UNIVERSIDADE FEDERAL DE ALAGOAS
INSTITUTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA UFAL-UFBA

JOSÉ EDUARDO MILTON DE SANTANA

EQUILIBRIUM STATES AND INDUCED SCHEMES WITH SPECIAL HOLES FOR NON-UNIFORMLY EXPANDING MAPS

Maceió/AL
October 2018

JOSÉ EDUARDO MILTON DE SANTANA

EQUILIBRIUM STATES AND INDUCED SCHEMES WITH SPECIAL HOLES FOR NON-UNIFORMLY EXPANDING MAPS

Thesis presented to IM-UFAL, Institute of Mathematics of the Federal University of Alagoas, as partial fulfillment of requirements for the degree of Doctor in Mathematics and approved on October 05, 2018.

Advisor: Krerley Irraciel Martins Oliveira

Maceió/AL

October 2018

Catalogação na fonte
Universidade Federal de Alagoas
Biblioteca Central
Divisão de Tratamento Técnico
Bibliotecária Responsável: Helena Cristina Pimentel do Vale CRB4 - 661

S232e Santana, José Eduardo Milton de.
Equilibrium states and induced schemes with special holes for non- uniformly expandanding maps / José Eduardo Milton de Santana. – 2019.
46 f.

Orientador: Krerley Irraciel Martins Oliveira.
Tese (doutorado em Matemática) – Universidade Federal de Alagoas. Instituto de Matemática. Programa de Pós-Graduação de Doutorado em Matemática Interinstitucional UFBA/UFAL. Maceió, 2018.

Bibliografia: f. 43-46.

1. Sistemas dinâmicos. 2. Estados de equilíbrio. 3. Formalismo termodinâmico. 4. Esquemas induzidos (Matemática). I. Título.

CDU: 517.9

AUTHOR: JOSÉ EDUARDO MILTON DE SANTANA

EQUILIBRIUM STATES AND INDUCED SCHEMES WITH SPECIAL HOLES FOR NON-UNIFORMLY EXPANDING MAPS

Thesis presented to IM-UFAL, Institute of Mathematics of the Federal University of Alagoas, as partial fulfillment of requirements for the degree of Doctor in Mathematics and approved on October 05, 2018.

Krerley Oliveira

Dr. Krerley Irraciel Martins Oliveira, UFAL (Advisor)

Stefano Luzzatto

Dr. Stefano Luzzatto, ICTP (External Member)

José Ferreira Alves

Dr. José Ferreira Alves, University of Porto (External Member)

Dr. Vilton Jeovan Viana Pinheiro, UFBA (External Member)

Wagner Ranter

Dr. Wagner Ranter Gouveia da Silva, UFAL

ACKNOWLEDGMENTS

This is not such an easy part of the thesis because I am grateful to so many people and I do not want to forget anyone.

Firstly, according to my Catholic faith, I thank God for blessing me since my birth, by giving me a marvelous family, friends and teachers. They supported me in times of troubles and encouraged me all the time. Also, I thank God for giving me health and opportunities to study and to do in life what I really love to do: Mathematics!

I wish to specially thank some people:

Family: My mother Maria, for always being present in my life and light on my path. My father João, for always being an example of courage and strength. My brother Júnior and sister Ionara for their friendship and for taking care of our parents when I am not with them. My aunt Milta, for being an example of fight and dedication in studies. They all are examples of fight in life. I thank all the support and tender all the time!

Friends: I thank Antônio Larangeiras, Isnaldo Barbosa, Carllos Eduardo Holanda, Davi Lima, Micael Dantas, Rodrigo Lima, Wagner Xavier, Jamerson Douglas, Diogo Santos, Ailton Campos, André Timótheo, Sergio Ibarra, Ítalo Melo, Abraão Mendes, Rafael Alvarez, Anderson Lima, Rafael Lucena, Pedro Carvalho, Matheus Martins, Giovane Ferreira, Marlon Oliveira, Vitor Alves, Alexander Arbieto, Carlos Matheus Santos, Mauricio Polletti, Sandoel Vieira, Alex Zamudio, Wanderson Costa, Hugo Araújo, Ermerson Araújo, Fernando Lenarduzzi, Anuar Montalvo, Gabriela Estevez, Yaya Tall, Cássia Monalisa, Isania Soares, Catarina Silva, Juliana Theodoro de Lima, Cícero Calheiros, Alyson Pereira, Denilson Tenório, Abinaldo Aluizio. Thank you all for the support and moments of fun!

Professors: Firstly, I thank my advisor, Krerley Oliveira, for his friendship and constant support and incentive (it was a great pleasure having him as an advisor). I thank the thesis jury members (Stefano Luzzatto, José F. Alves, Vilton Pinheiro and Wagner Ranter) for their valious comments and suggestions. I thank Enrique Pujals for letting me participate of his seminars at IMPA and for his advices; Mikhail Belolipetsky, for his support and encouragement; Carlos Gustavo Moreira (Gugu) for the financial support during a staying at IMPA and Paulo Ribenboim for his friendship, support and advices.

I also thank Valdenberg Silva, Ali Golmakani, Fernando Micena, André Contiero, Yuri Lima, Feliciano Vitório, Márcio Batista, Marcos Petrúcio, Amauri Barros and Luis Guillermo. In general, I thank all the people who taught me every single thing in my life to make me who I am.

I thank Institute of Mathematics (IM / UFAL) for letting me to be away to work on my thesis at IMPA and people of IMPA for well receive me in all my staying there. Finally, I thank all the people that contributed in any way to help me in this journey. Thank you so much, you all are amazing!

Finally, I thank CAPES for the financial support.

"Everything happens in the right time!"

ABSTRACT

We prove existence of equilibrium states and induced schemes with special holes for continuous non-uniformly expanding maps. The potential we consider is the hyperbolic one, which means that the pressure emanates from the expanding set. For zooming maps, we consider the open dynamics with hole obtained from small balls by erasing intersections with regular pre-images and construct induced schemes with the property of respecting the hole.

Keywords: Equilibrium state; Induced scheme; Open Dynamics; Zooming Map; Hole; Hyperbolic Potential.

RESUMO

Provamos a existência de estados de equilíbrio e esquemas induzidos com buracos especiais para mapas não-uniformemente expansores contínuos. O potencial considerado é o hiperbólico, o que significa que a pressão emana do conjunto expansor. Para mapas zooming, nós consideramos a dinâmica aberta com buraco obtido a partir de bolas suficientemente pequenas, ao retirar-se interseções com pré-imagens regulares e construímos esquemas induzidos com a propriedade de respeitar o buraco.

Palavras-chave: Estado de equilíbrio; Esquema induzido; Dinâmica Aberta; Mapa Zooming; Buraco; Potencial Hiperbólico.

Contents

Chapter 1

Introduction

In this thesis, we prove two results. In chapter 2 we prove the existence of finitely many ergodic equilibrium states that are expanding measures, in the context of continuous non-uniformly expanding maps and hyperbolic potentials. This class of maps extends the notion of differentiable non-uniformly expansion, where some expansion is obtained in the presence of hyperbolic times. The hyperbolic potentials are important because the pressure emanates from the expanding set, which is the set of points with positive frequence of hyperbolic times. This means that is where we obtain most of expansion. So, in the study of the thermodynamic formalism, we pay attention to the expanding measures, which give total mass for the expanding set.

The strategy is to use induced schemes constructed by Pinheiro in [25] to code the dynamics. Pinheiro proves that there are finetely many induced schemes for which we can lift an ergodic expanding measure. Once we code the dynamics, we use the results by Sarig in [28], [29], [30], [31] to obtain equilibrium states for the associated countable shift. In this context, the equilibrium state is unique and also it is a Gibbs measure. Then we can get at most finitely many ergodic equilibrium states that are expanding measures. It is obtained by showing that the inducing time is integrable, by following ideas of Iommi and Todd in [18]. Finally, we get equilibrium states for the hyperbolic potentials and they are, at most, finitely many.

In chapter 3 we consider open zooming maps and construct induced schemes that respects a certain type of holes. This idea appears in [13] for Collet-Eckmann maps.

A zooming map generalizes the notion of non-uniformly expansion by generalizing the notion of hyperbolic times, obtaining the zooming times. In the same way, we get expansion of certain type, in the context of continuous maps. It is the key ingredient to construct the induced schemes. We follow ideas of Pinheiro in [25]. The idea is considering disjoint balls sufficiently small located in certain regions of the phase space and erase intersections with regular pre-images, getting a special open set. The presence of points with zooming returns allows us to take regular pre-images inside this open set, constructing elements of the partition in one of the connected components. At the end, we have and induced scheme and a special hole, where the induced scheme respects the hole.

The class of maps introduced by Viana in [34], and called Viana maps, is an important example for our settings.

Chapter 2

Equilibrium States for Hyperbolic Potentials

2.1 Introduction

The theory of equilibrium states was firstly developed by Sinai, Ruelle and Bowen in the sixties and seventies. It was based on applications of techniques of Statistical Mechanics to smooth dynamics.

Given a continuous map $f : M \to M$ on a compact metric space M and a continuous potential $\phi : M \to \mathbb{R}$, an ***equilibrium state*** is an invariant measure that satisfies an Variational Principle, that is, a measure μ such that

$$h_\mu(f) + \int \phi d\mu = \sup_{\eta \in \mathcal{M}_f(M)} \left\{ h_\eta(f) + \int \phi d\eta \right\},$$

where $\mathcal{M}_f(M)$ is the set of f-invariant probabilities on M and $h_\eta(f)$ is the so-called metric entropy of η.

In the context of uniform hyperbolicity, which includes uniformly expanding maps, equilibrium states do exist and are unique if the potential is Hölder continuous and the map is transitive. In addition, the theory for finite shifts was developed and used to achieve the results for smooth dynamics.

Beyond uniform hyperbolicity, the theory is still far from complete. It was studied by several authors, including Bruin, Keller, Demers, Li, Rivera-Letelier, Iommi and Todd [11],[10],[13]),[17],[18],[19] for interval maps; Denker and Urbanski [14] for rational maps; Leplaideur, Oliveira and Rios [20] for partially hyperbolic horseshoes; Buzzi, Sarig and Yuri [12],[36], for countable Markov shifts and for piecewise expanding maps in one and higher dimensions.

For local diffeomorphisms with some kind of non-uniform expansion, there are results due to Oliveira [21]; Arbieto, Matheus and Oliveira [7]; Varandas and Viana [33]. All of whom proved the existence and uniqueness of equilibrium states for potentials with low oscillation. Also, for this type of maps, Ramos and Viana [27] proved it for potentials so-called ***hyperbolic***, which includes the previous ones. The hyperbolicity of the potential is characterized by the fact that the pressure emanates from the hyperbolic region.

Our result is similar to the Ramos and Viana, but for maps presenting some kind of non-uniform expansion and critical points. However, our strategy is completely different, since we do not use the analytical approach of the Transfer Operator in order to obtain conformal measures. We use results on countable Markov shifts by Sarig for the "coded" dynamics in induced schemes constructed by Pinheiro in [25], where a Markov structure is constructed. We prove that there exist finitely many ergodic equilibrium states that are expanding measures.

