pass through $\overline{F_1F_2}$, we have by [18, Theorem 4] that all the segments of that closed k-polygon is tangent to the same ellipse $\partial(E'_k)$, where E'_k is a planar full ellipse inside of E^2 and with the same focus of E^2. The Poncelet theorem (see for instance [32, Theorem 4]) says that if a closed k-polygon $\mathcal{P}$ is tangent to $\partial E'_k$, then any other polygon $\mathcal{Q}$ that is tangent to $\partial E'_k$ is also a closed k-polygon with the same perimeter of $\mathcal{P}$. Moreover, for each $k \geqslant 3$ there exists a unique E'_k such that all the convex closed k-polygons P_k^E have its trajectory tangent to $\partial(E'_k)$ (see for example [33, Section 4]). In particular for a fixed $k \geqslant 3$, all the polygons P_k^E have the same perimeter. In the Figure 6 (a), (b) and (c) we see examples of P_3^E, P_4^E and P_5^E, respectively. Also, we see E'_1, E'_2 and E'_3, respectively.

Figure 6



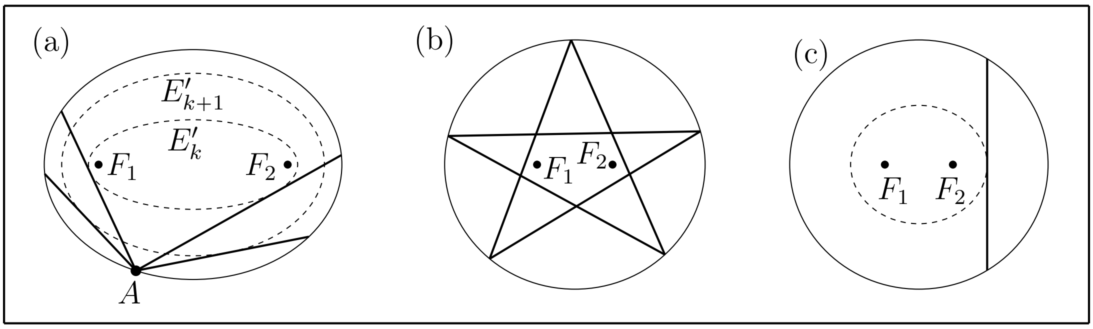
Source: Own construction

Note that, due to uniqueness existence of each E'_k for P_k^E, we have that $|\partial(E'_k)| < |\partial(E'_{k+1})|$ for E'_k and E'_{k+1} associated to P_k^E and P_{k+1}^E, respectively. Indeed, let $A \in \partial E^2$ and take a segment starting at A and tangent to $\partial(E'_k)$. This segment generates a billiard trajectory that is always tangent to $\partial(E'_k)$ ([18, Theorem 4]), so by the Poncelet theorem this billiard trajectory is also a k-polygon $\widetilde{P}_k^E$. In the same way we obtain a $(k+1)$-polygon $\widetilde{P}_{k+1}^E$ through A and tangent to $\partial(E'_{k+1})$. As the intern angle of $\widetilde{P}_{k+1}^E$ at A is bigger than the intern angle of $\widetilde{P}_k^E$ at A, we see that E'_{k+1} is bigger than E'_k (see Figure 7 (a)).

We also require $E^2 \approx B^2$ such that $2D < |P_3^E|, |P_4^E|, |P_5^E| < 6$, $|P_k^E| > R$ for $k = 6, \cdots, 11$, and $|\partial(E'_{12})| > 6$ since $|P_k| > |P_6| = 6$ for $k > 6$, and $|C_{12}| > 6$. As $|P_k^E| > |\partial(E'_k)| > |\partial(E'_{k-1})|$, we have that $|P_k^E| > R$ for all $k \geqslant 6$. So the only candidates for V in the set of closed convex k-polygons are P_3^E, P_4^E and P_E^5 (Figure 6).

Finally, with the same argument as in the proof of the theorem above, we see that all closed non-convex polygons in E^2 have perimeter bigger than 6, so there is not candidates for V in that set. Indeed, the estimates there are strictly, so for $E^2 \approx B^2$ and replace C_k by E'_k with average radius approximately 0.5, we conclude that the perimeters are bigger than 6. Compare the Figures 4 (c) and 7 (c). In the Figure 7 (b) below, we have an example of a closed non-convex 5-polygon.

Figure 7



Source: Own construction

□

Remark 3.10. As we saw above, for a fixed $k \geqslant 3$, all the polygons P_k^E have the same perimeter. So, the corollary above says more: there are only six possibles values for $\|V\|(E^2) : d, d + D, 2D, |P_3^E|, |P_4^E|$ or $|P_5^E|$.

3.4 Replacement and Regularity

The regularity of one dimensional stationary integral varifolds for open sets was proven by Allard and Almgren ([4], Section 3). As noted by Aiex ([2], Theorem 3.5), the regular structure described in [4] is exactly our definition of geodesic network. Precisely:

Theorem 3.11. ([4]; [2], Theorem 3.5). *Let M be a Riemannian manifold, $U \subset \mathring{M}$ open and $K \subset U$ compact. If $V \in I\mathcal{V}_1(M)$ is a stationary varifold in U, then $V \llcorner K$ is a geodesic network.*

Definition 3.12. Let $T \in \mathcal{Z}_k(M; \mathbb{Z}_2)$ and $U \subset M$ be a relatively open subset. We say that T is *locally mass minimizing in U* if for every $p \in \operatorname{spt}(T) \cap U$ there exists $r_p > 0$ such that $B_{r_p}(p) \cap M \subset U$ and for all $S \in \mathcal{Z}_k(M; \mathbb{Z}_2)$ such that $\operatorname{spt}(T - S) \subset B_{r_p}(p) \cap M$ we have

$$\mathbf{M}(S) \geqslant \mathbf{M}(T).$$

The definition above is equivalent if we take Fermi half-balls $\widetilde{\mathcal{B}}^+_{r_p}(p)$ instead of Euclidean balls $B_{r_p}(p)$ restricted to M.

The following theorem is about replacements of almost minimizing varifolds, which is one of the most important property of this kind of varifolds. Roughly speaking, we can replace an almost minimizing varifold V by another almost minimizing varifold V^*, which has better regularity properties.

Theorem 3.13. *Let $U \subset M$ be a relatively open set, $K \subset U$ compact and $V \in \mathcal{V}_k(M)$ be an $\mathbb{Z}_2$-almost minimizing varifold in U with free boundary. There exists $V^* \in \mathcal{V}_k(M)$ such that*

(i) $V^* \llcorner (M\backslash K) = V \llcorner (M\backslash K)$;

(ii) $\|V^*\|(M) = \|V\|(M)$;

(iii) V^* *is $\mathbb{Z}_2$-almost minimizing in U with free boundary;*

(iv) $V^* \in I\mathcal{V}(U \cap \mathring{M})$;

(v) $V^* \llcorner U = \lim_{i\to\infty} |T_i|$ *as varifolds for some* $\{T_i\} \in \mathcal{Z}_{k,rel}(M, (M\backslash U) \cup \partial M; \mathbb{Z}_2)$ *such that each T_i^0 is locally mass minimizing in $int_M(K)$.*

Proof. The proof follows as in Proposition 5.3 from [19], replacing Lemmas 3.10 and 3.7 by Theorem 2.3 and Proposition 2.4 from [22], respectively. See also Theorems 3.11 and 3.13 from [31] to get (iv) from (iii). □

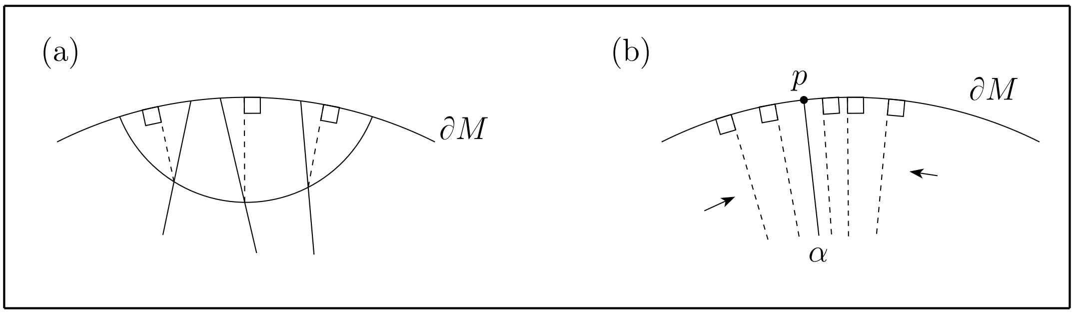
The varifold V^* in the above theorem is called of a *replacement of V in K*.

In the next lemma we will prove a weak regularity for varifolds that are $\mathbb{Z}_2$-almost minimizing with free boundary in an open set. As we will see in the Theorem A, we have actually a more strong regularity in the one-dimensional case. Restrict to $\mathring{M}$, that strong regularity was expected since V is $\mathbb{Z}_2$-almost minimizing, thus V is stationary and then holds the interior regularity (Theorem 3.11).

Lemma 3.14. (Weak Regularity of Replacements) *Under the same hypotheses of Theorem 3.13, assume that ∂M is strictly convex and take V an one-dimensional varifold. Then $spt\|V^*\| \cap int_M(K)$ is a free boundary geodesic network (possibly infinite) without junctions in $(K \cap \mathring{M})\backslash \partial_{rel} K$, such that each geodesic segment has to touch $\partial_{rel} K \cup \partial M$, and they can only touch $\partial M \cap int_M(K)$ orthogonally.*

Proof. From [2, Proposition 4.6] we know that if T is an one-cycle that is locally mass minimizing in an open set $W \subset \mathring{M}$ and $Z \subset W$ is compact, then $T \llcorner Z$ is a geodesic network (finite) such that each geodesic segment has endpoints in ∂Z and those segments do not intersect each other, since the coefficients are in $\mathbb{Z}_2$. So, for a relatively compact $K \subset M$ and T_i^0 locally mass minimizing in $\text{int}_M(K)$ (as in Theorem 3.13, (v)), we have that $T_i^0 \llcorner \text{int}_M(K)$ is given by geodesic segments without intersecting each other and each segment that touches $\partial M \cap \text{int}_M(K)$ is orthogonal to ∂M, in particular $|T_i^0| \llcorner \text{int}_M(K)$ is a free boundary geodesic network (possibly infinite). Indeed, as T_i^0 is locally mass minimizing, each segment of T_i^0 that touches ∂M is locally the shortest path, so it is orthogonal to ∂M (see Figure 8 (a)).

Figure 8



Source: Own construction

As V^* is given by the limit as in the Theorem 3.13 (v), we using the properties from geodesics and the fact that ∂M is strictly convex to see that V^* is given by geodesic segments that can only touch $\partial M \cap \text{int}_M(K)$ orthogonally. In fact, the strict convexity implies that any geodesic segment can only touch ∂M only in its endpoints. So, given a limit segment α that touches ∂M at $p \in \partial M$, we have that there exists a sequence of geodesic segments converging to α such that each segment of that sequence touches ∂M in a neighborhood of p. Therefore, α will be orthogonal to ∂M at p (see Figure 8 (b)). Moreover, as the segments of T_i^0 do not intersect each other, we get that in the limit the geodesic segments of V^* can have multiplicity, but two distinct segments can not intersect each other.

$\square$

We called the result above of weak regularity, because we do not know if the number of geodesic segments could be infinite. However, the above result is true for any codimension.

Let $p \in \mathbb{R}^2$ and let $C \in \mathcal{V}(\mathbb{R}^2)$ be a varifold such that $C = \sum_{i=1}^{l} v(r_i, m_i)$ for some

$l, m_1, \cdots, m_l \in \mathbb{N}$, and each r_i is some semi-straight line from p. We will call C of a *cone with vertex at* p.

The next proposition will be very important to prove our main result about regularity (Theorem A). Essentially, we will use it to glue replacements on overlapping annuli (see Step 2 in the proof of Theorem A)

Proposition 3.15. *Let $C \in I\mathcal{V}_1(\mathbb{R}^2)$ be a stationary cone with vertex at the origin $0 \in \mathbb{R}^2$, and such that it is $\mathbb{Z}_2$-almost minimizing in $B_2(0) \subset \mathbb{R}^2$. Then $C = v(r, m)$, for some r a straight line passing through the origin 0, and for some $m \in \mathbb{N}$.*

Proof. We will use the following fact: as C is $\mathbb{Z}_2$-almost minimizing in $B_2(0)$, then each varifold tangent is also an integral and stationary varifold on $T_x\mathbb{R}^2 \equiv \mathbb{R}^2$ such that it is $\mathbb{Z}_2$-almost minimizing in any bounded open subset of $\mathbb{R}^2$ [31, Theorems 3.11 and 3.12(1)].

As C is a cone, we have $C = \sum_{i=1}^{l} v(r_i, m_i)$ for some $l, m_1, \cdots, m_l \in \mathbb{N}$, and each r_i is some semi-straight line from the origin 0. So,

$$\Theta^1(C, 0) = \Theta^1_0(C, \infty) = \sum_{i=1}^{l} \frac{m_i}{2}.$$

By Theorem 3.5 we actually have that $\Theta^1(C, 0) = k$ for some $k \in \mathbb{N}$.

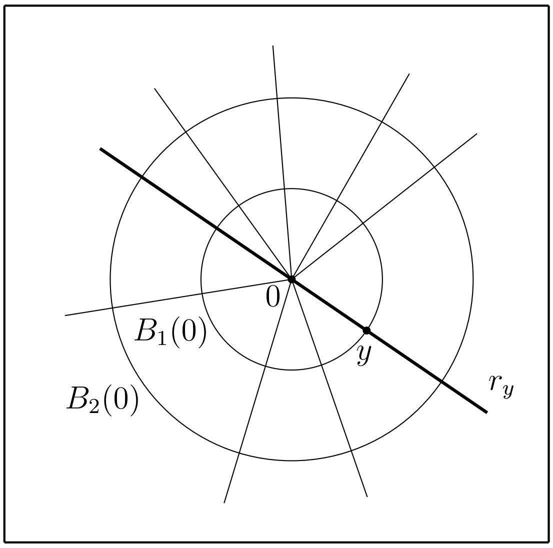
It is sufficient to prove the result restricted to $B_2(0)$, that is, to prove that C is a straight line with possibly with multiplicity on $B_2(0)$. We will prove it by induction on $\Theta^1_0(C, \infty)$. Indeed, the result is obvious for $\Theta^1_0(C, \infty) \leqslant 1$. Suppose that $\Theta^1_0(C, \infty) = k + 1$, and that the result is true for $\Theta^1_0(C, \infty) \leqslant k$, $k \geqslant 1$. Let C^* be a replacement of C on $\overline{B}_1(0)$, we know that C^* is integral, stationary and $\mathbb{Z}_2$-almost minimizing in $B_2(0)$. Also, $\|C^*\|(\overline{B}_2(0)) = \|C\|(\overline{B}_2(0))$, $C^* \llcorner (\overline{B}_2(0) \backslash \overline{B}_1(0)) = C \llcorner (\overline{B}_2(0) \backslash \overline{B}_1(0))$, and together with the monotonicity formula we get

$$\Theta^1_y(\mathrm{VarTan}(C^*, y), \infty) = \Theta^1(C^*, y) \leqslant \Theta^1_y(C^*, \infty) = \Theta^1_0(C, \infty),$$

where $y \in \partial B_1(0) \cap \operatorname{spt}\|C^*\|$.

We have two cases: $\Theta^1(C^*, y) = \Theta^1_y(C^*, \infty)$ for some $y \in \partial B_1(0) \cap \operatorname{spt}\|C^*\|$, or $\Theta^1(C^*, y) < \Theta^1_y(C^*, \infty)$ for any $y \in \partial B_1(0) \cap \operatorname{spt}\|C^*\|$. In the first case, C^* will be a cone with vertex at y. This implies that $C = mr_y$, for some $m \in \mathbb{N}$ and r_y is the straight line that passes through y and the origin, since $C^* \llcorner (\overline{B}_2(0) \backslash \overline{B}_1(0)) = C \llcorner (\overline{B}_2(0) \backslash \overline{B}_1(0))$ (see Figure 9).

Figure 9



Source: Own construction

In the second case, $\Theta^1_x(\mathrm{VarTan}(C^*, y), \infty) \leqslant k$ for any $y \in \partial B_1(0)$, since $\Theta^1_0(C, \infty) = k + 1$. So, as $\mathrm{VarTan}(C^*, y)$ is $\mathbb{Z}_2$-almost minimizing in $B_2(0)$, we can use the induction hypothesis for each y to get that $\mathrm{VarTan}(C^*, y) = m_y r_y$ for some $m_y \in \mathbb{N}$ and r_y is the straight line that passes through y and the origin. As C^* does not have junctions in $B_1(0)$ (previous Lemma), we conclude that $C^* = C$ and then $C = mr$ for some $m \in \mathbb{N}$, and r is a straight line through the origin (see Figure 9).

$\square$

The next result is a boundary maximum principle for stationary varifolds with free boundary in codimension one case.

Theorem 3.16. (Boundary maximum principle [19, Theorem 2.5]). *Let $U \subset M^{n+1}$ be a relatively open subset and $V \in \mathcal{V}_n(M)$ be stationary with free boundary in U. Suppose $N \subset\subset U$ is a relatively open connected subset in M such that*

(i) $\partial_{rel} N$ meets ∂M orthogonally, if $\partial_{rel} N \cap \partial M \neq \emptyset$;

(ii) N is relatively strict convex in M;

(iii) $spt\|V\| \subset \overline{N}$.

Then we have $spt\|V\| \cap \partial_{rel} N = \emptyset$.

Proof. It follows from the interior maximum principle of White [38, Theorem 1] and the boundary maximum principle for stationary varifolds with free boundary of Li-Zhou [20, Theorem 1.4]. □

Now we will prove our main theorem about regularity of stationary $\mathbb{Z}_2$-almost minimizing with free boundary.

Main Theorem A. *Let M^2 be a compact Riemannian manifold with non-empty strictly convex boundary. If $V \in I\mathcal{V}_1(M)$ is a stationary varifold with free boundary such that it is integral in M and $\mathbb{Z}_2$-almost minimizing in small anulli with free boundary, then V is a free boundary geodesic network.*

Proof. Here we follow similarly to the proof of [19, Theorem 5.2] and [8, Proposition 6.3], with the necessary modifications.

Given $p \in \operatorname{spt}\|V\| \cap \mathring{M}$, we know by the Theorem 3.11 that in a compact neighborhood around p we have that V is a geodesic network. So, assume that $p \in \operatorname{spt}\|V\| \cap \partial M$ and fix $r > 0$ such that

$$r < \frac{1}{4}\min\{r_{Fermi}, r_{am}(p), r_{ort}(p)\}, \tag{9}$$

where $r_{am}(p) > 0$ is such that V is $\mathbb{Z}_2$-almost minimizing in $\mathcal{A}_{s,t}(p)$ with free boundary for all $0 < s < t < r_{am}$, and $r_{ort}(p) > 0$ is such that two distinct geodesics that are orthogonal to $\partial M \cap \widetilde{\mathcal{B}}^+_\delta(p)$ do not intersect each other in $\widetilde{\mathcal{B}}^+_\delta(p)$ for all $0 < \delta < r_{ort}(p)$.

Note that, as a consequence of the maximum principle (Theorem 3.16), we have the following: if $W \in \mathcal{V}_1(M)$ is stationary in $\widetilde{\mathcal{B}}^+_r(p)$ with free boundary for $p \in \operatorname{spt}\|W\|$ and r as above, then

$$\operatorname{spt}\|W\| \cap \widetilde{\mathcal{S}}^+_t(p) \neq \varnothing \quad \text{ for all } 0 < t \leqslant r. \tag{10}$$

In fact, suppose that there exists $\tilde{t}_1 \in (0, r]$ such that $\operatorname{spt}\|W\| \cap \widetilde{\mathcal{S}}^+_{\tilde{t}_1}(p) = \varnothing$, then $\operatorname{spt}\|W \llcorner \widetilde{\mathcal{B}}^+_{\tilde{t}_1}(p)\| \subset \widetilde{\mathcal{B}}^+_{\tilde{t}_2}(p)$ for some $0 < \tilde{t}_2 < \tilde{t}_1$. By the maximum principle (Theorem 3.16) we conclude that $\operatorname{spt}\|W \llcorner \widetilde{\mathcal{B}}^+_{\tilde{t}_1}(p)\| \cap \widetilde{\mathcal{S}}^+_{\tilde{t}_2}(p) = \varnothing$ and we could repeat this argument indefinitely, which contradicts the fact that $p \in \operatorname{spt}\|W\|$. Using the same argument and suppose only that $W \neq 0$ in $\widetilde{\mathcal{B}}^+_r(p)$ for some $p \in \partial M$, we conclude that there exists $0 < \tilde{t} < r$ such that

$$\operatorname{spt}\|W\| \cap \widetilde{\mathcal{S}}^+_t(p) \neq \varnothing \quad \text{ for all } 0 < \tilde{t} < t \leqslant r. \tag{11}$$