2.2 Definitions and Statement of the Result

We begin by defining non-uniformly expanding maps in a non-differentiable context.

2.2.1 Non-Uniformly Expanding Maps

Let M be a connected compact metric space, $f : M \to M$ a continuous map and μ a reference Borel measure on M. Firstly, we will define hyperbolic times.

Fix $\sigma \in (0,1), \delta > 0$ and $x \in M$.

Definition 2.2.1 *We say that $n \in \mathbb{N}$ is a (σ, δ)-**hyperbolic time** for x if*

- *There exists a neighbourhood $V_n(x)$ of x such that $f^n_{|V_n(x)} : V_n(x) \to B_\delta(f^n(x))$ is a homeomorphism;*

- $d(f^i(y), f^i(z)) \leq \sigma^{n-i} d(f^n(y), f^n(z)), \forall\ y, z \in V_n(x), \forall\ 0 \leq i \leq n-1.$

The sets $V_n(x)$ are called ***hyperbolic pre-balls*** *and their images $f^n(V_n(x)) = B_\delta(f^n(x))$,* ***hyperbolic balls****.*

We say that $x \in M$ has ***positive frequency*** *of hyperbolic times if*

$$\limsup_{n\to\infty} \frac{1}{n} \#\{0 \le j \le n-1 \mid j \text{ is a hyperbolic time for } x\} > 0$$

We denote by

$$H = \{x \in M \mid \text{the frequency of hyperbolic times of } x \text{ is positive}\}.$$

We call H the ***expanding set*** and we say that the reference measure μ is ***expanding*** if $\mu(H) = 1$.

Definition 2.2.2 *Given a measure μ on M, its* ***jacobian*** *is a function $J_\mu f : M \to [0, +\infty)$ such that $\mu(f(A)) = \int_A J_\mu f d\mu$ for every A* ***domain of injectivity****, that is, a measurable set such that $f(A)$ is measurable and $f_A : A \to f(A)$ is a bijection.*

Definition 2.2.3 *Given a measure μ with a jacobian $J_\mu f$, we say that the measure has* ***bounded distortion*** *if there exists $\rho > 0$ such that*

$$\left| \log \frac{J_\mu f^n(y)}{J_\mu f^n(z)} \right| \le \rho d(f^n(y), f^n(z))$$

for every $y, z \in V_n(x)$, μ-almost everywhere $x \in M$, for every hyperbolic time n of x.

A map with an expanding measure with bounded distortion associated is called **non-uniformly expanding**.

Remark 2.2.1 *We observe that a hyperbolic time in our context is a type of zooming time in the context of [25].*

2.2.2 Induced Schemes

We recall the definition of an induced scheme. It is useful to code the dynamics in order to use results concerning symbolic dynamics.

Definition 2.2.4 *Given an open set $\mathcal{U} \subset M$ and $\mathcal{P} = \{P_1, \dots, P_n, \dots\}$ a partition of open subsets in $\mathcal{U}$, if there exists a map $F : \mathcal{U} \to \mathcal{U}$ such that $F_{|P_i} : P_i \to \mathcal{U}$ is an homeomorphism for all $i \in \mathbb{N}$ and if for every element P_i there exists $\tau_i \in \mathbb{N}$ such that $F_{|P_i} = f^{\tau_i}_{|P_i}$, F is said to be an* ***induced map*** *and we call the pair $(F, \mathcal{P})$ an* ***induced***

scheme *on* $\mathcal{U}$. *The function* $\tau : \mathcal{U} \to \mathbb{N}$ *such that* $\tau_{|P_i} := \tau_i$ *is called the* ***inducing time***.

Definition 2.2.5 *Given an induced scheme* $(F, \mathcal{P})$ *and an invariant probability* μ, *we say that* μ *is* ***liftable*** *to* $(F, \mathcal{P})$ *if there exists a measure* $\overline{\mu}$ *such that for every measurable set* A,

$$\mu(A) = \sum_{k=1}^{\infty} \sum_{j=0}^{\tau_k - 1} \overline{\mu}(f^{-j}(A) \cap P_k)$$

The next result, due to Pinheiro, guarantee that every ergodic expanding measure can be lifted to some induced scheme.

Theorem 2.2.1 *(Pinheiro [25]) There exist finitely many induced schemes*

$$(F_1, \mathcal{P}_1), \ldots, (F_s, \mathcal{P}_s),$$

such that every ergodic probability μ *for which* $\mu(H) = 1$ *is liftable to some of these induced schemes, with uniformly bounded integral of inducing time.*

2.2.3 Markov Shifts

Now we recall the basic definitions of symbolic dynamics. Given a countable set S, we define the ***space of symbols*** $\Sigma := \{(x_1, x_2, \ldots, x_n, \ldots) \mid x_i \in S, \forall i \in \mathbb{N}\}$. The ***shift map*** $\sigma : \Sigma \to \Sigma$ is defined by $\sigma((x_1, x_2, \ldots, x_n, \ldots)) = (x_2, x_3, \ldots, x_n, \ldots)$. A ***cylinder*** is a set of the form $C_n := \{x \in \Sigma : x_1 = a_1, \ldots, x_n = a_n\}$.

When an induced scheme $(F, \mathcal{P})$ is given, we can define a space of symbols by the following rule. Let $x \in \mathcal{U}$ be a point such that $F^k(x)$ is well defined for all $k \in \mathbb{N}$. To obtain a sequence $(x_1, x_2, \ldots, x_n, \ldots)$, we put $x_i = j$ if $F^i(x) \in P_j$. So, we can see that the map F is conjugate to the shift map. The advantage here is that we can use the theory of symbolic dynamics to obtain results for our original map.

2.2.4 Topological Pressure

We recall the definition of relative pressure for non-compact sets by dynamical balls, as it is given in [4].

Let M be a compact metric space. Consider $f : M \to M$ and $\phi : M \to \mathbb{R}$. Given $\delta > 0$, $n \in \mathbb{N}$ and $x \in M$, we define the ***dynamical ball*** $B_\delta(x, n)$ as the set

$$B_\delta(x, n) := \{y \in M \mid d(f^i(x), f^i(y)) < \delta, \text{ for } 0 \leq i \leq n\}.$$

Consider for each $N \in \mathbb{N}$, the set

$$\mathcal{F}_N = \{B_\delta(x,n) \mid x \in M, n \geq N\}.$$

Given $\Lambda \subset M$, denote by $\mathcal{F}_N(\Lambda)$ the finite or countable families of elementes in $\mathcal{F}_N$ that cover Λ. Define for $n \in \mathbb{N}$

$$S_n\phi(x) = \phi(x) + \phi(f(x)) + \cdots + \phi(f^{n-1}(x)).$$

and

$$R_{n,\delta}\phi(x) = \sup_{y \in B_\delta(x,n)} S_n\phi(y).$$

Given a f-invariant set $\Lambda \subset M$, not necessarily compact, define for each $\gamma > 0$

$$m_f(\phi, \Lambda, \delta, N, \gamma) = \inf_{\mathcal{U} \in \mathcal{F}_N(\Lambda)} \left\{ \sum_{B_\delta(y,n) \in \mathcal{U}} e^{-\gamma n + R_{n,\delta}\phi(y)} \right\}.$$

Define

$$m_f(\phi, \Lambda, \delta, \gamma) = \lim_{N \to +\infty} m_f(\phi, \Lambda, \delta, N, \gamma).$$

and

$$P_f(\phi, \Lambda, \delta) = \inf\{\gamma > 0 \mid m_f(\phi, \Lambda, \delta, \gamma) = 0\}.$$

Finally, define the ***relative pressure*** of ϕ on Λ as

$$P_f(\phi, \Lambda) = \lim_{\delta \to 0} P_f(\phi, \Lambda, \delta).$$

The ***topological pressure*** of ϕ is, by definition, $P_f(\phi) = P_f(\phi, M)$ and satisfies

$$P_f(\phi) = \sup\{P_f(\phi, \Lambda), P_f(\phi, \Lambda^c)\} \tag{2.1}$$

where Λ^c denotes the complement of Λ on M. We refer the reader to [24] for the proof of 2.1 and for additional properties of the pressure. See also [35] for a proof of the fact that

$$P_f(\phi) = \sup_{\mu \in \mathcal{M}_f(M)} \left\{ h_\mu(f) + \int \phi d\mu \right\}.$$

2.2.5 Hyperbolic Potentials

Definition 2.2.6 *We say that a real continuous function $\phi : M \to \mathbb{R}$ is a **hyperbolic potential** if the topological pressure $P_f(\phi)$ is located on H, i.e.,*

$$P_f(\phi, H^c) < P_f(\phi).$$

In [19] H. Li and J. Rivera-Letelier consider other type of hyperbolic potentials for one-dimensinal dynamics that is weaker than ours. In their context ϕ is a hyperbolic potential if

$$\sup_{\mu\in\mathcal{M}_f(M)}\int \phi d\mu < P_f(\phi).$$

2.2.6 Equilibrium States

Given a continuous map $f : M \to M$ and a Hölder potential $\phi : M \to \mathbb{R}$, an ***equilibrium state*** is an invariant measure that satisfies an Variational Principle, that is, a measure μ such that

$$h_\mu(f) + \int \phi d\mu = \sup_{\eta\in\mathcal{M}_f(M)}\left\{h_\eta(f) + \int \phi d\eta\right\},$$

where $\mathcal{M}_f(M)$ is the set of invariant probabilities on M and $h_\eta(f)$ is the so-called metric entropy of η.

2.2.7 Statement of Main Result

Now, we state our main result on the existence of equilibrium states.

Theorem A *Given a non-uniformly expanding map $f : M \to M$ and a Hölder hyperbolic potential $\phi : M \to \mathbb{R}$ with finite topological pressure $P_f(\phi)$, there exist finitely many ergodic equilibrium states and they are expanding measures.*

2.3 Equilibrium States for the Lifted Dynamics

In this section we begin the proof of Theorem A. The strategy is lifting the dynamics, finding equilibrium states and then projecting them.

2.3.1 Hyperbolic Potentials and Expanding Measures

The next proposition and the Theorem 2.2.1 guarantee that every ergodic expanding measure with *"high free energy"* and, in particular, those which are candidates to be equilibrium states can be lifted to some induced scheme.

Proposition 2.3.1 *If ϕ is a hyperbolic potential, given an ergodic probability μ such that $h_\mu(f) + \int \phi d\mu > P_f(\phi, H^c)$, then $\mu(H) = 1$.*

Proof. Since H is an invariant set and μ is an ergodic probability, we have $\mu(H)=0$ or $\mu(H)=1$. But,

$$h_\mu(f)+\int\phi d\mu > P_f(\phi,H^c)\geq \sup_{\nu(H^c)=1}\left\{h_\nu(f)+\int_{H^c}\phi d\nu\right\}$$

(For the second inequality, see Pesin [24], Theorem A2.1)

So, we can not have $\mu(H^c)=1$ and we obtain $\mu(H)=1$,i.e., μ is an expanding measure.