Step 1: *Constructing successive replacements on two overlapping annuli.*

Fix any $0 < s < t < r$. As $r < (1/4)r_{am}$ and V is $\mathbb{Z}_2$-almost minimizing in $\mathcal{A}_{\tilde{s},r_{am}/2}(p)$ with free boundary for all $0 < \tilde{s} < t < r_{am}/2$, we can use the Theorem 3.13 to get a first replacement V^* of V on $K = \overline{\mathcal{A}_{s,t}(p)}$. The Lemma 3.14 says that

$$\Sigma_1 := \operatorname{spt}\|V^*\| \cap \mathcal{A}_{s,t}(p)$$

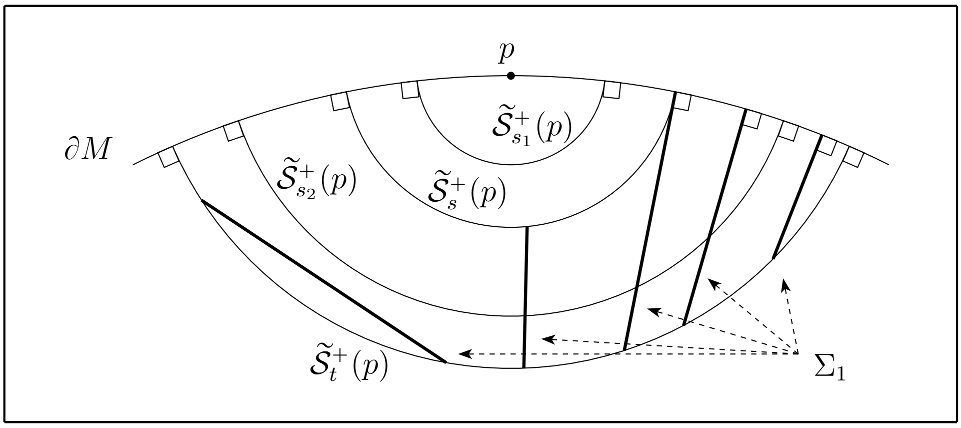
is a free boundary geodesic network (possibly infinite). By Theorem 3.13 (iv) we have that V^* is still $\mathbb{Z}_2$-almost minimizing in $\mathcal{A}_{\tilde{s},r_{am}/2}(p)$ with free boundary for all $0 < \tilde{s} < t < r_{am}/2$, so we can apply again the Theorem 3.13 to get a second replacement V^{**} of V^* on $K = \overline{\mathcal{A}_{s_1,s_2}(p)}$ for $0 < s_1 < s < s_2 < t$. Again,

$$\Sigma_2 := \operatorname{spt}\|V^{**}\| \cap \mathcal{A}_{s_1,s_2}(p)$$

is a free boundary geodesic network (possibly infinite). Let us consider the following choices: we fix any $s_1 \in (0, s)$, and we choose $s_2 \in (s, t)$ such that $\operatorname{VarTan}(\Sigma_1, x)$ is a straight line transversal to $\tilde{\mathcal{S}}^+_{s_2}(p)$ for all $x \in (\tilde{\mathcal{S}}^+_{s_2}(p)\backslash\partial M)$, and $(\alpha \cap \tilde{\mathcal{S}}^+_{s_2}(p))\backslash\partial M \neq \varnothing$ for every geodesic segment $\alpha \in \Sigma_1$. Indeed, fixing $s_2 \in (s, t)$, we know by the regularity of replacements (Lemma 3.14) that $\operatorname{VarTan}(\Sigma_1, x)$ is a straight line for any $x \in \mathcal{A}_{s,t}(p)$. Also, we will have only a finite number of geodesic segments $\{\alpha_i\} \subset \Sigma_1$ in $\mathcal{A}_{s,\tilde{t}}(p)$ for any $0 < s < \tilde{t} < t$. Too see the last one, note that any geodesic segment (with possible multiplicity) $\alpha_i \in \Sigma_1 \cap \mathcal{A}_{s,t}(p)$ has to touch $\tilde{\mathcal{S}}^+_t(p)$. Indeed, by the Lemma 3.14 each α_i has to touch $\tilde{\mathcal{S}}^+_s(p) \cup \tilde{\mathcal{S}}^+_t(p) \cup (\partial M \cap \mathcal{A}_{s,t}(p))$ and it can only touch $\partial M \cap \mathcal{A}_{s,t}(p)$ orthogonally. Using that any two orthogonal geodesic segments to ∂M do not intersect each other in $\tilde{\mathcal{B}}^+_r(p)$, together with the fact that $\tilde{\mathcal{S}}^+_s(p)$ is strictly convex and orthogonal to ∂M, we conclude that if $\alpha_i \in \Sigma_1$ touches $\partial M \cap \mathcal{A}_{s,t}(p)$, then $\alpha_i \cap \tilde{\mathcal{S}}^+_s(p) = \varnothing$, unless α_i touches $\tilde{\mathcal{S}}^+_s(p) \cap \partial M$ (see Figure 10). Also, if α_i does not touch $\tilde{\mathcal{S}}^+_t(p)$, then its endpoints can not be on $\partial M \cap \mathcal{A}_{s,t}(p)$, because α_i would be a stationary varifold with free boundary, contracting (11). Then, any α_i that touches $\tilde{\mathcal{S}}^+_s(p)$ or $\partial M \cap \mathcal{A}_{s,t}(p)$, should touch $\tilde{\mathcal{S}}^+_t(p)$. Therefore, if there were an infinite number of geodesic segments $\{\alpha_i\} \subset \Sigma_1$ in $\mathcal{A}_{s,\tilde{t}}(p)$, then there would be an infinite number of geodesic segments from $\tilde{\mathcal{S}}^+_{\tilde{t}}(p)$ to $\tilde{\mathcal{S}}^+_t(p)$, contradicting the fact that Σ_1 has finite mass. Thus the set $\{\alpha_i\}$ is finite. Finally, using again the strict convexity of $\tilde{\mathcal{S}}^+_{s_2}(p)$, each geodesic segment that is tangent to $\tilde{\mathcal{S}}^+_{s_2}(p)$ can not touch $\tilde{\mathcal{S}}^+_{\tilde{s}_2}(p)$

for all $0 < \tilde{s}_2 < s_2$. So, by the finitude of the geodesic segments and by (10), we can choose $0 < s_2 < t$ as requested. See Figure 10.

Figure 10



Source: Own construction

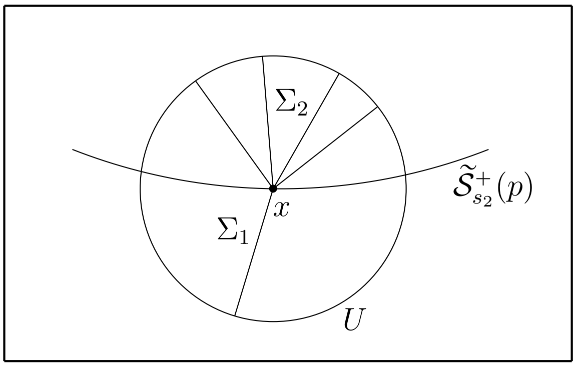
Note that each $\alpha_i \subset \Sigma_1$ have to touch $\widetilde{\mathcal{S}}^+_t(p)$ at points in $\mathring{M}$, since $\widetilde{\mathcal{S}}^+_t(p)$ is orthogonal to ∂M.

Step 2: *Gluing* Σ_1 *and* Σ_2 *across* $\widetilde{\mathcal{S}}^+_{s_2}(p)$.

As before, any geodesic segment (with possible multiplicity) $\beta_i \in \Sigma_2 \cap \mathcal{A}_{s,s_2}(p)$ has to touch $\widetilde{\mathcal{S}}^+_{s_2}(p)$ in points belonging to $\mathring{M}$. Since V^{**} is stationary and integral in $\mathcal{A}_{s_1,t}(p)$, we have by the interior regularity (Theorem 3.11) that each $x \in \operatorname{spt}\|V^{**}\| \cap \mathring{M} \cap \mathcal{A}_{s,t}(p)$ belongs to a finite number of geodesic segments (including multiplicity). In particular, if $x \in \operatorname{spt}\|V^{**}\| \cap \mathring{M} \cap \widetilde{\mathcal{S}}^+_{s_2}(p)$ then x belongs to $\overline{\Sigma}_1 \cap \overline{\Sigma}_2$, since each geodesic segment of Σ_1 touches $\widetilde{\mathcal{S}}^+_{s_2}(p)$ transversally. So, Σ_1 and Σ_2 glue continuously across $\widetilde{\mathcal{S}}^+_{s_2}(p)$. Note that $\operatorname{spt}\|V^{**}\| \cap \widetilde{\mathcal{S}}^+_{s_2}(p) = \overline{\Sigma}_1 \cap \widetilde{\mathcal{S}}^+_{s_2}(p) = \overline{\Sigma}_2 \cap \widetilde{\mathcal{S}}^+_{s_2}(p) \subset \mathring{M}$. As we will see below, that gluing is actually C^1. To prove this, we will show that the varifold tangent $\operatorname{VarTan}(V^{**}, x)$ is a straight line for every $x \in \operatorname{spt}\|V^{**}\| \cap \widetilde{\mathcal{S}}^+_{s_2}(p)$, that is, x is not a junction.

In fact, as $x \in \mathring{M}$, we can choose by the interior regularity an open neighborhood $U \subset \mathring{M}$ of x such that $V^{**} \llcorner U = \sum_{i=1}^{l} v(\alpha_i \cap U, m_i)$ for some $l, m_1, \cdots, m_l \in \mathbb{N}$ and $\{\alpha_1, \cdots, \alpha_l\}$ geodesic segments in M from x to ∂U. So, $\operatorname{VarTan}(V^{**}, x) = \sum_{i=1}^{l} v(r_i, m_i) \subset \mathbb{R}^2$, where each r_i is a semi-straight line from the origin $0 \in \mathbb{R}^2$ (see Figure 11). In other words, $\operatorname{VarTan}(V^{**}, x) \in I\mathcal{V}_1(\mathbb{R}^2)$ is a stationary cone satisfying the Proposition 3.15, then $\operatorname{VarTan}(V^{**}, x)$ will be a straight line with possible multiplicity.

Figure 11



Source: Own construction

Step 3: *Unique continuation up to the point p.*

By Step 2 and property (i) (Theorem 3.13) of replacements, we can extend Σ_2 to $\widetilde{\Sigma}_2$ in $\mathcal{A}_{s_1,t}(p)$ such that $\widetilde{\Sigma}_2 = \Sigma_1$ on $\mathcal{A}_{s,t}(p)$, $\widetilde{\Sigma}_2$ is given by geodesic segments possibly with multiplicity and without interior junctions that can only touch $\mathcal{A}_{s_1,t}(p) \cap \partial M$ orthogonally, $\widetilde{\Sigma}_2 \llcorner \mathcal{A}_{s,s_2}(p)$ has a finite number of geodesic segments, and each geodesic segment of $\widetilde{\Sigma}_2$ has to touch $\widetilde{\mathcal{S}}^+_t(p)$. Using (10), we can continue to take replacements in this way for all $0 < s_1 < s$. For each $0 < s_1 < s$ as before, denote $\widetilde{\Sigma}_2$ by Σ_{s_1}. If $0 < s_1' < s_1 < 0$, then we have that $\Sigma_{s_1'} = \Sigma_{s_1}$ on $\mathcal{A}_{s_1,t}(p)$. Thus

$$\Sigma := \bigcup_{0<s_1<s} \Sigma_{s_1}$$

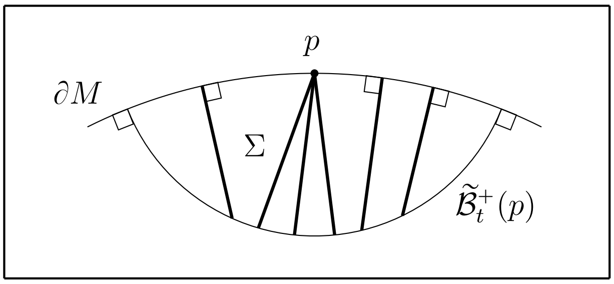
in $\widetilde{\mathcal{B}}^+_t(p)$ is given by geodesic segments possibly with multiplicity and without interior junctions that can touch $\partial M \cap (\widetilde{\mathcal{B}}^+_t(p) \backslash \{p\})$ orthogonally only, and each geodesic segment of Σ has to touch $\widetilde{\mathcal{S}}^+_t(p)$. Moreover, $\Sigma \llcorner \widetilde{\mathcal{B}}^+_{\tilde{t}}(p)$ has a finite number of segments for all $0 < \tilde{t} < t$. See Figure 12.

Claim: $\mathrm{spt}\|V\| = \Sigma$ *in the punctured ball* $\widetilde{\mathcal{B}}^+_s(p) \backslash \{p\}$.

Proof of Claim: Consider the set

$$T^V_p = \Big\{ y \in \mathrm{spt}\|V\| \ \ : \ \ \text{VarTan}(V, y) \text{ is a straight line or} \\ \text{a semi-straight line transversal to } \widetilde{\mathcal{S}}^+_{\tilde{r}_p(y)}(p) \Big\}.$$

Figure 12



Source: Own construction

We know by [8, Lemma B.2] (see also [19], Claim 3, p. 42) that the set T_p^V is a dense subset of $\operatorname{spt}\|V\| \cap \widetilde{\mathcal{B}}_s^+(p)$.

Given $y \in T_p^V \cap (\widetilde{\mathcal{B}}_s^+(p)\backslash\{p\})$, let $\rho = \widetilde{r}_p(y)$. Take V^* the replacement of V in $\mathcal{A}_{s,t}(p)$ and V^{**} the replacement of V^* in $\mathcal{A}_{\rho,s_2}(p)$ for $s_2 \in (s,t)$ chosen as in Step 1. By the property (i) from Theorem 3.13, we have $V^{**} = V^* = V$ in $\widetilde{\mathcal{B}}_\rho^+(p)$, then

$$y \in \overline{\operatorname{spt}\|V\| \cap \widetilde{\mathcal{B}}_\rho^+(p)} \cap \widetilde{\mathcal{S}}_\rho^+(p) = \overline{\operatorname{spt}\|V^{**}\| \cap \widetilde{\mathcal{B}}_\rho^+(p)} \cap \widetilde{\mathcal{S}}_\rho^+(p).$$

Since $\operatorname{spt}\|V^{**}\| = \Sigma$ in $\mathcal{A}_{\rho,t}(p)$ and $\operatorname{VarTan}(V^{**}, y)$ is transversal to $\widetilde{\mathcal{S}}_\rho^+(p)$, we have by (10) and above that $y \in \Sigma$. Thus, $T_p^V \cap (\widetilde{\mathcal{B}}_s^+(p)\backslash\{p\}) \subset \Sigma$, and hence $\operatorname{spt}\|V\| \cap (\widetilde{\mathcal{B}}_s^+(p)\backslash\{p\}) \subset \Sigma$. The last one is deduced using that T_p^V is a dense subset of $\operatorname{spt}\|V\| \cap \widetilde{\mathcal{B}}_s^+(p)$, and the fact that Σ is compact in $\widetilde{\mathcal{B}}_s^+(p)$.

To see the converse inclusion $\Sigma \subset \operatorname{spt}\|V\|$ in $\widetilde{\mathcal{B}}_s^+(p)$, note that by the Constancy Theorem [35, Theorem 41.1], we have $\operatorname{spt}\|V\| \cap (\widetilde{\mathcal{B}}_s^+(p)\backslash\{p\}) = \Sigma$ in $M\backslash\partial M$. For $y \in \Sigma \cap \partial M \cap (\widetilde{\mathcal{B}}_s^+(p)\backslash\{p\})$, we know that $\operatorname{VarTan}(\Sigma, y)$ is a straight line perpendicular to $T_y(\partial M)$, which implies that y is a limit point of $\Sigma \cap \mathring{M}$ and thus $y \in \operatorname{spt}\|V\|$. Therefore, $\operatorname{spt}\|V\| \cap (\widetilde{\mathcal{B}}_s^+(p)\backslash\{p\}) = \Sigma$.

Step 4: *V is a free boundary geodesic network*

From the interior regularity (Theorem 3.11) and the Step 3, V is a geodesic network (finite) in $\widetilde{\mathcal{B}}_s^+(p)$ and a free boundary geodesic network (finite) in $(\widetilde{\mathcal{B}}_s^+(p)\backslash\{p\})$. In particular, $\Theta^1(V \llcorner \partial M, p) = 0$. So, if there exist geodesic segments at p, as in the Figure 12, then those segments must satisfy (5), and then V is a free boundary geodesic network (finite) in $\widetilde{\mathcal{B}}_s^+(p)$.

Varying $p \in \partial M$, we see that V is a free boundary geodesic network (not necessarily finite) on M. Given any compact $K \subset \mathring{M}$, the interior regularity says that $V \llcorner K$ has a finite number of geodesic segments. So, we only need to show that there exists a compact $K \subset \mathring{M}$ such that $V \llcorner (M \backslash K)$ has also a finite number of geodesic segments. Indeed, take a cover of ∂M by open balls $\widetilde{\mathcal{B}}_s^+(p)$ as in the previous steps, extract a finite cover $\{\widetilde{\mathcal{B}}_j^+(p_j)\}_{j=1}^l$, and define $K := \overline{M \backslash \big(\cup_{j=1}^l \widetilde{\mathcal{B}}_j^+(p_j) \big)}$. This finishes the proof.

$\square$

Remark 3.17. We believe that in the above theorem the stationary condition is enough for V to be a geodesic network finite in $M \backslash \partial M$, like in the case without boundary ([4]). In a general case, if we suppose noting about the boundary and V is just stationary in M, then we believe that V is given by the union of a geodesic network finite V_1 in $M \backslash \partial M$ with another geodesic network finite V_2 on ∂M, this last one considered as a varifold in ∂M (the geodesics of V_2 are geodesic on ∂M, not necessary geodesic on M.)

4 THE WIDTH OF A FULL ELLIPSE

In this section we will prove our main theorem about p-widths: we will calculate the first p-widths of B^2 and E^2, where E^2 is a planar full ellipse C^∞-close to B^2. As in Aiex [2], we will take the p-sweepouts from Guth [15, Section 6]. We will consider some adaptations to get a convenient upper bound for the mass of the cycles. Also, we will need to take a better estimate than that given by the Cauchy-Crofton Formula. Indeed, to calculate the widths of the unit sphere in [2], the Cauchy-Crofton Formula given a sharp estimate, which does not happen in our case. Fortunately, by our regularity results, we do not need a sharp estimate a priori, but we will give a sharp estimate in the appendix.

4.1 Planar Cauchy-Crofton Formula

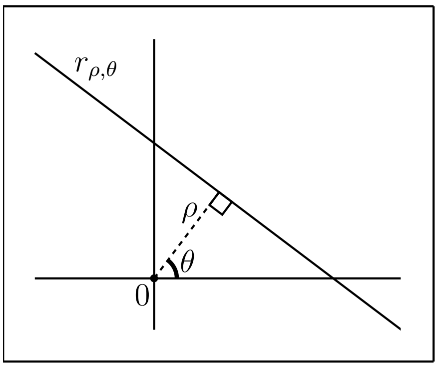
In this section we follow Do Carmo [10, Section 1.7 C].

Let $\mathcal{E}$ be the set of straight lines in the plane. A straight line r in the plane is determined by the distance $\rho \geqslant 0$ from r to the origin of the coordinates and by the angle θ, $0 \leqslant \theta < 2\pi$, which a half-line starting at the origin and normal to r makes with the x axis (see Figure 13). The equation of r in terms of these parameters is given by

$$x\cos(\theta) + y\sin(\theta) = \rho.$$

We refer to r as above by $r_{\rho,\theta}$.

Figure 13



Source: Own construction

Given a piecewise C^1 curve $C : [a, b] \to \mathbb{R}^2$, let $n(\rho, \theta)$ be the number of intersection points (with multiplicity) of the stratight line $r_{\rho,\theta}$ with C. The function $n(\rho, \theta)$ is finite almost everywhere. The Cauchy-Crofton formula [10, Theorem 3, p. 41] states that the length $L(C)$ of C is given by

$$L(C) = \frac{1}{2}\int_0^{2\pi}\int_{\mathbb{R}^+} n(\rho,\theta)d\rho d\theta. \tag{12}$$

4.2 A Sweepout for B^2

The sweepout that we will use to calculate the p-widths is given by a map whose image is given by real algebraic varieties. The properties of this map can be found in Guth [15, Section 6].

Let $Q_i : \mathbb{R}^2 \to \mathbb{R}$ denote the following polynomials for $i = 1, \ldots, 4$:

$$Q_1(x,y) = x, \quad Q_2(x,y) = y, \quad Q_3(x,y) = x^2 \quad \text{and} \quad Q_4(x,y) = xy.$$

Also, put $A_p = \operatorname{span}(1 \cup_{i=1}^p Q_i)\backslash\{0\}$ and define the relation $Q \sim \lambda Q$, for $\lambda \neq 0$ and $Q \in A_p$. The quotient $(A_p, \sim)$ can be identified with $\mathbb{RP}^p$ and by this identification we can define the map $F_p : \mathbb{RP}^p \to \mathcal{Z}_{1,rel}(B^2, \partial B^2; \mathbb{Z}_2)$, which send a class $[Q]$ to the real algebraic variety defined by $Q(x,y) = 0$ restricted to B^2, considered as a mod 2 relative Lipschitz cycle (see [15, Section 6]).

As proved in [15, Section 6], F_p is a flat continuous map and it defines a p-sweepout. Also, in the next lemma we will use the Cauchy-Crofton formula to prove that F_p has no concentration of mass, thus $F_p \in \mathcal{P}_p(B^2)$.