□

2.3.2 Equilibrium States for Markov Shifts

Definition 2.3.1 *Given a potential $\phi: M\to\mathbb{R}$ and an induced scheme $(F,\mathcal{P})$, we define the* ***induced potential*** *as $\overline{\phi}(x)=\sum_{j=0}^{\tau(x)-1}\phi(f^j(x))$.*

Definition 2.3.2 *Given a potential $\overline{\phi}:\Sigma\to\mathbb{R}$, we say that $\overline{\phi}$ is* ***locally Hölder*** *if there exist $A>0$ and $\theta\in(0,1)$ such that for all $n\in\mathbb{N}$ the following holds*

$$V_n(\overline{\phi}):=\sup\{\mid\overline{\phi}(x)-\overline{\phi}(y)\mid,\forall x,y\in C_n\}\leq A\theta^n.$$

Proposition 2.3.2 *If $\phi: M\to\mathbb{R}$ is a Hölder potential, then $\overline{\phi}:\Sigma\to\mathbb{R}$ is a locally Hölder potential.*

Proof. As ϕ is Hölder, there are constants $\rho,\alpha>0$ such that $\mid\phi(x)-\phi(y)\mid\leq \rho d(x,y)^\alpha$. We must show that there are constants $A>0$ and $\theta\in(0,1)$ such that $\mid\overline{\phi}(x)-\overline{\phi}(y)\mid\leq A\theta^n,\forall x,y\in C_n$. In fact, given $x,y\in C_n$, there are $P_{i_0},P_{i_1},\ldots,P_{i_n}$ such that $F^k(x),F^k(y)\in P_{i_k}$. Then, we have

$$\mid\overline{\phi}(x)-\overline{\phi}(y)\mid=\left|\sum_{j=0}^{\tau_{i_0}-1}\phi(f^j(x))-\phi(f^j(y))\right|\leq\sum_{j=0}^{\tau_{i_0}-1}\mid\phi(f^j(x))-\phi(f^j(y))\mid\leq$$

$$\leq\rho\sum_{j=0}^{\tau_{i_0}-1}d(f^j(x),f^j(y))^\alpha\leq\rho\sum_{j=0}^{\tau_{i_0}-1}(\sigma^{\tau_{i_0}-j}d(F(x),F(y))^\alpha\leq$$

$$\leq\rho\sum_{j=0}^{\infty}(\sigma^\alpha)^j d(F(x),F(y))^\alpha=A\theta^n.$$

where $A:=\rho(\sum_{j=0}^{\infty}(\sigma^\alpha)^j/\sigma^\alpha)\delta$, $\theta:=\sigma^\alpha$. □

As a consequence of Theorem 2.2.1, there exist an induced scheme $(F,\mathcal{P})$ and a sequence μ_n of liftable ergodic probabilities such that $h_{\mu_n}(f)+\int\phi d\mu_n\to P_f(\phi)$.

Proposition 2.3.3 *(Abramov's Formulas [37]) Given a measure μ liftable to $\overline{\mu}$, we have the formulas*

$$h_\mu(f) = \frac{h_{\overline{\mu}}(F)}{\int \tau d\overline{\mu}}, \int \phi d\mu = \frac{\int \overline{\phi} d\overline{\mu}}{\int \tau d\overline{\mu}}.$$

As a consequence, we have

$$h_{\overline{\mu}}(f) + \int \overline{\phi} d\overline{\mu} = \Big(\int \tau d\overline{\mu} \Big) \Big(h_\mu(f) + \int \phi d\mu \Big).$$

Definition 2.3.3 *Given a Markov shift (Σ, σ), we define the **Gurevich Pressure** as*

$$P_G(\overline{\phi}) := \lim_{n\to\infty} \frac{1}{n} \log \Big(\sum_{\sigma^n(x)=x, x_0=a} e^{\overline{\phi}_n(x)} \Big), \text{ where } \overline{\phi}_n(x) = \sum_{j=0}^{n-1} \overline{\phi}(f^j(x)).$$

Theorem 2.3.1 *(Sarig [28]) If (Σ, σ) is topologically mixing and $\overline{\phi}$ is locally Hölder, then the Gurevich Pressure is well defined and independent of a.*

Theorem 2.3.2 *(Approximation Property [28])(Sarig) If (Σ, σ) is topologically mixing and $\overline{\phi}$ is locally Hölder, we have*

$$P_G(\overline{\phi}) = \sup\{P_{top}(\overline{\phi}_{|Y}), Y \subset \Sigma \text{ is a topologically mixing finite Markov shift}\}.$$

Theorem 2.3.3 *(Variational Principle)(Iommi-Jordan-Todd [16]) Let (Σ, σ) be a topologically mixing countable Markov shift and $\overline{\phi} : \Sigma \to \mathbb{R}$ a potential with summable variations (i.e., $\sum_{n=1}^{\infty} V_n(\overline{\phi}) < \infty$). Then we have*

$$P_G(\overline{\phi}) = \sup \Big\{ h_\nu(F) + \int \overline{\phi} d\nu : - \int \overline{\phi} d\nu < \infty \Big\}.$$

Given $\phi : M \to \mathbb{R}$ a hyperbolic potential, set $\varphi := \phi - P_f(\phi)$. So, φ is a hyperbolic potential such that $P_f(\varphi) = 0$.

Proposition 2.3.4 *Given a Hölder hyperbolic potential φ such that $P_f(\varphi) = 0$, we have that $P_G(\overline{\varphi}) = 0$.*

Proof. $P_G(\overline{\varphi}) \geq 0$: By Pinheiro's Theorem, there exist a sequence μ_n of liftable ergodic probabilities. By the Abramov's Formulas, we have that

$$h_{\overline{\mu}_n}(f) + \int \overline{\phi} d\overline{\mu}_n = \Big(\int \tau d\overline{\mu}_n \Big) \Big(h_{\mu_n}(f) + \int \phi d\mu_n \Big).$$

Since $h_{\mu_n}(f) + \int \varphi d\mu_n \to P_f(\varphi) = 0$ and the sequence $\{ \int \tau d\overline{\mu}_n \}_n$ is uniformly bounded, we obtain $h_{\overline{\mu_n}}(F) + \int \overline{\varphi} d\overline{\mu_n} \to 0$, which means $0 \leq P(\overline{\varphi})$.

$P_G(\varphi) \leq 0$: By taking the finite Markov subshift with symbols $P_1, P_2, ..., P_N$, denoted by (Σ_N, σ_N), we obtain an equilibrium state $\overline{\nu}_N$ such that $h_{\overline{\nu_N}}(F) + \int \overline{\varphi} d\overline{\nu_N} = P_{top}(\varphi_{|\Sigma_N}) \to P_G(\overline{\varphi})$ (By the approximation property). By contradiction, if $P_G(\overline{\varphi}) > 0$, for some N big enough we obtain

$$0 < h_{\overline{\nu}_N}(f) + \int \overline{\varphi} d\overline{\nu}_N = \Big(\int \tau d\overline{\nu}_N \Big) \Big(h_{\nu_N}(f) + \int \varphi d\nu_N \Big).$$

that implies

$$0 < h_{\nu_N}(f) + \int \varphi d\nu_N < P_{top}(\varphi) = 0.$$

a contradiction. So, $P_G(\overline{\varphi}) \leq 0$.

Finally, by the Variational Principle, we obtain $0 \leq P_{top}(\overline{\varphi}) = P_G(\overline{\varphi}) \leq 0$ and $P_G(\overline{\varphi}) = 0$. □

Definition 2.3.4 *If we denote the matrix transition of (Σ, σ) by $T = (t_{ij})$, the **Big Images and Preimages (BIP) Property** is*

$$\exists\ b_1, \ldots, b_N \in S \text{ such that } \forall a \in S\ \exists\ i, j \in \{1, \ldots, N\} \text{ such that } t_{b_i a} t_{a b_j} = 1.$$

Clearly, if (Σ, σ) is a full shift, it has the BIP property.

We are able to use the Sarig's results on the existence and uniqueness of conformal measures, Gibbs measures and equilibrium states for countable Markov shifts. The original works are [28], [29], [30], but the results can all be found in [31]. They can be summarized as follows.

Theorem 2.3.4 *(Sarig [31]) Let (Σ, σ) be topologically mixing and $\overline{\varphi}$ of summable variations. Then $\overline{\varphi}$ has an invariant Gibbs measure $\mu_{\overline{\varphi}}$ if, and only if, it has the BIP property and $P_G(\overline{\varphi}) < \infty$. Moreover, the Gibbs measure has the following properties:*

- *If $h_{\mu_{\overline{\varphi}}}(\sigma) < \infty$ or $-\int \overline{\varphi} d\mu_{\overline{\varphi}} < \infty$, then $\mu_{\overline{\varphi}}$ is the unique equilibrium state (in particular, $P_G(\overline{\varphi}) = h_{\mu_{\overline{\varphi}}}(\sigma) + \int \overline{\varphi} d\mu_{\overline{\varphi}}$).*
- *$\mu_{\overline{\varphi}}$ is finite and $d\mu_{\overline{\varphi}} = h_{\overline{\varphi}} dm_{\overline{\varphi}}$, where $L(h_{\overline{\varphi}}) = \lambda h_{\overline{\varphi}}$ and $L^*(m_{\overline{\varphi}}) = \lambda m_{\overline{\varphi}}$, $\lambda = e^{P_G(\overline{\varphi})}$. It means that $m(\sigma(A)) = \int_A e^{\overline{\varphi} - \log \lambda} dm_{\overline{\varphi}}$.*
- *This $h_{\overline{\varphi}}$ is unique and $m_{\overline{\varphi}}$ is the unique $(\overline{\varphi} - \log \lambda)$-conformal probability measure.*

From this Theorem, we obtain a unique $(\overline{\varphi} - \log \lambda)$-conformal probability measure $m_{\overline{\varphi}}$ and a unique equilibrium state $\mu_{\overline{\varphi}}$, which is also a Gibbs measure.

We now need to show that the inducing time is integrable in order to project the Gibbs measure.

2.4 Finiteness of Ergodic Equilibrium States

We will prove, by following ideas of Iommi and Todd, that the inducing time is integrable with respect to the Gibbs measure $\mu_{\overline{\varphi}}$. As a consequence, we can project it to a measure μ_φ, which will be shown to be an equilibrium state for the original system (f, φ).

2.4.1 Measures with Low Free Energy

We state the results we adapted from ([18]) and whose proofs are exactly the same. We reproduce the main steps here.