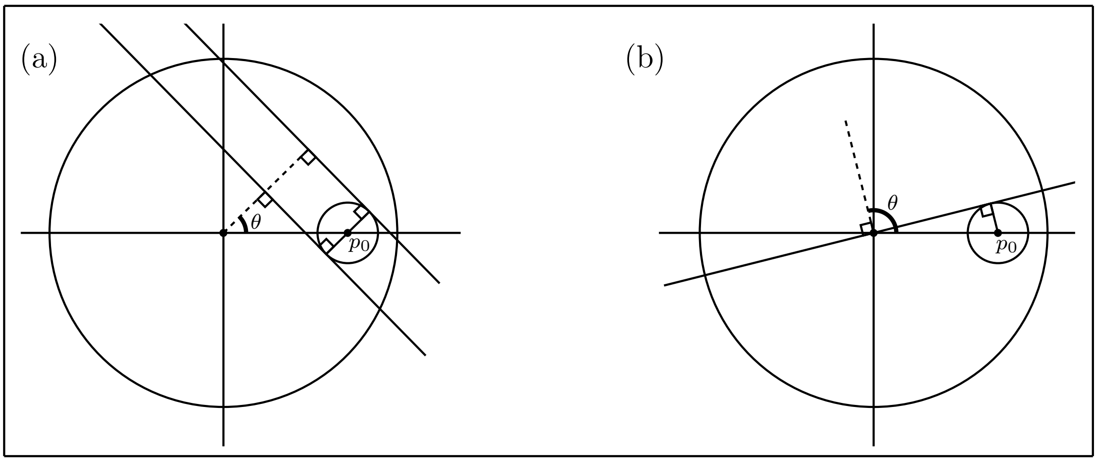
Lemma 4.1. *The map $F_p : \mathbb{RP}^p \to \mathcal{Z}_{1,rel}(B^2, \partial B^2; \mathbb{Z}_2)$ has no concentration of mass for $p = 1, \ldots, 4$.*

Proof. Without loss of generality, consider $P_0 = (p_0, 0) \in B^2$ for $p_0 \geqslant 0$, and the ball $B_r(P_0)$ for small $r > 0$.

If $p_0 > 0$, note that for $\theta \in [0, \pi/2]$ the straight line $r_{\rho,\theta}$ intersects $B_r(P_0)$ if and only if $\rho \in [p_0\cos(\theta) - r, p_0\cos(\theta) + r]$ (see Figure 14 (a)). On the other hand, if $\theta \in (\pi/2 + \sin^{-1}(r/p_0), \pi]$, then $r_{\rho,\theta}$ does not intersect $B_r(P_0) \cap B^2$ for all ρ (see Figure 14 (b)).

For $p = 1, \ldots, 4$, we have that $F_p([Q])$ is an algebraic variety of degree at most 2, so $F_p([Q])$ intersects $r_{\rho,\theta}$ at most 2 times. By the Cauchy-Crofton Formula and symmetry,

Figure 14



Source: Own construction

we obtain

$$\begin{aligned}\|F_p([Q])\|(B_r(P_0)\backslash\partial B^2) &\leqslant \frac{2}{2}\int_0^{\pi/2+\sin^{-1}(r/p_0)}\int_{p_0\cos(\theta)-r}^{p_0\cos(\theta)+r} 2d\rho d\theta \\ &= 4r\left(\frac{\pi}{2}+\sin^{-1}\left(\frac{r}{p_0}\right)\right).\end{aligned}$$

In the same way we have that $\|F_p([Q])\|(B_r(P_0)\backslash\partial B^2) \leqslant 4r\pi$, when $p_0 = 0$. Then, in all the cases we conclude that $\|F_p([Q])\|(B_r(P_0)\backslash\partial B^2) \to 0$ as $r \to 0$.

$\square$

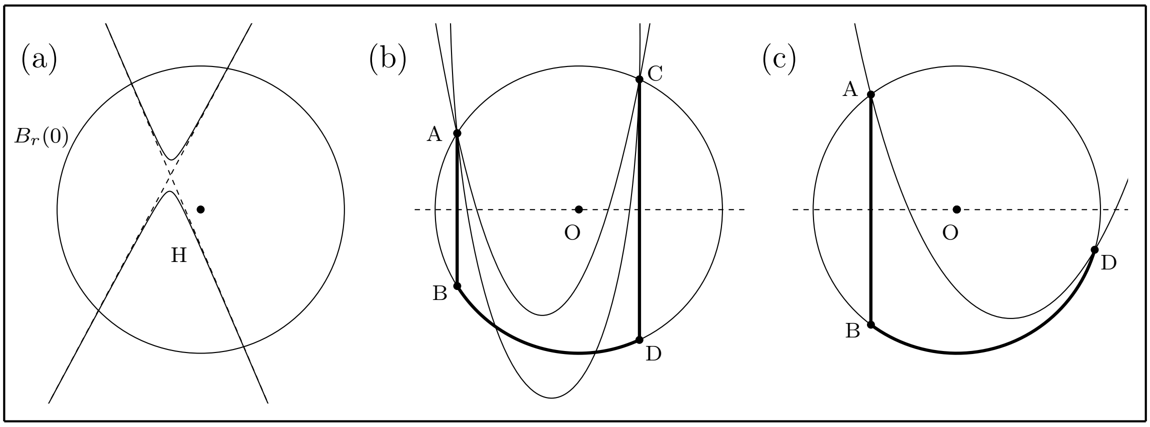
In the following, we estimate an upper bound for $\|F_p([Q])\|$, $p = 1,\ldots,4$. In other words, we estimate the maximum length of the algebraic variety $F_p([Q])$. By the definitions above, $F_p([Q])$ is degenerate or is the restriction to B^2 of a straight line, or of two straight lines, or of a parabola, or of a hyperbola. Actually, $F_p([Q])$ is a quadratic curve which is not an ellipse, since we excluded the polynomial $Q_5(x,y) = y^2$. In the Appendix we will give a sharp upper bound for $\|F_p([Q])\|$, $p = 1,\cdots,4$.

Lemma 4.2. *For any* $[Q] \in \mathbb{RP}^p$ *we have that* $\|F_p([Q])\| \leqslant 2$, $p = 1,2$, *and* $\|F_p([Q])\| < 4.52$, $p = 3,4$.

Proof. Clearly, for $p = 1,2$ the algebraic variety $F_p([Q])$ is degenerate or the restriction to B^2 of a straight line, thus $\|F_p([Q])\| \leqslant 2$ for $p = 1,2$ and for all $[Q] \in \mathbb{RP}^k$.

For $p = 1,\ldots,4$ note that if $F_p([Q])$ is degenerate or the intersection to B^2 of a straight line, or two straight lines, then $\|F_p([Q])\| \leqslant 4$. Also, this estimate holds when $F_p([Q])$ is the restriction to B^2 of a hyperbola H such that each branch of the hyperbola intersects B^2. Indeed, if we take $B_r(0)$ for r large, each arm of the branches tends to their respective asymptotes, so the length of the two asymptotes restricted to $B_r(0)$ is bigger than the length $L(H)$ of this hyperbola restricted to $B_r(0)$, then $L(H) \leqslant 4r$ in $B_r(0)$ (see Figure 15 (a)). Decreasing r, we note that the reduction of length is at least the reduction of $4r$, since there exist four points in $H \cap \partial B^2$ during the reduction $r \to 1^+$. We conclude that $L(H) \leqslant 4$ in B^2.

Figure 15



Source: Own construction

In the other cases (hyperbolas with a unique branch intersecting B^2, or parabolas intersecting B^2), we choose an orientation such that the axis of symmetry of the curve is orthogonal to x-axis. Hence, $F_p([Q])$ will be a convex downward curve intersected with B^2 and we have two cases: there exist two points A, C in the intersection of the curve with ∂B^2 such that $y(A), y(C) > 0$; or there exists at most one such point. In the first case, as in the examples of the Figure 15 (b), take $B = (x(A), -y(A)), D = (x(C), -y(C)) \in \partial B^2$ (AB and CD are perpendicular to x-axis), and the circular arc BD. The length of this convex curve in B^2 is at most the length of $AB + BD + CD$. Let α (resp. β) be the angle between OA (resp. OC) and x-axis for $\alpha, \beta \in (0, \pi/2]$, then

$$L(F_p([Q])) \leqslant AB + CD + BD = 2\sin(\alpha) + 2\sin(\beta) + \pi - (\alpha + \beta) < 4.52. \qquad (13)$$

In the second case, as in the example of the Figure 15 (c), where does not exist A or C as in the first case, we take $\alpha = 0$ or $\beta = 0$ in the above estimate, respectively. Without loss of generality suppose $\beta = 0$, then

$$L(F_p([Q])) \leqslant 2\sin(\alpha) + \pi - (\alpha) < 3.83, \tag{14}$$

for $\alpha \in [0, \pi/2]$.

$\square$

4.3 The First Widths of B^2 and E^2

In this section we will prove our main result about k-widths: we will calculate the low p-widths of the unit ball B^2, and of full ellipses C^∞-close to B^2.

The next theorem is similar to Aiex [2, A.1] and a weaker version of the results in Marques and Neves [25].

Theorem 4.3. *Let M^2 be a Riemannian manifold with non-empty strictly convex boundary. If $\omega_p(M) = \omega_{p+1}(M)$ for some p, then there exist infinitely many free boundary geodesic networks whose masses tend to $\omega_p(M)$.*

Proof. As proved in [2, Proposition A.1], the proof follows from [25, Theorem 6.1] for the case without boundary. For our case take the following modifications: note that the results of [25, Section 3.3] can be extended from the results of [22, Section 2]; the conclusion of [25, Proposition 4.8] holds for free boundary geodesic networks in consequence of Theorem 2.22 and Theorem A ; take Theorem 2.21 in place of [25, Proposition 2.4]; for the sets $\mathcal{S}$ and $\mathcal{T}$ we take the supports on free boundary geodesic networks.

As $\operatorname{spt}\|V\|$ is a geodesic network with free boundary, we can use the Constancy Theorem in [25, Claim 6.2] as in [2, Proposition A.1]. Finally, [25, Theorem 2.8] follows from Theorem 2.22, as noted in the proof of Theorem 4.21 from [19], and the conclusion about the masses follow from the fact that the infinitely many free boundary geodesic networks are taken from the proof of [25, Proposition 4.8]. $\square$

Now, we will prove our Main Theorem B. Compare with Theorems 5.2 and 5.6 from [2].

Main Theorem B. *For B^2 we have*

(i) $\omega_1(B^2) = \omega_2(B^2) = 2$;

(ii) $\omega_3(B^2) = \omega_4(B^2) = 4$.

Also, if E^2 is a full ellipse C^∞-close to B^2 with small diameter d and large diameter D, then

(iii) $\omega_1(E^2) = d$ *and* $\omega_2(E^2) = D$;

(iv) $\omega_3(E^2), \omega_4(E^2) \in \{2d, d + D, 2D\}$ *such that* $\omega_3(E^2) \neq \omega_4(E^2)$. *In particular, one of those widths is reached by a 1-varifold with multiplicity two.*

Proof. (i) Let $p = 1, 2$ and take the p-sweepout $F_p \in \mathcal{P}_p(B^2)$. By Lemma 4.2 we know that $\|F_p([Q])\| \leqslant 2$ for all $[Q] \in \mathbb{RP}^p$, thus $\omega_1(B^2), \omega_2(B^2) \leqslant 2$.

Now, given $\epsilon > 0$ we can find by the Corollary 2.23 a special varifold V such that $0 < \omega_p(B^2) \leqslant \|V\|(B^2) \leqslant \omega_p(B^2) + \epsilon \leqslant 2 + \epsilon$. By Theorem A and Theorem 3.7 we actually have that V is a diameter of B^2 and $\|V\|(B^2) = 2$. Therefore, $\omega_1(B^2) = \omega_2(B^2) = 2$.

(iii) Still consider $p = 1, 2$. As E^2 is close to B^2, we deduce by continuity (as did in [2, Proposition 5.4 (iv)]) that $\omega_p(E^2)$ is close to $\omega_p(B^2)$. Then, $\omega_p(E^2) \leqslant 2 + \delta$ for some small $\delta > 0$. By the same argument above using Corollary 2.23, Theorem A and Corollary 3.9, we conclude that the only possible values for $\omega_1(E^2)$ and $\omega_2(E^2)$ are d or D. Finally, by Theorem 4.3 and Corollary 3.9 we know that $\omega_1(E^2) \neq \omega_2(E^2)$, then $\omega_1(E^2) = d$ and $\omega_2(E^2) = D$.

(ii) As $\omega_2(E^2) = D$, by the same argument above we have that $\omega_2(E^2) \neq \omega_3(E^2)$. It follows from Corollary 3.9 that $\omega_3(E^2) \geqslant 2d = 4 - \delta$ for some small $\delta > 0$. Therefore, by continuity $\omega_3(B^2) \geqslant 4 - \epsilon$ for some small $\epsilon > 0$, which implies by Theorem 3.7 that $\omega_3(B^2) \geqslant 4$ (two diameters). Now, by Lemma 4.2 we obtain $4 \leqslant \omega_3(B^2), \omega_4(B^2) < 4.52 <$ (length of P_3), and so by Theorem 3.7 we actually have that $4 \leqslant \omega_3(B^2), \omega_4(B^2) \leqslant 4$, which concludes this case.

(iv) We use again the continuity and Corollary 3.9 to conclude that the only possible values to $\omega_3(E^2)$ and $\omega_4(E^2)$ are $2d, d + D$ or $2D$. Finally, by Theorem 4.3 and Corollary 3.9 we know that $\omega_3(E^2) \neq \omega_4(E^2)$.

$\square$

Remark 4.4. An alternative way to see that $\omega_3(B^2) > 2$, without using Theorem 4.3, is to use the Lusternik-Schnirelmann theory as in Guth [15], p. 1923-24. Indeed, we can take three disjoint balls B_i in $B^2 \backslash \partial B^2$ with radius 0.4 each ball. Each 3-sweepout Φ of

B^2 is also an 1-sweepout of B^2, in particular it is an 1-sweepout of each B_i. Lusternik-Schnirelmann theory says that Φ contains a cycle such that its mass is at least the sum of the first width of each B_i. By the item (i) above we know that the first width of a ball is equal to the diameter of that ball, so $\omega_3(B^2) \geqslant 3 \times 0.8 > 2$. Therefore, we could calculate the widths above of B^2 without use Theorem 4.3. However, we could not determine the widths above of E^2.

Remark 4.5. For $p = 1, 2$, note that F_p is an optimal p-sweepout in the sense that

$$\omega_p(B^2) = \sup\{\mathbf{M}(F_p([Q])) : [Q] \in \mathrm{dmn}(F_p)\}.$$

The estimate obtained in the Lemma 4.2 is not enough to know if F_p is also an optimal p-sweepout for $p = 3, 4$. Actually, in the Appendix we calculate that

$$4 < \sup\{\mathbf{M}(F_p([Q])) : [Q] \in \mathrm{dmn}(F_p)\} \approx 4.0027,$$

for $p = 3, 4$. So, F_p is almost an optimal p-sweepout for $p = 3, 4$.

Remark 4.6. Notice how similar is our result comparing with the results in Aiex [2, Theorems 5.2 and 5.6] about the p-widths for the unit sphere and the ellipsoid in $\mathbb{R}^3$. In fact, as in that work, we obtained in (iv) above an example of a min-max critical varifold with multiplicity. So, as in the closed case [2], we see that in the case with boundary the Multiplicity One Conjecture [24] is also false for min-max critical curves.

5 FURTHER WORKS

Both in the case of the sphere, both in the case of the unit ball, we note certain standards for the values of the widths, so we could conjecture a general formula for the width of the unit ball in a similar way as suggested in [2, Section 6] for S^2. Unfortunately, we did not proof a general formula for the widths of the unit ball. Indeed, the sweepout that we used does not guarantee good estimates that are sufficient to calculate the higher widths.

With some more care, we can use the Lusternik-Schnirelmann inequality [22, Section 3] and the Theorem 3.7 to get that $3\sqrt{3} \leqslant \omega_5(B^2) \leqslant 6$ and then, by Theorem 3.7, we can estimate that $\omega_5(B^2) = 3\sqrt{3}, 4\sqrt{2}, 10\sin(\pi/5)$, or 6. We expected that $\omega_5(B^2) = 6$, if we could prove a more strong regularity theorem than Theorem A. However, so far we did not conclude the details of that argument.

As we commented in Remark 3.17, we expected that holds a more general theorem than Theorem A: if M^{n+1} is a Riemannian manifold and $V \in I\mathcal{V}_1(M)$ is a stationary varifold, then V is a geodesic network finite Σ_1 in $M\backslash\partial M$ union with another geodesic network finite Σ_2 in ∂M, this last one considered as a varifold in ∂M, that is, the geodesics of Σ_2 are geodesics on ∂M, not necessary geodesics on M.

If we have a general formula for the widths, automatically we deduce the constant in the Weyl law for the volume spectrum. But, since to find a general formula seems difficult, it is to be expected that it will be possible to deduce that constant without finding a such formula.

Finally, to find the maximum length of a real algebraic curve of degree $d > 0$ restricted to B^2 can be a little complicated, even in the case with degree two, as we did in the Appendix. Moreover, the result did not follow the intuition of [15, p. 1974]. For higher degree, or for a general degree, we do not know the way to calculate these maximum length.

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APPENDIX

In the Lemma 4.2 we calculated an upper bound for the length of $F_p([Q])$ for $p = 1, \ldots, 4$. Note that we did not calculate a sharp upper bound. In this appendix we will do it, that is, we will calculate the maximum length of a parabola or a hyperbola restricted to B^2, since the others cases were did in the Lemma 4.2. As we will see, this sharp upper bound will imply that F_p is not an optimal p-sweepout of B^2 for $p = 3, 4$.

The length of a real algebraic curve C can be bounded in terms of its degree using the Cauchy-Crofton formula (12). In fact, if that curve has degree d, then it intersects a straight line at most d times, so by (12) the length of C restricted to B^2 is at most $d \cdot \text{area}(B^2) = \pi d$. Obviously, this upper bound is not sharp. For example, if $d = 1$ we have that C is a straight line and the length of the intersection of a straight line with B^2 is at most 2. It is intuitive, and it was conjectured in Guth [15, p. 1974], that the sharp upper bound is similar to the case $d = 1$, that is, $L(C \cap B^2) \leqslant 2d$ for all $d \in \mathbb{N}$. Contradicting that result, we will see that for $d = 2$ we can find C such that $L(C \cap B^2) > 4$. Our counterexample in the case $d = 2$ is a parabola $P(x)$ such that its length in B^2 is bigger than any other parabola restricted to B^2, and $L(P(x) \cap B^2) > 4$. Moreover, $\|F_p([Q])\| \leqslant L(P(x) \cap B^2)$ for $p = 1, \ldots, 4$. See the arguments below.

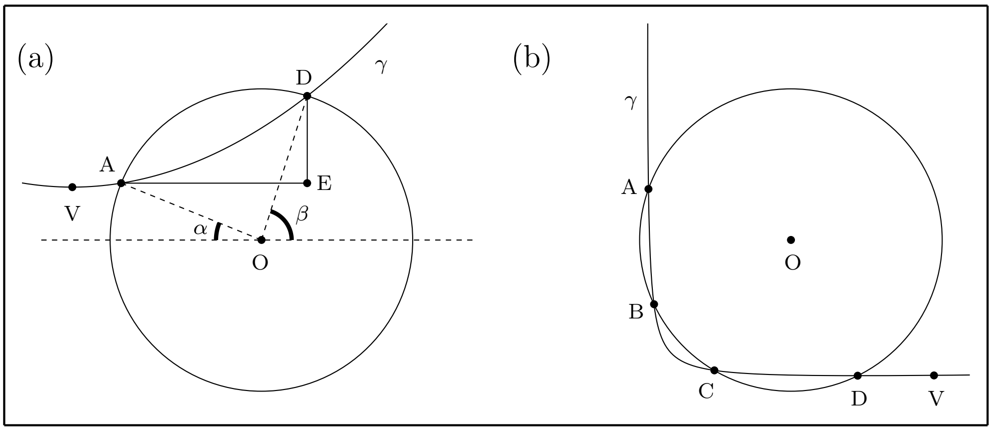
Theorem. *Let L_{max} such that $L(P(x) \cap B^2) \leqslant L_{max}$ for any parabola $P(x)$. Then $\|F_p([Q])\| \leqslant L_{max}$ for $p = 1, \ldots, 4$ and all $[Q] \in \mathbb{RP}^p$. Furthermore, that estimate is sharp and $L_{max} \sim 4.0027$.*

Proof. By the proof of Lemma 4.2, we know that if $F_p([Q])$ is not the restriction to B^2 of a hyperbola with a unique branch intersecting B^2, or is not the restriction to B^2 of a parabola, then $\|F_p([Q])\| \leqslant 4$ for $i = 1, \cdots, 4$. So we focus on these two exceptional cases to improve the estimate (13). We will use the fact that in these cases the curvature of the curve $F_p([Q]) = \gamma(x)$ is strictly increasing in the direction of the axis of symmetry, and it has at most four points in the intersection with ∂B^2. Also, as in Lemma 4.2, we choose an orientation such that the axis of symmetry of that curve is orthogonal to x-axis, then the curve will be downward convex with vertex V such that, $\gamma(x)$ is increasing for $x > x(V)$, and decreasing for $x < x(V)$. We will fix a such curve with $L(\gamma \cap B^2) > 0$, and by translation we will find the positions such that the length of that curve restricted to B^2 increases, next we will change the parameters of that curve to get the maximum length in B^2.