Proposition 2.4.1 *Given a hyperbolic potential φ with $P_f(\varphi) = 0$ and an induced scheme $(\tilde{F}, \mathcal{P})$, there exists $k \in \mathbb{N}$ such that replacing $(\tilde{F}, \mathcal{P})$ by $(F, \mathcal{P})$, where $F = \tilde{F}^k$, the following holds. There exists $\gamma_0 \in (0,1)$ and for any cylinder $C_n \in \mathcal{P}_n^F$, any $n \in \mathbb{N}$, a constant $\delta_n < 0$ such that any measure $\mu_F \in \mathcal{M}_F$ with*

$$\mu_F(C_n) \leq (1-\gamma_0) m_{\overline{\varphi}}(C_n) \text{ or } m_{\overline{\varphi}}(C_n) \leq (1-\gamma_0)\mu_F(C_n),$$

where $m_{\overline{\varphi}}$ is the conformal measure for the system $(F, \overline{\varphi})$, must have $h_{\mu_F}(F) + \int \overline{\varphi} d\mu_F < \delta_n$.

The following lemma will allow us to choose k in the above proposition.

Lemma 2.4.1 *Suppose that we have an induced scheme $(F, \mathcal{P})$ and a locally Hölder potential $\overline{\varphi}$ with distortion constant $K := \exp\left(\sum_{j=1}^{\infty} V_j(\overline{\varphi})\right)$ and $P_G(\overline{\varphi}) = 0$. We let $m_{\overline{\phi}}$ denote the conformal measure for the system $(F, \overline{\varphi})$. Then, for any $C_n \in \mathcal{P}_n^F$ and any $n \in \mathbb{N}$,*

$$m_{\overline{\varphi}}(C_n) \leq e^{-\lambda n}$$

where $\lambda := -\log(K \sup_{C_1 \in \mathcal{P}_1^F} m_{\overline{\varphi}}(C_1))$.

Suppose that the distortion of the potential $\tilde{\overline{\varphi}}$ for the scheme $(\tilde{F}, \mathcal{P})$ is bounded by $K \geq 1$. We first prove that measures giving cylinders very small mass compared to $m_{\overline{\varphi}}$ must have low free energy. Note that for any $k \in \mathbb{N}$, the potential $\overline{\varphi}$ for the scheme $(F, \mathcal{P})$, where $F = \tilde{F}^k$ also has distortion bounded by K. We will choose k later so that λ for $(F, \mathcal{P})$, as defined in Lemma 2.4.1, is large enough to satisfy the conditions associated to 2.3,2.4 and 2.6. Note that as in [[32], Lemma 3] we also have $P_G(\overline{\varphi}) = 0$.

In Lemma 2.4.3 below, we will use the Variational Principle to bound the free energy of measures for the scheme which, for some γ, have $\mu(C_n^i) \leq Km_{\overline{\varphi}}(C_n^i)(1-\gamma)/(1-m_{\overline{\varphi}}(C_n^i))^n$ in terms of the Gurevich pressure. However, instead of using $\overline{\varphi}$, which, in the computation of Gurevich pressure weights points $x \in C_n^i$ by $e^{\overline{\varphi}(x)}$, we use a potential which weighs points in C_n^i by $(1-\gamma)e^{\overline{\varphi}(x)}$.

We define the potential $\overline{\varphi}^\flat$ as

$$\overline{\varphi}^\flat(x) = \begin{cases} \overline{\varphi}(x) + \log(1-\gamma), & \text{if } x \in C_n^i \\ \overline{\varphi}(x) & \text{if } x \in C_n^j, \text{for } j \neq i \end{cases}$$

Lemma 2.4.2 *We have that $P_G(\overline{\varphi}^\flat) = \log(1-\gamma m_{\overline{\varphi}}(C_n))$.*

We define $\mathcal{M}_F(\overline{\varphi})$ as the set of F-invariant measures such that $-\int \overline{\varphi} d\mu < \infty$.

Lemma 2.4.3 *We have that $\mathcal{M}_F(\overline{\varphi}) = \mathcal{M}_F(\overline{\varphi}^\flat)$ and for any cylinder $C_n^i \in \mathcal{P}_n^F$ the following holds.*

$$\begin{aligned} &\sup\left\{h_F(\mu) + \textstyle\int \overline{\varphi} d\mu : \mu \in \mathcal{M}(\overline{\varphi}), \mu(C_n^i) < \frac{K(1-\gamma)}{1-m_{\overline{\varphi}}(C_n^i)^n}\mu_{\overline{\varphi}}(C_n^i)\right\} \\ &\leq \sup\left\{h_F(\mu) + \textstyle\int \overline{\varphi}^\flat d\mu : \mu \in \mathcal{M}(\overline{\varphi}^\flat), \mu(C_n^i) < \frac{K(1-\gamma)}{1-m_{\overline{\varphi}}(C_n^i)^n}\mu_{\overline{\varphi}}(C_n^i)\right\} \\ &\qquad - \left[\frac{K(1-\gamma)\log(1-\gamma)}{1-m_{\overline{\varphi}}(C_n^i)^n}\right]\mu_{\overline{\varphi}}(C_n^i) \\ &\leq P_G(\overline{\varphi}^\flat) - \left[\frac{K(1-\gamma)\log(1-\gamma)}{1-m_{\overline{\varphi}}(C_n^i)^n}\mu_{\overline{\varphi}}(C_n^i)\right]. \end{aligned}$$

Note that we can prove that the final inequality is actually an equality, but we don't require this here.

Lemmas 2.4.2 and 2.4.3 imply that any measure μ_F with $\mu_F(C_n^i) < K(1-\gamma)m_{\overline{\varphi}}(C_n^i/(1-m_{\overline{\varphi}}(C_n^i)^n$ must have

$$h_F(\mu) + \int \overline{\varphi}^\flat d\mu \leq P_G(\overline{\varphi}^\flat) - \left[\frac{K(1-\gamma)\log(1-\gamma)}{1-m_{\overline{\varphi}}(C_n^i)^n}\mu_{\overline{\varphi}}(C_n^i)\right] \tag{2.2}$$

$$= \log(1-\gamma m_{\overline{\varphi}}(C_n)) - \left[\frac{K(1-\gamma)\log(1-\gamma)}{1-m_{\overline{\varphi}}(C_n^i)^n}\mu_{\overline{\varphi}}(C_n^i)\right]. \tag{2.3}$$

If $m_{\overline{\varphi}}(C_n^i)$ is very small then $\log(1-\gamma m_{\overline{\varphi}}(C_n)) \approx -\gamma m_{\overline{\varphi}}(C_n)$ and so choosing $\gamma \in (0,1)$ close to 1 the above is strictly negative. By Lemma 2.4.1, $\mu_{\overline{\varphi}}(C_n^i) < e^{-\lambda n}$ so

C_n^i is small if λ large. Hence, if λ is sufficiently large then we can set $\gamma = \tilde{\gamma}^\flat \in (0,1)$ so that

$$\log(1-\tilde{\gamma}^\flat e^{-\lambda n}) - \left[\frac{K(1-\tilde{\gamma}^\flat)\log(1-\tilde{\gamma}^\flat)}{(1-e^{-\lambda n})^n} e^{-\lambda n}\right]$$

is strictly negative for all $n \in \mathbb{N}$. This implies that 2.3 with $\gamma = \tilde{\gamma}^\flat$ is strictly negative for any $C_n^i \in \mathcal{P}_n^F$ and any n, so we set 2.3 to be the value $\delta_n^{i,\flat}$.

For the upper bound on the free energy of measures giving C_n^i relatively large mass, we follow a similar proof, but with the potential $\overline{\varphi}^\sharp$ defined as

$$\overline{\varphi}^\sharp(x) = \begin{cases} \overline{\varphi}(x) - \log(1-\gamma), & \text{if } x \in C_n^i \\ \overline{\varphi}(x) & \text{if } x \in C_n^j, \text{for } j \neq i \end{cases}$$

Similarly to above, one can show that

Lemma 2.4.4 *We have that* $P_G(\overline{\varphi}^\sharp) = \log\left(1 + m_{\overline{\varphi}}(C_n)\left(\frac{\gamma}{1-\gamma}\right)\right) \leq m_{\overline{\varphi}}(C_n)\left(\frac{\gamma}{1-\gamma}\right)$.

Also, one can show that

Lemma 2.4.5 *We have that* $\mathcal{M}_F(\overline{\varphi}) = \mathcal{M}_F(\overline{\varphi}^\sharp)$ *and for any cylinder* $C_n^i \in \mathcal{P}_n^F$ *the following holds.*

$$\begin{aligned}
&\sup\left\{h_F(\mu) + \int \overline{\varphi} d\mu : \mu \in \mathcal{M}(\overline{\varphi}), \mu(C_n^i) > \frac{1}{K(1-\gamma)\left(1+m_{\overline{\varphi}}(C_n)\left(\frac{\gamma}{1-\gamma}\right)\right)^n}\mu_{\overline{\varphi}}(C_n^i)\right\} \\
&\leq \sup\left\{h_F(\mu) + \int \overline{\varphi}^\sharp d\mu : \mu \in \mathcal{M}(\overline{\varphi}^\sharp), \mu(C_n^i) > \frac{1}{K(1-\gamma)\left(1+m_{\overline{\varphi}}(C_n)\left(\frac{\gamma}{1-\gamma}\right)\right)^n}\mu_{\overline{\varphi}}(C_n^i)\right\} \\
&\quad + \left[\frac{\log(1-\gamma)\mu_{\overline{\varphi}}(C_n^i)}{K(1-\gamma)\left(1+m_{\overline{\varphi}}(C_n)\left(\frac{\gamma}{1-\gamma}\right)\right)^n}\right] \\
&\leq P_G(\overline{\varphi}^\sharp) + \left[\frac{\log(1-\gamma)\mu_{\overline{\varphi}}(C_n^i)}{K(1-\gamma)\left(1+m_{\overline{\varphi}}(C_n)\left(\frac{\gamma}{1-\gamma}\right)\right)^n}\right].
\end{aligned}$$

Therefore, if

$$\mu(C_n^i) > \frac{m_{\overline{\varphi}}(C_n)}{K(1-\gamma)\left(1+m_{\overline{\varphi}}(C_n)\left(\frac{\gamma}{1-\gamma}\right)\right)^n}$$

we have

$$h_F(\mu) + \int \overline{\varphi} d\mu \le m_{\overline{\varphi}}(C_n^i)\left(\frac{\gamma}{1-\gamma}\right) + \frac{\log(1-\gamma)\mu_{\overline{\varphi}}(C_n^i)}{K(1-\gamma)\left(1+m_{\overline{\varphi}}(C_n)\left(\frac{\gamma}{1-\gamma}\right)\right)^n}. \tag{2.4}$$

If λ is sufficiently large then we can choose $\gamma = \tilde{\gamma}^\sharp \in (0,1)$ so that this is strictly negative and can be fixed to be our value $\delta_n^{i,\sharp}$. This can be seen as follows: let $\gamma = p/(p+1)$ for some p to be chosen later. Then the right hand side of 2.4 becomes

$$m_{\overline{\varphi}}(C_n^i)(p+1)\left[\frac{p}{p+1} - \frac{log(p+1)}{K(1+pe^{-\lambda n})^n}\right]. \tag{2.5}$$

If λ is sufficiently large, then there exists some large $\lambda' \in (0,\lambda)$ such that $(1+pe^{-\lambda n})^n \le 1 + pe^{-\lambda' n}$ for all $n \in \mathbb{N}$. Hence with this suitable choice of λ we can choose p so that the quantity in the square brackets in 2.5 is negative for all n. So we can choose $\delta_n^{i,\sharp} < 0$ to be 2.4 with $\gamma = \tilde{\gamma}^\sharp$.