First of all, suppose that $V \notin B^2$. As $L(\gamma \cap B^2) > 0$, then when $\gamma \cap B^2$ is connected there will be one point $A \in \partial B^2$ where the curve goes inside B^2, and another point $D \in \partial B^2$ where the curve goes outside. Consider $x(A) < x(D)$ and, because of symmetry, we can assume without loss of generality that $x(V) < x(A)$. For $x > x(V)$ the curve is strictly increasing, so $y(A) < y(D)$. If $y(A) \leqslant 0$, the estimate (14) says that $L(\gamma \cap B^2) < 3.83$. For the case $y(A) > 0$, we have that $y(A) < 1$ and $L(\gamma \cap B^2)$ is bounded by $|\overline{AE}| + |\overline{DE}|$, where $E = (x(D), y(A))$, see Figure A1 (a). Let β be the clockwise angle between the x-axis and $\overline{OD}$, and let α be the angle between $\overline{OA}$ and the x-axis. So $\alpha \in (0, \pi/2)$, $\beta \in (0, \pi)$, and

$$L(\gamma \cap B^2) \leqslant |\overline{AE}| + |\overline{DE}| = \cos(\alpha) + \cos(\beta) + \sin(\beta) - \sin(\alpha) < 2.42. \tag{A1}$$

Figure A1



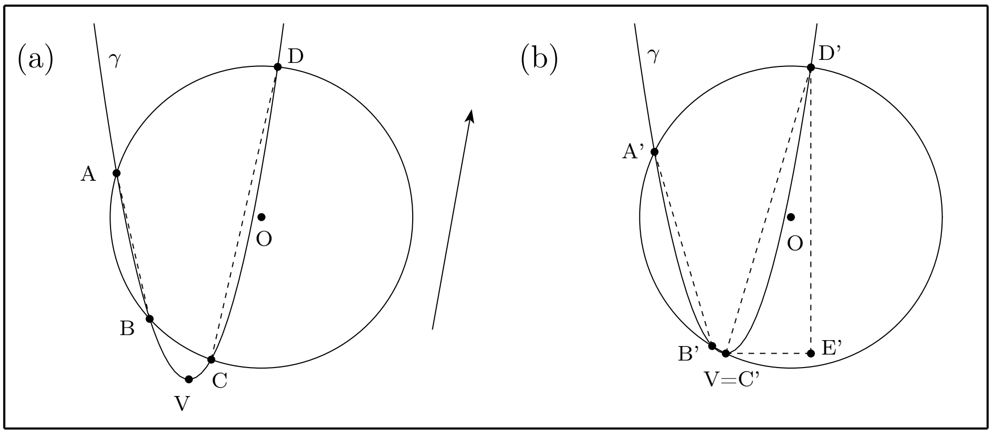
Source: Own construction

Is not difficult to see that to get the maximum length of $\gamma \cap B^2$ by translations, it is necessary that $V \in B^2$. We will see this fact in the next case below, and we could have done this in the previous case, however the estimate above is enough for what we need.

Suppose now the case $V \notin B^2$ and $\gamma \cap B^2$ is not connected. As the intersection $\gamma \cap \partial B^2$ has at most four points and $L(\gamma \cap B^2) > 0$, we have two possibilities for this case: there exists at least one and at most two points of γ, which are tangent to ∂B^2; or does not exist such tangent points. In the first case, $\gamma \cap B^2$ has a connected component which length is the length of $\gamma \cap B^2$, so we can estimate this case as in the previous case. In the second case, there exist two points $A, C \in \partial B^2$ where the curve goes inside the B^2, and two points $B, D \in \partial B^2$ where the curve goes outside. Consider $x(A) < x(B) < x(C) < x(D)$. We claim that $x(B) < x(V) < x(C)$. Otherwise, as $V \notin B^2$, we should have $x(V) < x(A)$, or $x(D) < x(V)$. By symmetry, it is enough to verify that the second inequality can not be true (see Figure A1 (b)). Indeed, the curvature $\kappa_\gamma(x)$ of the $\gamma(x)$ is increasing for $x < x(V)$, so as γ goes out of B^2 at B and goes inside of B^2 at C, we have that $\kappa_\gamma(x(C)) > \kappa_{\partial B^2}(x(C)) = 1$. On the other hand, as γ goes inside of B^2 at C and goes out of B^2 at D, we have that $\kappa_\gamma(x(C)) \leqslant \kappa_{\partial B^2}(x(C)) = 1$, which is a contradiction. Therefore, $x(B) < x(V) < x(C)$. Now we have (by symmetry) two cases: $\overline{CD}$ passes through or is on the left of O; or $\overline{AB}$ is on the left of O, and $\overline{CD}$ is on the right of O. See Figures A2 and A3.

In the first case, note that $y(D) > 0$, $x(B), x(C) < 0$, $-1 < y(B), y(C) < 0$, and $y(A) < 1, x(A) < 0$, since $\gamma(x)$ is decreasing for $x < x(V)$, increasing for $x > x(V)$, $V \notin B^2$, and $\overline{CD}$ is not on the right of O (see Figure A2 (a)). If $y(A) \leqslant 0$, again by the estimate (14) we know that $L(\gamma \cap B^2) < 3.83$. In the case $y(A) > 0$, take a short translation of γ and get news points $A', B', C', D' \in \gamma \cap \partial B^2$, as above. To keep the properties above for the news points, we translating γ such that $C' \to V$, and $|\overline{C'D'}| = |\overline{CD}|$ is constant. We note that $L(\gamma)|_{C'}^{D'}$ increase because C' is approaching to V, and then the curvature of γ is increasing between C' and D'. Also, we have that γ remains in B^2 between C' and D', since $\gamma(x)$ is increase for $x > x(V)$, $|C'D'| \leqslant 1$, $\overline{C'D'}$ is not on the right of O, and then $\triangle C'E'D' \subset B^2$, where $E' = (x(D'), y(C'))$ (see Figure A2 (b)). Furthermore, note that the graphic of γ is going up and to the right, so $x(A'), y(A'), x(B')$ increase and $y(B')$ decrease, by the properties above. In particular, $|\overline{A'B'}|$ is increasing, B' is approaching to V, and then $L(\gamma)|_{A'}^{B'}$ is increasing. In the end get that $L(\gamma \cap B^2)$ increased, and $V \in B^2$ (see Figure A2 (b)).

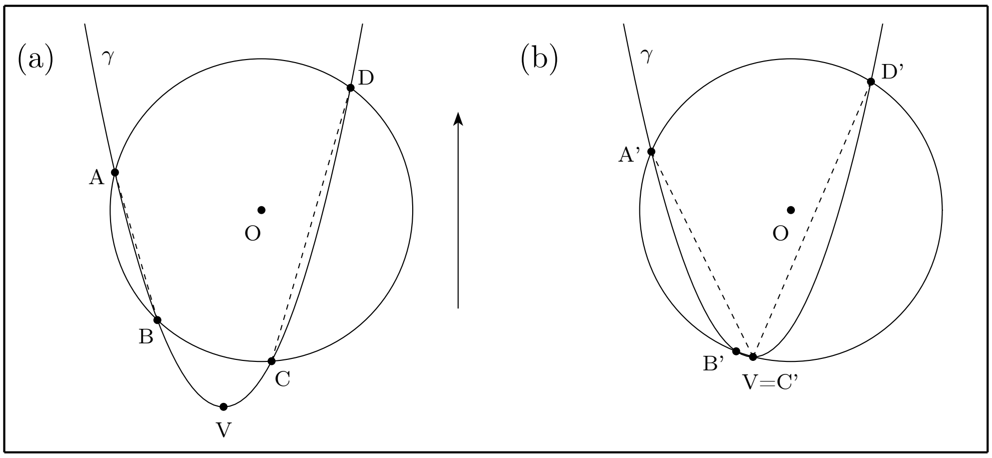
Figure A2



Source: Own construction

In the second case, we just know that $x(A), y(B), y(C) < 0$, $y(A), y(D), x(D) > 0$ (see Figure A3 (a)). In this case, we take a short translation of γ for $x(V)$ fixed and such that $y(V)$ increasing to get news points A', B', C', D' with the same properties. We take this translation as long as $\overline{A'B'}$ and $\overline{C'D'}$ do not pass through the origin O, and $V \notin \partial B^2$. In particular, these properties apply to the news points. Note that $x(A'), x(B'), y(A'), y(D')$ increase, $x(C'), x(D'), y(B'), y(C')$ decrease, then $|\overline{A'B'}|, |\overline{C'D'}|$ increase. Also, as B', C' are approaching to V, we have that $L(\gamma)|_{A'}^{B'}$ and $L(\gamma)|_{C'}^{D'}$ are increasing. We stop that translation when V touches ∂B^2, or when $\overline{A'B'}$ or $\overline{C'D'}$ pass through the origin, in the last case we continue with the translation as in the previous case until V touches ∂B^2. In the end, we get again that $L(\gamma \cap B^2)$ increased and $V \in B^2$ (see Figure A3 (b)).