We let

$$\gamma^\sharp = 1 - (1-\tilde{\gamma}^\sharp)\left(1 + e^{-\lambda n}\left(\frac{\tilde{\gamma}^\sharp}{1-\tilde{\gamma}^\sharp}\right)\right)^n. \tag{2.6}$$

For appropriately chosen λ this is in $(0,1)$.

We set $\gamma_0' := \max\{\gamma^\flat, \gamma^\sharp\}$ and for each $C_n^i \in \mathcal{P}_n^F$ we let $\delta_n^i := \max\{\delta_n^{i,\flat}, \delta_n^{i,\sharp}$. The proof of the proposition is completed by setting $\gamma_0 := 1 - K(1-\gamma_0')$, which we may assume is in $(0,1)$.

2.4.2 The Inducing Time is Integrable

To finish the proof of Theorem A we will use the following proposition.

Proposition 2.4.2 *Given a hyperbolic potential φ with $P_f(\varphi) = 0$, there exist finitely many ergodic equilibrium states for the system (f,φ) and they are expanding.*

In order to prove the above proposition, we take a sequence of f-invariant measures $\{\mu_n\}_n$ such that $h_{\mu_n}(f) + \int \varphi d\mu_n \to 0$ and liftable with respect to the same induced scheme. We will show that the set of lifted measure is tight (see definition below and [22] for details) and has the Gibbs measure as its unique accumulation point with respect to which the inducing time is integrable. Finally, the Gibbs measure is projected to an equilibrium state for (f,φ).

Definition 2.4.1 *We say that a set of measures $\mathcal{K}$ on X is **tight** if for every $\epsilon > 0$ there exists a compact set $K \subset X$ such that $\eta(K^c) < \epsilon$ for every measure $\eta \in \mathcal{K}$.*

Lemma 2.4.6 *Let us consider the lifted sequence $\{\overline{\mu}_n\}_n$. The set $\{\overline{\mu}_n\}$ is tight.*

Proof.

By Pinheiro's Theorem, if $\tilde{\tau}$ is the inducing time of $\tilde{F}$, there exists $\theta > 0$ such that $\int \tilde{\tau} d\overline{\mu}_n < \theta, \forall n \in \mathbb{N}$. We claim that this implies that the set $\{\overline{\mu}_n\}$ is tight. It is enough to show that, given $j \in \mathbb{N}$, we can find a compact set K_j such that $\overline{\mu}_n(K_j^c) < \frac{\theta}{j}, \forall n \in \mathbb{N}$. If fact,

$$j\overline{\mu}_n(\{\tilde{\tau} > j\}) < \int_{\{\tilde{\tau} > j\}} \tilde{\tau} d\overline{\mu}_n < \theta \Rightarrow \overline{\mu}_n(\{\tilde{\tau} > j\}) < \frac{\theta}{j}, \forall n \in \mathbb{N}.$$

It remains to show that $K_j = \{\tilde{\tau} \leq j\}$ is compact. In fact, K_j is the union of finitely many cylinders, which are compact. So, the set $\{\overline{\mu}_n\}$ is tight, as we claimed.

□

Proof. *(of Proposition 2.4.2):* Proposition 2.4.1 implies that there exists $K' > 0$ such that, given a cylinder $C_n \in \mathcal{P}_n^F$, there exists $k_n \in \mathbb{N}$ such that for $k \geq k_n$ we have

$$\frac{1}{K'} \leq \frac{\overline{\mu}_k(C_n)}{e^{S_k \overline{\varphi}(x)}} \leq K', \ \forall x \in C_n.$$

From Lemma 2.4.6 the set $\{\overline{\mu}_n\}$ is tight and we obtain a convergent subsequence, which we keep writing $\{\overline{\mu}_n\}_n$. We can see that the limit is a Gibbs measure and, by uniqueness, is the measure $\mu_{\overline{\varphi}}$. Now, we can see that the inducing time $\tilde{\tau}$ is integrable with respect to $\mu_{\overline{\varphi}}$.

First of all, we remind that $F = \tilde{F}^k$ and denote $\overline{\mu}_n$ as $\overline{\mu}_{F,n}$ if we look at the map F and $\overline{\mu}_{\tilde{F},n}$ if we look at the map $\tilde{F}$. Then, note that $\int \tau d\overline{\mu}_{F,n} = \int \tilde{\tau}^k d\overline{\mu}_{\tilde{F},n} \leq \theta k$. For the purpose of this proof, we let $\tau_N := \min\{\tau, N\}$. By the Monotone Convergence Theorem, we obtain.

$$\int \tau d\mu_{\overline{\varphi}} \leq \lim_{N\to\infty} \int \tau_N d\mu_{\overline{\varphi}} \leq \lim_{N\to\infty} \limsup_{n\to\infty} \int \tau_N d\overline{\mu}_{F,n} \leq \theta k.$$

Finally, since τ is integrable, we can project the Gibbs measure $\mu_{\overline{\varphi}}$ and obtain an invariant measure μ for f. By the Abramov's formulas we can see that μ is an equilibrium state for the system (f, φ).

As there exists an equilibrium state, we also can find an ergodic one. Also, if ν is an ergodic equilibrium state, we can lift it to an equilibrium state for the shift, which is the Gibbs measure $\mu_{\overline{\varphi}}$. So, the projection of it is ν. It shows that there exists at most one ergodic equilibrium state for each induced scheme. Then, they are, at most, finitely many. $\square$

2.5 Applications

In order to give examples of maps that satisfies our hypothesis, we begin with some definitions as they are given in [1].

Definition 2.5.1 *Let M be a compact Riemannian manifold of dimension $d \geq 1$ and $f : M \to M$ a continuous map defined on M. The map f is called* ***non-flat*** *if it is a local $C^{1+\alpha}, (\alpha > 0)$ diffeomorphism in the whole manifold except in a non-degenerate set $\mathcal{C} \subset M$. We say that $\mathcal{C} \subset M$ is a* ***non-degenarate set*** *if there exist $\beta, B > 0$ such that the following two conditions hold.*

- $\frac{1}{B} d(x, \mathcal{C})^{\beta} \leq \frac{\|Df(x)v\|}{\|v\|} \leq Bd(x, \mathcal{C})^{-\beta}$ *for all $v \in T_x M$.*

  For every $x, y \in M \backslash \mathcal{C}$ with $d(x, y) < d(x, \mathcal{C})/2$ we have

- $| \log \| Df(x)^{-1} \| - \log \| Df(y)^{-1} \| | \leq \frac{B}{d(x,\mathcal{C})^{\beta}} d(x, y)$.

In what follows, we give an example of a non-flat map.

2.5.1 Viana maps

Example 2.5.1 *(Viana maps) We recall the definition of the open class of maps with critical sets in dimension 2, introduced by Viana in [34]. We skip the technical points. It can be generalized for any dimension (See [1]).*

Let $a_0 \in (1, 2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ a Morse function, for instance $b(\theta) = \sin(2\pi\theta)$. For fixed small $\alpha > 0$, consider the map

$$\begin{aligned} f_0 : S^1 \times \mathbb{R} &\longrightarrow S^1 \times \mathbb{R} \\ (\theta, x) &\longmapsto (g(\theta), q(\theta, x)) \end{aligned}$$

where g is the uniformly expanding map of the circle defined by $g(\theta) = d\theta (mod \mathbb{Z})$ for some $d \geq 16$, and $q(\theta, x) = a(\theta) - x^2$ with $a(\theta) = a_0 + \alpha b(\theta)$. It is easy to check that for $\alpha > 0$ small enough there exists an interval $I \subset (-2, 2)$ for which $f_0(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map f sufficiently close to f_0 in the C^0 topology has $S^1 \times I$ as a forward invariant region. We consider from here on these maps f close to f_0 restricted to $S^1 \times I$. Taking into account the expression of f_0 it is not difficult to check that for f_0 (and any map f close to f_0 in the C^2 topology) the critical set is non-degenerate.

The main properties of f in a C^3 neighbourhood of f are summarized below (See [1], [6], [25]):

(1) f is ***differentiable non-uniformly expanding***, that is, there exist $\lambda > 0$ and a Lebesgue full measure set $H \subset S^1 \times I$ such that for all point $p = (\theta, x) \in H$, the following holds

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df(f^i(p))^{-1} \|^{-1} < -\lambda.$$

(2) Its orbits have ***slow approximation to the critical set***, that is, for every $\epsilon > 0$ the exists $\delta > 0$ such that for every point $p = (\theta, x) \in H \subset S^1 \times I$, the following holds

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} - \log \mathrm{dist}_\delta(p, \mathcal{C}) < \epsilon.$$

where

$$\mathrm{dist}_\delta(p, \mathcal{C}) = \begin{cases} dist(p, \mathcal{C}), & if \quad dist(p, \mathcal{C}) < \delta \\ 1 & if \quad dist(p, \mathcal{C}) \geq \delta \end{cases}$$

(3) f is ***topologically mixing***: for every open set $U \subset M$, there exists $n(U) \in \mathbb{N}$ such that $f^{n(U)}(U) = \cap_{n=0}^{\infty} f^n(M)$.

(4) f is ***strongly topologically transitive***: for all open set $U \subset M$, we have $M = \cup_{n=0}^{\infty} f^n(U)$.

(5) it has a unique ergodic absolutely continuous invariant (thus SRB) measure;

(6) the density of the SRB measure varies continuously in the L^1 norm with f.

The idea of hyperbolic times for differentiable maps is a key notion on the study of non-uniformly hyperbolic dynamics and it was introduced by J. Alves et al. This is a powerful tool in order to get expansion in the context of non-uniform expansion.

In the following, we recall the definition of a hyperbolic time for differentiable maps see([3], [25]).

Definition 2.5.2 *(Hyperbolic times). Let us fix $0 < b = \frac{1}{3}\min\{1, 1/\beta\} < \frac{1}{2}\min\{1, 1/\beta\}$. Given $0 < \sigma < 1$ and $\epsilon > 0$, we will say that n is a (σ, ϵ)-**hyperbolic time** for a point $x \in M$ (with respect to the non-flat map f with a β-non-degenerate critical/singular set $\mathcal{C}$) if for all $1 \le k \le n$ we have*

$$\prod_{j=n-k}^{n-1} \|(Df \circ f^j(x)^{-1}\| \le \sigma^k \text{ and } dist_\epsilon(f^{n-k}(x), \mathcal{C}) \ge \sigma^{bk}.$$

We denote de set of points of M such that $n \in \mathbb{N}$ is a (σ, ϵ)-hyperbolic time by $H_n(\sigma, \epsilon, f)$.

Proposition 2.5.1 *(Positive frequency). Given $\lambda > 0$ there exist $\theta > 0$ and $\epsilon_0 > 0$ such that, for every $x \in U$ and $\epsilon \in (0, \epsilon_0]$,*

$$\#\{1 \le j \le n;\ x \in H_j(e^{-\lambda/4}, \epsilon, f)\} \ge \theta n,$$

whenever $\frac{1}{n}\sum_{i=0}^{n-1} \log \|(Df(f^i(x)))^{-1}\|^{-1} \ge \lambda$ and $\frac{1}{n}\sum_{i=0}^{n-1} -\log dist_\epsilon(x, \mathcal{C}) \le \frac{\lambda}{16\beta}$.

If f is non-uniformly expanding with slow approximation to the critical set, it follows from the Proposition 2.5.1 that the points of U have infinitely many moments with positive frequency of hyperbolic times. In particular, they have infinitely many hyperbolic times.

Proposition 2.5.2 *Given $\sigma \in (0, 1)$ and $\epsilon > 0$, there is $\delta, \rho > 0$, depending only on σ and ϵ and on the map f, such that if $x \in H_n(\sigma, \epsilon, f)$ then there exists a neighbourhood $V_n(x)$ of x with the following properties:*

(1) f^n maps $\overline{V_n(x)}$ diffeomorphically onto the ball $\overline{B_\delta(f^n(x))}$;

(2) $dist(f^{n-j}(y), f^{n-j}(z)) \le \sigma^{j/2} dist(f^n(y), f^n(z)), \forall y, z \in V_n(x)$ and $1 \le j < n$.

(3) $\log \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \le \rho d(f^n(y), f^n(z))$.

for all $y, z \in V_n(x)$.

The sets $V_n(x)$ are called ***hyperbolic pre-balls*** and their images $f^n(V_n(x)) = B_\delta(f^n(x))$, ***hyperbolic balls***.

From the above facts we can see that the Viana maps are included in our setting. Here the Lebesgue measure is expanding.

2.5.2 Benedicks-Carleson Maps

We study a class of non-hyperbolic maps of the interval with the condition of exponential growth of the derivative at critical values, called ***Collet-Eckmann Condition***. We also ask the map to be C^2 and topologically mixing and the critical points to have critical order $2 \leq \alpha < \infty$.

Given a critical point $c \in I$, the ***critical order*** of c is a number $\alpha_c > 0$ such that $f(x) = f(c) \pm |g_c(x)|^{\alpha_c}$, $\forall x \in \mathcal{U}_c$ where g_c is a diffeomorphism $g_c : \mathcal{U}_c \to g(\mathcal{U}_c)$ and $\mathcal{U}_c$ is a neighbourhood of c.

Let $\delta > 0$ and denote $\mathcal{C}$ the set of critical points and $B_\delta = \cup_{c \in \mathcal{C}}(c - \delta, c + \delta)$. Given $x \in I$, we suppose that

- **(Expansion outside B_δ)**. There exists $\kappa > 1$ and $\beta > 0$ such that, if $x_k = f^k(x) \notin B_\delta$, $0 \leq k \leq n-1$ then $|Df^n(x)| \geq \kappa \delta^{(\alpha_{\max}-1)} e^{\beta n}$, where $\alpha_{\max} = \max\{\alpha_c, c \in \mathcal{C}\}$. Moreover, if $x_0 \in f(B_\delta)$ or $x_n \in B_\delta$ then $|Df^n(x)| \geq \kappa e^{\beta n}$.

- **(Collet-Eckmann Condition)**. There exists $\lambda > 0$ such that

$$|Df^n(f(c))| \geq e^{\lambda n}.$$

- **(Slow Recurrence to $\mathcal{C}$)**. There exists $\sigma \in (0, \lambda/5)$ such that

$$dist(f^k(x), \mathcal{C}) \geq e^{-\sigma k}.$$

2.5.3 Rovella Maps

There is a class of non-uniformly expanding maps known as ***Rovella Maps***. They are derived from the so-called *Rovella Attractor*, a variation of the *Lorenz Attractor*. We proceed with a brief presentation. See [5] for details.

Contracting Lorenz Attractor

The geometric Lorenz attractor is the first example of a robust attractor for a flow containing a hyperbolic singularity. The attractor is a transitive maximal invariant set for a flow in three-dimensional space induced by a vector field having a singularity at the origin for which the derivative of the vector field at the singularity has real eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$ with $\lambda_1 + \lambda_3 > 0$. The singularity is accumulated by regular orbits which prevent the attractor from being hyperbolic.

The geometric construction of the contracting Lorenz attractor (Rovella attractor) is the same as the geometric Lorenz attractor. The only difference is the condition (A1)(i) below that gives in particular $\lambda_1 + \lambda_3 < 0$. The initial smooth vector field X_0 in $\mathbb{R}^3$ has the following properties:

(A1) X_0 has a singularity at 0 for which the eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ of $DX_0(0)$ satisfy:

(i) $0 < \lambda_1 < -\lambda_3 < 0 < -\lambda_2$,

(ii) $r > s + 3$, where $r = -\lambda_2/\lambda_1, s = -\lambda_3/\lambda_1$;

(A2) there is an open set $U \subset \mathbb{R}^3$, which is positively invariant under the flow, containing the cube $\{(x, y, z) :| x |\leq 1, | y |\leq 1, | x |\leq 1\}$ and supporting the *Rovella attractor*

$$\Lambda_0 = \bigcap_{t\geq 0} X_0^t(U).$$

The top of the cube is a Poincaré section foliated by stable lines $\{x = \text{const}\} \cap \Sigma$ which are invariant under Poincaré first return map P_0. The invariance of this foliation uniquely defines a one-dimensional map $f_0 : I\backslash\{0\} \to I$ for which

$$f_0 \circ \pi = \pi \circ P_0,$$

where I is the interval $[-1, 1]$ and π is the canonical projection $(x, y, z) \mapsto x$;

(A3) there is a small number $\rho > 0$ such that the contraction along the invariant foliation of lines $x =$const in U is stronger than ρ.

See [5] for properties of the map f_0.

Rovella Parameters

The Rovella attractor is not robust. However, the chaotic attractor persists in a measure theoretical sense: there exists a one-parameter family of positive Lebesgue measure of C^3 close vector fields to X_0 which have a transitive non-hyperbolic attractor. In the proof of that result, Rovella showed that there is a set of parameters $E \subset (0, a_0)$ (that we call *Rovella parameters*) with a_0 close to 0 and 0 a full density point of E, i.e.

$$\lim_{a \to 0} \frac{\mid E \cap (0, a) \mid}{a} = 1,$$

such that:

(C1) there is $K_1, K_2 > 0$ such that for all $a \in E$ and $x \in I$

$$K_2 \mid x \mid^{s-1} \leq f_a'(x) \leq K_1 \mid x \mid^{s-1},$$

where $s = s(a)$. To simplify, we shall assume s fixed.

(C2) there is $\lambda_c > 1$ such that for all $a \in E$, the points 1 and -1 have *Lyapunov exponents* greater than λ_c:

$$(f_a^n)'(\pm 1) > \lambda_c^n, \ \forall n \geq 0;$$

(C3) there is $\alpha > 0$ such that for all $a \in E$ the *basic assumption* holds:

$$\mid f_a^{n-1}(\pm 1) \mid > e^{-alphan}, \ \forall n \geq 1;$$

(C4) the forward orbits of the points ± 1 under f_a are dense in $[-1, 1]$ for all $a \in E$.

Definition 2.5.3 *We say that a map f_a with $a \in E$ is a **Rovella Map**.*

Theorem 2.5.1 *(Alves-Soufi [5]) Every Rovella map is non-uniformly expanding.*

Chapter 3

Induced Schemes with Special Holes

3.1 Introduction

In this chapter we deal with open zooming maps. A ***zooming map*** is a map which extends the notion of non-uniform expansion, where we have a type of expansion, obtained in the presence of hyperbolic times. The zooming times extend this notion beyond exponential contractions. In our context, a map is said to be ***open*** when the phase space is not invariant. In other words, we begin with a closed map $f : M \to M$ and consider a Borel set $H \subset M$, in order to study the orbits with respect to H (called the ***hole*** of the system): if a point $x \in M$ is such that its orbit pass through H, we consider the point x escapes from the system. In particular, we study the set of points that never pass through H. Also, once a reference measure m is fixed, we study the **escape rate** defined by:

$$\mathcal{E}(f, m, H) = -\lim_{n\to\infty} \frac{1}{n} \log m\Big(\cap_{j=0}^{n-1} f^{-j}(M\backslash H)\Big).$$

The study of open dynamics began with Yorke and Pianigiani in the late 1970s. The abstract concept of an open system leads immediately to the notion of a conditionally invariant measure and escape rate along with a host of detailed questions about how mass escapes or fails to escape from the system under time evolution. For interval maps, induced schemes are studied in [15] and changes are made in [9] to study induced schemes for interval maps with holes.

In order to study the escape rate, Young towers are quite useful and the induced schemes as well. In the context of open dynamics, the induced schemes need to have the property of ***respecting the hole***, as considered in [13], for example . It means that an element of the partition can only escape entirely. It is an important property to construct the Young tower with hole.

In this work, we prove the existence of induced schemes respecting holes of a special type. The context is the zooming maps and the hole is obtained from small balls, by erasing intersections with regular pre-images. Our construction mimics the work of Pinheiro in [25], where the main ingredient is the zooming times.

3.2 Definitions and Statement of the Result

In this section we give some definitions and state our main result. We define induced schemes and what we mean by an induced scheme respecting a hole.

3.2.1 Markov Maps and Induced Schemes

We recall the definitions of Markov partition, Markov map and induced Markov map (See [25] for details).

Consider a measurable map $f : M \to M$ defined on the metric space M endowed with the Borel σ-algebra.

Definition 3.2.1 *(Markov partition). Let $f : U \to U$ be a measurable map defined on a Borel set U of a compact, separable metric space M. A countable collection $\mathcal{P} = \{P_1, P_2, P_3, \dots\}$ of Borel subsets of U is called a* ***Markov partition*** *if*

(1) $int(P_i) \cap int(P_j) = \emptyset$, if $i \neq j$;

(2) if $f(P_i) \cap int(P_j) \neq \emptyset$ then $f(P_i) \supset int(P_j)$;

(3) $\#\{f(P_i); i \in \mathbb{N}\} < \infty$;

(4) $f_{|P_i}$ is a homeomorphism and it can be extended to a homeomorphism sending $\overline{P_i}$ onto $\overline{f(P_i)}$;

(5) $\lim_{n\to\infty} diam(\mathcal{P}_n(x)) = 0, \ \forall\ x \in \cap_{n\geq 0} f^{-n}(\cup_i P_i)$,

where $\mathcal{P}_n(x) = \{y;\ \mathcal{P}(f^j(y)) = \mathcal{P}(f^j(x)),\ \forall\ 0 \le j \le n\}$ *and* $\mathcal{P}(x)$ *denotes the element of* $\mathcal{P}$ *that contains* x*.*

Definition 3.2.2 *(Markov map). The pair* $(F, \mathcal{P})$*, where* $\mathcal{P}$ *is a Markov partition of* $F : U \to U$*, is called a* ***Markov map*** *defined on* U*. If* $F(P) = U,\ \forall\ P \in \mathcal{P}, (F, \mathcal{P})$ *is called a* ***full Markov map****.*

Note that if $(F, \mathcal{P})$ is a full Markov map defined on an open set U then the elements of $\mathcal{P}$ are open sets (because $F(P) = U$ and $F\mid_P$ is a homeomorphism, $\forall\ P \in \mathcal{P}$).