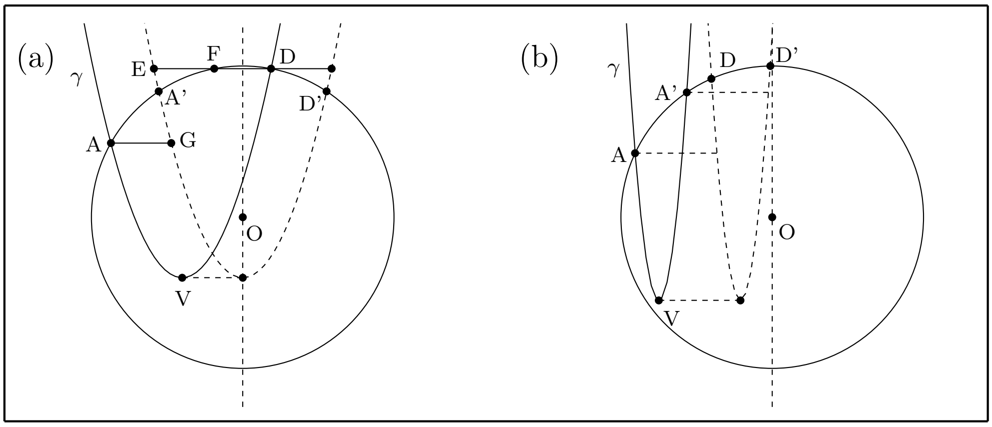
Figure A3



Source: Own construction

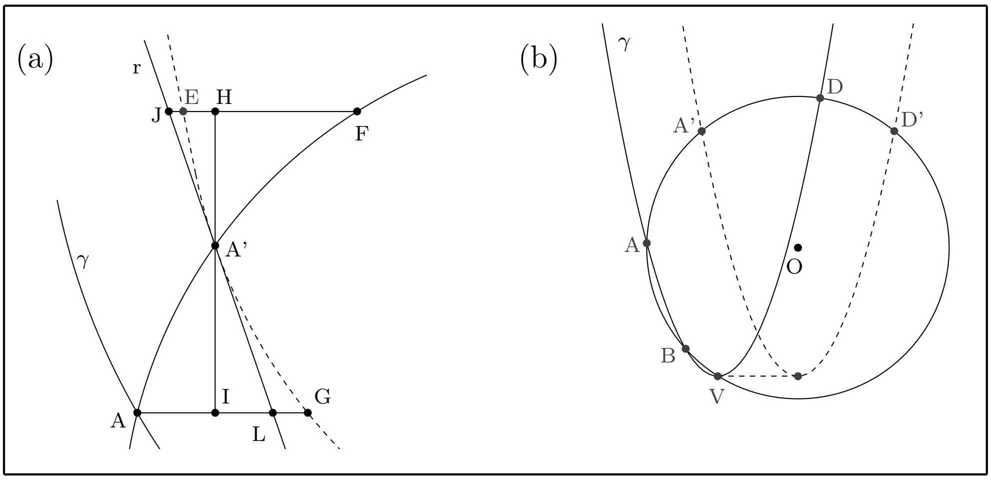
By the above arguments and by the estimates (14) and (A1), we have that $L(\gamma \cap B^2) < 3.83$, or the maximum length of $\gamma \cap B^2$ is reached for some γ such that $V \in B^2$, and such that there exist two points $A, D \in \gamma \cap \partial B^2$ with $y(A), y(D) > 0$. We will see that the second situation happens, and we will calculate it. First, we will see that if we fix γ in the last situation, then the maximum length of $\gamma \cap B^2$ is reached when $x(V) = 0$, in other words, when we translate γ such that $y(V)$=constant and $x(V) \to 0$. Indeed, suppose for now that $\gamma \cap B^2$ is connected, $x(A) < x(V) < x(D), |x(D)| < |x(A)|$, and consider a such translation so that $A \to A', D \to D'$. If $x(D) < 0$, then during the translation and as long as $x(D') < 0$, we have that the curve goes inside B^2, and in particular $L(\gamma \cap B^2)$ increase (see Figure A4 (b)). Now, consider the case $x(D) \geqslant 0$, as in the Figure A4 (a). In this case, let $|x(V)| = \epsilon > 0$. Take $E = (-(x(D) + \epsilon), y(D)), F = (-x(D), y(D)), G = (x(A) + \epsilon, y(A)), H = (x(A'), y(D))$, and $I = (x(A'), y(A))$, we claim that $|\overline{EH}| < |\overline{IG}|$. To see this, we take a straight line $r(x)$, which is tangent to $\gamma(x)$ in A', and take $J = (r^{-1}(y(D)), y(D)), L = ((r^{-1}(y(A)), y(A))$. Note that $\overline{HI}$ is orthogonal to $\overline{JF}$ and $\overline{AG}$, also $E, H \in \overline{JF}$, $I, L \in \overline{AG}$ and $\epsilon = |\overline{EF}| = |\overline{AG}|$ (see Figures A4 (a) and A5 (a)). Since $A, F \in \partial B^2$, and $|\overline{EF}| = |\overline{AG}|$, if $|\overline{EH}| \geqslant |\overline{IG}|$ then we would have $|\overline{HF}| \leqslant |\overline{AI}|$ and, therefore, $|\overline{HA'}| < |\overline{A'I}|$, because $y(A') > 0$. The latter and the fact that γ is convex imply that $\overline{EH} < \overline{JH} < \overline{IL} < \overline{IG}$, which is a contradiction. Let $\overline{\gamma}$ be the curve γ after the translation. The inequality $\overline{EH} < \overline{IG}$ means that $L(\overline{\gamma})\big|_{E}^{A'} < L(\overline{\gamma})\big|_{A'}^{G}$, since the curvature of γ is strictly increasing in the direction of the vertex V. Thus, the length of $\gamma \cap B^2$ increased after the translation $A \to A'$, $B \to B'$, because $L(\overline{\gamma})\big|_{E}^{A'}$ is the amount of the curve that comes out of B^2, and $L(\overline{\gamma})\big|_{A'}^{G}$ is the amount that comes into B^2.

Figure A4



Source: Own construction

Figure A5

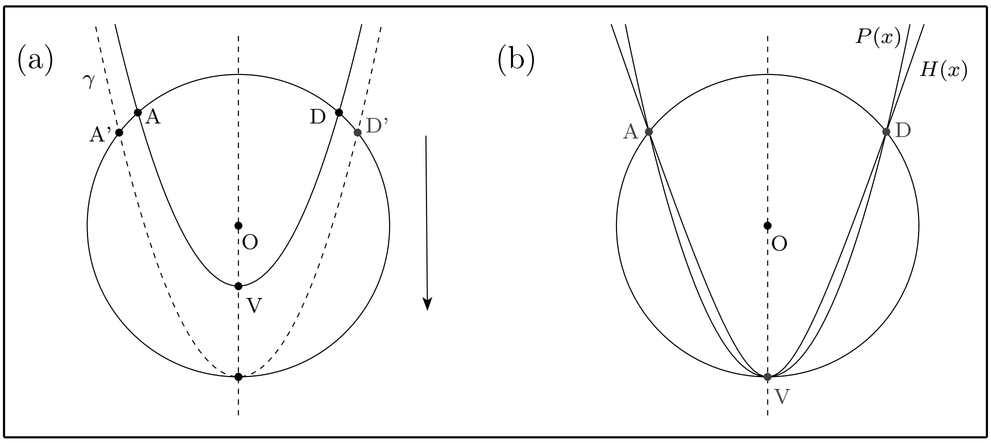


Source: Own construction

For the case $\gamma \cap B^2$ not connected, the length of $\gamma \cap B^2$ also increase after that translation. Indeed, in this case we already know that there exist $A, B, C, D \in \gamma \cap B^2$ such that γ is inside B^2 between A and B, and between C and D; otherwise is outside. Without loss of generality we can suppose $x(A) < x(B) < x(C) < x(D)$, $x(V) < 0$, and, as we are supposing that $y(A), y(D) > 0$, we have that $y(B), y(C) < 0$. Also, as $V \in B^2$, then $x(A) < x(V) \leqslant x(B)$ or $x(C) \leqslant x(V) < x(D)$. Actually, the case $x(A) < x(V) \leqslant x(B)$, $x(V) < 0$ implies that $x(A) < x(V) < x(B)$, so we can see in an analogous way as in the Figure A1 (b) that the latter can not happen, then $x(C) \leqslant x(V) < x(D)$. In particular, $x(A) < x(B) < x(C) < 0$. The extreme case $C = V$ is sketchy in the Figure A5 (b). Note that during the translation such that $x(V) \to 0$, we have that $x(A), y(A)$ increase since the graphic of γ goes to right. Moreover, $x(A) < 0$ during that translation. In the end, we get new points A', D' such that $-x(D') = x(A') < 0$, $y(D') = y(A') > 0$, and $V \in B^2$, the latter is because the vertex V is the global minimum of $\gamma(x)$. Finally, observe that $\gamma \cap B^2$ is now connected after the translation, since $y(A'), y(D') > 0$, $x(V) = 0$, and $\gamma \cap B^2$ has at most four points. In particular, $L(\gamma \cap B^2)$ increased, since the curve went inside B^2 for $\gamma(x) \leqslant 0$, and the previous paragraph for $\gamma(x) > 0$.

By the last two paragraphs, we need to find an upper bound for $L(\gamma \cap B^2)$, when $x(V) = 0$, $L(\gamma \cap B^2)$ is connected, and $\{\gamma \cap B^2\} \sim \{V\}$ is given by two points A, D such that $-x(A) = x(D)$, and $y(A) = y(D) > 0$. In this situation, if we translate γ such that $y(V) \to -1$, then $L(\gamma \cap B^2)$ increase. So, we will consider the last hypothesis above with $V = (0, -1)$, in other words, γ will be tangent to ∂B^2 in V (see Figure A6 (a)).

Figure A6



Source: Own construction

For us, the curve $\gamma(x)$ can be a hyperbola $H(x)$ with a unique branch intersecting B^2, or a parabola $P(x)$. To satisfy our situation, the equations become

$$H(x) = \frac{c}{d}\sqrt{d^2 + x^2} - (1 + c) \quad \text{and} \quad P(x) = ax^2 - 1,$$

where $a, c, d > 0$, $H(1) = (c/d)\sqrt{d^2 + 1} - (1 + c) > 0$, and $P(1) = a - 1 > 0$. Note that if the branch of the hyperbola and the parabola above pass through the same points A, D, as in the Figure A6 (b), then $H(x) \cap P(x) = \{A, D, V\}$, because of the symmetry and because that intersection can be at most four points. As $-x(A) = x(D) < 1$, we conclude that the graphic of $H(x)$ is above of $P(x)$ for $x(A) \leqslant x \leqslant x(D)$. In particular, $L(H(x) \cap B^2) < L(P(x) \cap B^2)$, so we only need to bound $L(P(x) \cap B^2)$ for $a > 1$. As $a > 1$, the points A, D can be determined uniquely by the value of the parameter a. In fact, $-x(A) = x(D) = x(a) = \sqrt{2a - 1}/a$, where $x(a)$ is the positive solution of $x^2 + (ax^2 - 1)^2 = 1$. Then, we can calculate $L(P(x) \cap B^2)$ in the parameter a :

$$\begin{aligned} L(P(x) \cap B^2) = L(a) &= 2\int_0^{x(a)} \sqrt{1 + 4a^2x^2}\, dx \\ &= \frac{\ln\left(\sqrt{4a^2x(a)^2 + 1} + 2ax(a)\right) + 2ax(a)\sqrt{4a^2x(a)^2 + 1}}{2a} \\ &= \frac{\ln\left(\sqrt{8a - 3} + 2\sqrt{2a - 1}\right) + 2\sqrt{2a - 1}\sqrt{8a - 3}}{2a}. \end{aligned}$$

By the expression above we have $L(P(x) \cap B^2) \to 4$ as $a \to \infty$ ($P(x) \cap B^2$ becomes two diameters). We will prove that $L(a)$ has a global maximum point in $a_0 < \infty$, and then $L(a_0) > 4$. Indeed, taking the derivative of the expression above, we obtain

$$L'(a) = \frac{8a - 3 - (1/2)\ln\left(\sqrt{8a - 3} + 2\sqrt{2a - 1}\right)\sqrt{8a - 3}\sqrt{2a - 1}}{a^2\sqrt{2a - 1}\sqrt{8a - 3}}.$$

Put $z = 2a - 1$, the denominator above becomes

$$4z + 1 - \frac{1}{2}\ln\left(\sqrt{4z + 1} + 2\sqrt{z}\right)\sqrt{4z + 1}\sqrt{z}.$$

So, the sign of $L'(a)$ is the sign of

$$\frac{2\sqrt{4z + 1}}{\sqrt{z}} - \ln\left(\sqrt{4z + 1} + 2\sqrt{z}\right). \tag{A2}$$

Note that the expression above starts positive for $a > 1$ and tends to $-\infty$ when $a \to \infty$, moreover it is strictly decreasing for $a > 1$. The last one is because the derivative of the last expression is given by

$$-\frac{z + z^2}{z^{5/2}\sqrt{4z+1}} < 0, \quad \text{for} \quad a > 1.$$

Therefore, there exists a unique $a_0 > 1$ such that $L'(a_0) = 0$. Moreover, $L(a)$ is strictly increasing for $1 < a < a_0$, and it is strictly decreasing for $a > a_0$. In particular, $L(a_0) > 4$ and $L(a_0)$ is the global maximum of $L(a)$, since $L(a) \to 4$ as $a \to \infty$. We can estimate a_0 such that (A2) becomes zero, and we obtain $a_0 \approx 94.091282$, and then $L_{max} = L(a_0) \approx 4.002671$.

$\square$