Definition 3.2.3 *(Induced Markov map). A Markov map* $(F, \mathcal{P})$ *defined on* U *is called an* ***induced Markov map*** *for* f *on* U *if there exists a function* $R : U \to \mathbb{N} = \{0, 1, 2, 3, \ldots\}$*(called* ***inducing time****) such that* $\{R \ge 1\} = \cup_{P \in \mathcal{P}} P$*,* $R\mid_P$ *is constant* $\forall\, P \in \mathcal{P}$ *and* $F(x) = f^{R(x)}(x), \forall\ x \in U$*.*

If an induced Markov map $(F, \mathcal{P})$ is a full Markov map, we call $(F, \mathcal{P})$ an ***induced full Markov map***. We will also call the pair $(F, \mathcal{P})$ an ***induced scheme***.

3.2.2 Open Dynamics

For the classical dynamical systems the phase spaces are invariant and called *closed*. When we consider systems where the phase space is not invariant, they are called *open*. It is done by considering holes in a closed dynamical system.

Definition 3.2.4 *(Hole). Given a dynamical system* $f : M \to M$*, we say that an open set with finitely many connected components is a* ***hole*** *for the system.*

Definition 3.2.5 *(Dynamics that respects the hole). Given a hole* $H \subset M$*, we say that an induced scheme* $(F, \mathcal{P})$ ***respects the hole*** H *if we have* $f^k(P) \cap H \ne \emptyset \Rightarrow f^k(P) \subset H, \forall\ P \in \mathcal{P}, \forall\ 0 \le k < R(P)$*.*

3.2.3 Statement of Main Result

Now, we state our main result on the existence of induced schemes respecting holes of a special type.

Theorem B *Let* $f : M \to M$ *be a zooming map. There exists a ball* $B_r(p_0)$ *with radius* $r > 0$ *sufficiently small such that for any pairwise disjoint finite collection of balls*

$\mathcal{A} = \{B_r(p_i), i = 1, 2, \ldots, k\}$ contained in $B_r(p_0)^c$, there are an open set $U \subset B_r(p_0)$, an induced scheme $(F, \mathcal{P})$ constructed in U and a hole H, such that

$$\cup_{i=1}^{k} B_{\frac{r_i}{2}}(p_i) \subset H \subset \cup_{i=1}^{k} B_{r_i}(p_i)$$

and the induced scheme respects the hole H.

3.3 Preliminary Results

In order to construct an induced scheme that respect holes of a certain type, we need the notions of *linked sets*, *nested collections* and *zooming times*, introduced by V. Pinheiro in [25]. Zooming times generalize the notion of *hyperbolic times*, which are fundamental for our construction. The elements of the partition will be regular pre-images of a certain open set and the hole will be obtained from small balls by deleting linked regular pre-images of the considered balls.

3.3.1 Nested Collections

We recall some definitions and results from [25], that will help us to show that the induced map we will build respects a certain type of hole.

Definition 3.3.1 *(Linked sets). We say that two open sets A and B are* ***linked*** *if both $A - B$ and $B - A$ are not empty.*

We introduce the following notation.

Notation: We write $A \leftrightarrow B$ to mean that A and B are linked and $A \not\leftrightarrow B$ to mean that A and B are not linked.

Definition 3.3.2 *(Regular pre-images). Given V an open set, we say that P is a* ***regular pre-image*** *of order n of V if f^n sends P homeomorphically onto V. Denote by $ord(P)$ the order of P (with respect to V).*

Let us fix a collection $\mathcal{E}_0$ of open sets. For each n we consider $\mathcal{E}_n(V)$ as the collection of pre-images of order n of V. Set $\mathcal{E}_n = (\mathcal{E}_n(V))_{V \in \mathcal{E}_0}$. We call the sequence $\mathcal{E} = (\mathcal{E}_n)_n$ a ***dynamically closed family*** of regular pre-images if $f^k(E) \in \mathcal{E}_{n-k}, \forall\, E \in \mathcal{E}_n, \forall\, 0 \leq k \leq n$.

Let $\mathcal{E} = (\mathcal{E}_n)_n$ be a dynamically closed family of regular pre-images. A set P is called an $\mathcal{E}$-pre-image of a set W if there is $n \in \mathbb{N}$ and $Q \in \mathcal{E}_n$ such that $\overline{W} \subset f^n(Q)$ and $P = f_{|Q}^{-n}(W)$.

Definition 3.3.3 *(Nested sets). An open set V is called* ***$\mathcal{E}$-nested*** *if it is not linked with any $\mathcal{E}$-pre-image of it.*

Definition 3.3.4 *(Nested collections). A collection $\mathcal{A}$ of open sets is called an* ***$\mathcal{E}$-nested collection*** *of sets if every $A \in \mathcal{A}$ is not linked with any $\mathcal{E}$-pre-image of an element of $\mathcal{A}$ with order bigger than zero. Precisely, if $A_1 \in \mathcal{A}$ and P is an $\mathcal{E}$-pre-image of some $A_2 \in \mathcal{A}$, then either $A_1 \not\leftrightarrow P$ or $P = A_2$.*

It follows from the definition of an $\mathcal{E}$-nested collection of sets that every sub-collection of an $\mathcal{E}$-nested collection is also an $\mathcal{E}$-nested collection. In particular, each element of an $\mathcal{E}$-nested set collection is an $\mathcal{E}$-nested set.

Lemma 3.3.1 *(Main property of a nested collection). If $\mathcal{A}$ is an $\mathcal{E}$-nested collection of open sets and P_1 and P_2 are $\mathcal{E}$-pre-images of two elements of $\mathcal{A}$ with $ord(P_1) \neq ord(P_2)$ then $P_1 \not\leftrightarrow P_2$.*

Corollary 3.3.1 *(Main property of a nested set). If V is an $\mathcal{E}$-nested set and P_1 and P_2 are $\mathcal{E}$-pre-images of V then $P_1 \not\leftrightarrow P_2$. Furthermore,*

(1) if $P_1 \cap P_2 \neq \emptyset$ then $ord(P_1) \neq ord(P_2)$;

(2) if $P_1 \subset_{\neq} P_2$ with $ord(P_1) < ord(P_2)$ then V is contained in an $\mathcal{E}$-pre-image of itself with order bigger than zero, $f^{ord(P_2)-ord(P_1)}(V) \subset V$.

3.3.2 Constructing Nested Sets and Collections

Let $\mathcal{A}$ be a collection of connected open subsets such that the elements of $\mathcal{A}$ are not contained in any $\mathcal{E}$-pre-image of order bigger than zero of an element of $\mathcal{A}$.

Definition 3.3.5 *(Chains of pre-images). A finite sequence $\mathcal{K} = \{P_0, P_1, \dots, P_n\}$ of $\mathcal{E}$-pre-images of $\mathcal{A}$ is called a* ***chain*** *of $\mathcal{E}$-pre-images beginning in $A \in \mathcal{A}$ if*

- $0 < ord(P_0) \leq ord(P_1) \leq \cdots \leq ord(P_n)$;

- *$A \leftrightarrow P_0$;*
- *$P_{j-1} \leftrightarrow P_j$, $1 \le j \le n$;*
- *$P_i \neq P_j$ if $i \neq j$.*

For each $A \in \mathcal{A}$ define the open set

$$A^\star = A \backslash \overline{\bigcup_{(P_j)_j \in ch_{\mathcal{E}}(A)} \bigcup_j P_j}$$

where $ch_{\mathcal{E}}(A)$ is the set of chains.

Proposition 3.3.1 *(An abstract construction of a nested collection). For each $A \in \mathcal{A}$ such that $A^\star \neq \emptyset$ choose a connected component A' of $A^\star$. If $\mathcal{A}' = \{A'; A \in \mathcal{A}$ and $A^\star \neq \emptyset\}$ is not an empty collection then $\mathcal{A}'$ is an $\mathcal{E}$-nested collection of sets.*

3.3.3 Zooming Sets and Measures

For differentiable dynamical systems, hyperbolic times are a powerful tool to obtain a type of expansion in the context of non-uniform expansion. As it was given in the previous chapter, it may be generalized for systems considered in a metric space, also with exponential contractions. The zooming times generalizes it beyond the exponential context. Details can be seen in [25].

Let $f : M \to M$ be a measurable map defined on a connected, compact, separable metric space M.

Definition 3.3.6 *(Zooming contractions). A **zooming contraction** is a sequence of functions $\alpha_n : [0, +\infty) \to [0, +\infty)$ such that*

- $\alpha_n(r) < r, \forall\, n \in \mathbb{N}, \forall\, r > 0.$
- $\alpha_n(r) < \alpha_n(s),\ if\ 0 < r < s, \forall\, n \in \mathbb{N}.$
- $\alpha_m \circ \alpha_n(r) \le \alpha_{m+n}(r), \forall\, r > 0, \forall\, m, n \in \mathbb{N}.$
- $\sup_{r \in (0,1)} \sum_{n=1}^{\infty} \alpha_n(r) < \infty.$

Let $(\alpha_n)_n$ be a zooming contraction and $\delta > 0$.

Definition 3.3.7 *(Zooming times). We say that $n \in \mathbb{N}$ is an (α, δ)-**zooming time** for $p \in X$ if there exists a neighbourhood $V_n(p)$ of p such that*

- *f^n sends $\overline{V_n(p)}$ homeomorphically onto $\overline{B_\delta(f^n(p))}$;*
- *$d(f^j(x), f^j(y)) \leq \alpha_{n-j}(d(f^n(x), f^n(y)))$ for every $x, y \in V_n(p)$ and every $0 \leq j < n$.*

*We call $B_\delta(f^n(p))$ a **zooming ball** and $V_n(p)$ a **zooming pre-ball**.*

We denote by $Z_n(\alpha, \delta, f)$ the set of points in X for which n is an (α, δ)- zooming time.

Definition 3.3.8 *(Zooming measure) A f-non-singular finite measure μ defined on the Borel sets of M is called a **weak zooming measure** if μ almost every point has infinitely many (α, δ)-zooming times. A weak zooming measure is called a **zooming measure** if*

$$\limsup_{n \to \infty} \frac{1}{n} \{1 \leq j \leq n \mid x \in Z_n(\alpha, \delta, f)\} > 0,$$

μ almost every $x \in M$.

Definition 3.3.9 *(Zooming set) A positively invariant set $\Lambda \subset M$ is called a **zooming set** if the above inequality holds for every $x \in \Lambda$.*

Definition 3.3.10 *(Bounded distortion) Given a measure μ with a Jacobian $J_\mu f$, we say that the measure has **bounded distortion** if there exists $\rho > 0$ such that*

$$\left| \log \frac{J_\mu f(y)}{J_\mu f(z)} \right| \leq \rho d(f^n(y), f^n(z))$$

for every $y, z \in V_n(x)$, μ-almost everywhere $x \in M$, for every hyperbolic time n of x.

The map f with an associated zooming measure is called a ***zooming map***.

Let us introduce the following notation.

Notation: We denote by $\mathcal{E}_\mathcal{Z} = (\mathcal{E}_{\mathcal{Z},n})_n$ the collection of all (α, δ)-zooming pre-balls, where $(\mathcal{E}_{\mathcal{Z},n})_n = \{V_n(x) | x \in Z_n(\alpha, \delta, f)\}$. Observe that this collection is a dynamically closed family of pre-images.

With the notation of Corollary 3.4.1, let A_i be contained in a zooming ball for all $i = 0, 1, 2, \ldots, k$. This corollary implies that we have $\mathcal{A}'$ an $\mathcal{E}_\mathcal{Z}$-nested collection, if the chains are small enough.

Definition 3.3.11 *(Zooming nested collection). We call $\mathcal{A}'$ an (α, δ)-**zooming nested collection**.*

Definition 3.3.12 *(Backward separated map) We say that a map $f : M \to M$ is **backward separated** if for every finite set $F \subset M$ we have*

$$d\big(F, \cup_{j=1}^{n} f^{-j}(F) \backslash F\big) > 0, \forall\ n \geq 1.$$

Observe that if f is continuous and $\sup\{\# f^{-1}(x), x \in M\} < \infty$, then f is backward separated.

3.4 Induced Schemes Respecting Special Holes

In this section, we proceed with the proof of Theorem B.

We will use the notation and results of the previous sections to prove some preliminaries results. We will follow ideas from [25].

3.4.1 Existence of Nested Collections

The following Lemma is proved in [25] for the case of a collection with one ball as corollary of Proposition 3.3.1. We prove it here for a collection with finitely many open sets, pairwise disjoint.

Lemma 3.4.1 *(Existence of nested collections). Let $\epsilon \in (0,1)$ and $\mathcal{A} = \{A_0, A_1, A_2, \ldots, A_k\}$ be a finite collection of pairwise disjoint open sets. Given $p_i \in A_i, i = 0, 1, 2, \ldots, k$, set $r_i = d(p_i, \partial A_i), i = 0, 1, 2, \ldots, k$. If we have*

- *$f^n(A_i) \not\subset A_j, \forall\ n \geq 1, i, j = 0, 1, 2, \ldots, k$;*
- *Every chain of $\mathcal{E}$-pre-images of $\mathcal{A}$ has diameter less than $m_0 = \min\{\epsilon r_i, i = 0, 1, 2, \ldots, k\}$;*

Then the set $A_i^\star$ contains the ball $B_{r_i(1-\epsilon)}(p_i), i = 0, 1, 2, \ldots, k$. Moreover, setting A_i' as the connected component of $A_i^\star$ that contains p_i, we have that $\mathcal{A}' = \{A_i', i = 0, 1, 2, \ldots, k\}$ is an $\mathcal{E}$-nested collection.

Proof. Since $f^n(A_i) \not\subset A_j, \forall\, n \geq 1, i, j = 0, 1, 2, \ldots, k$, we have that A_i is not contained in any $\mathcal{E}$-pre-image of $\mathcal{A}$ (with order bigger than zero) $i = 0, 1, 2, \ldots, k$. Let Γ_{A_i} be the collection of all chains intersecting ∂A_i. If $(P_j)_j \in \Gamma_{A_i}$, then $\bigcup_j P_j$ is a connected open set intersecting ∂A_i with diameter less than $m \leq \epsilon r_i$. So, $\bigcup_j P_j \subset B_{\epsilon r_i}(\partial A_i), \forall (P_j)_j \in \Gamma_{A_i}$. As a consequence, we obtain $A_i^\star = A_i \backslash \overline{\bigcup_{(P_j)_j \in ch_{\mathcal{E}}(A_i)} \bigcup_j P_j} \supset A_i \backslash \overline{B_{\epsilon r_i}(\partial A_i)} \supset B_{r_i(1-\epsilon)}(p_i)$. Then, $A_i^\star \supset B_{r_i(1-\epsilon)}(p_i)$. Setting A_i' the connected component of $A_i^\star$ that contain p_i, by the previous proposition we have that $\mathcal{A}'$ is an $\mathcal{E}$-nested collection. □

3.4.2 Existence of Zooming Nested Collections

The following lemma gives a sufficient condition to guarantee that the chains are small enough, in order to show that $\mathcal{A}'$ in the previous section is an (α, δ)-zooming nested collection. It is proved in [25] for the case of a nested set.

Lemma 3.4.2 *(Existence of zooming nested collection). Let $\epsilon \in (0, 1)$ and set $M_0 = \max\{diam(A_i), i = 0, 1, 2, \ldots, k\}$ and $m_0 = \min\{\epsilon r_i, i = 0, 1, 2, \ldots, k\}$. Suppose that $diam(A_i) > \alpha_n(diam(A_j)), i, j = 0, 1, 2, \ldots, k, \forall n \in \mathbb{N}$*

- *If we have $\sum_{n\geq 1} \alpha_n(M_0) \leq m_0$, then $A_i^\star$ is well defined and $A_i^\star \supset B_{(1-\epsilon)r_i}(p_i), i = 0, 1, 2, \ldots, k$.*
- *If f is backward separated, $\sup_{r>0} \sum_{n\geq 1} \alpha_n(r)/r < \infty$ and M_0 is sufficiently small, $M_0\epsilon/2 \leq m_0$, then $A_i^\star$ is well defined and $A_i^\star \supset B_{(1-\epsilon)r_i}(p_i), i = 0, 1, 2, \ldots, k$.*

Proof. Firstly, observe that $f^n(A_i) \not\subset A_j, \forall\, n \geq 1, i, j = 0, 1, 2, \ldots, k$, since $diam(A_i) > \alpha_n(diam(A_j)), i, j = 0, 1, 2, \ldots, k, \forall n \in \mathbb{N}$. Moreover, every chain has diameter less than $\sum_{n\geq 1} \alpha_n(a_n)$ for some $(a_m)_m \in \{diam(A_0), diam(A_1), diam(A_2), \ldots, diam(A_k)\}^{\mathbb{N}}$ and also, $\sum_{n\geq 1} \alpha_n(a_n) < \sum_{n\geq 1} \alpha_n(M_0) \leq m_0$. By Lemma 3.4.1, the first part is done.

For the second part $\sup_{r>0} \sum_{n\geq 1} \alpha_n(r)/r < \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\sum_{n\geq n_0} \alpha_n(M_0)/M_0 < \epsilon/2$. If f is backward separated, let $\gamma > 0$ such that $d(F_0, \cup_{j=1}^{n_0} f^{-j}(F_0) \backslash F_0) > \gamma$, where $F_0 = \{p_0, p_1, \ldots, p_k\}$. Set $r_\gamma = \frac{1}{3}\min\{\epsilon, \gamma\}$. Suppose that $M_0 < 2r_\gamma$. Note that if $j < n_0$ then $A_i \cap P = \emptyset$, $\forall P \in \mathcal{E}_{\mathcal{Z},j}$ (because $P \cap \left(\cup_{j=1}^{n_0} f^{-j}(F_0) \backslash F_0\right) \neq \emptyset$ and $diam(P) < diam(A_i) < 2r_\gamma < \gamma, \epsilon,\ \forall\, i$). Thus, every chain of $\mathcal{E}_{\mathcal{Z}}$-pre-images of $\mathcal{A}$ begins with a pre-image of order bigger than n_0. Observe that the diameter of any chain is smaller than $\sum_{n\geq n_0} \alpha_n(M_0) < M_0\epsilon/2 \leq m_0 \leq \epsilon r_i, i = 0, 1, 2, \ldots, k$ and, as a chain intersects the boundary A_i, we can conclude that the chain cannot intersect $B_{(1-\epsilon)r_i}(p_i)$. So, we obtain $A_i^\star \supset B_{(1-\epsilon)r_i}(p_i)$. □

3.4.3 Finding Zooming Returns

Let $f : M \to M$ be a zooming map. Since the set of points which has infinitely many zooming times is full measure, we can consider such points. If a zooming time is also a return time we call it a ***zooming return***.

In order to prove the theorem we will need the following proposition.

Lemma 3.4.3 *Given $t \leq \frac{\delta}{2}$ and $x \in M$ with infinitely many zooming times, there is $p_0 \in M$ such that p_0 has infinitely many zooming returns to the ball $B_t(p_0)$.*

PROOF. Let $m_1 < m_2 < \cdots < m_k < \ldots$ be the zooming times of x. Since M is a compact set, there is a convergent subsequence of $\{f^{m_k}(x)\}_k$. So, there are $m_{k_1} < m_{k_2} < \cdots < m_{k_j} < \ldots$ and $y \in M$ such that $\lim\limits_{j\to\infty} f^{m_{k_j}}(x) = y$. It means that, given $t \leq \frac{\delta}{2}$, $\exists\ j_0 \in \mathbb{N}$ such that $d(f^{m_{k_{j_1}}}(x), f^{m_{k_{j_2}}}(x)) < t$, $\forall\ j_1, j_2 \geq j_0$. Let $p_0 = f^{m_{k_{j_0}}}(x)$ and $n_j = m_{k_j} - m_{k_{j_0}}$. We have that n_j is a zooming time of p_0, $\forall\ j > j_0$. Then, $\forall\ j > j_0, n_j$ is both a zooming time of p_0 and a return time to the ball $B_t(p_0)$. We got infinitely many zooming returns to $B_t(p_0)$. □

Remark 3.4.1 *From the proof we observe that $f^{m_{k_j}}(x), \forall\ j \geq j_0$ has infinitely many zooming returns to $B_t(p_0)$.*

3.4.4 Constructing an Induced Scheme

Let $r < \frac{\delta}{4}$ such that $A_0 = B_r(p_0) \subset B_\delta(f^{n_0}(p_0))$, where n_0 is a zooming return for p_0 and take $A_i = B_r(p_i), i = 1, 2, \ldots, k$, a finite pairwise disjoint collection of balls outside A_0. By considering $A_0, A_1, A_2, \ldots, A_k$ with diameters $diam(A_i = 2r, i = 0, 1, 2, \ldots, k$, we can take $M_0 = 2r$, in the Lemma 3.4.2 to obtain a zooming nested collection $\mathcal{A}' = \{A'_0, A'_1, A'_2, \ldots, A'_k\}$ such that $B_{\frac{r}{2}}(p_i) \subset A'_i \subset B_r(p_i)$. We will construct the induced full Markov map in A'_0, respecting the hole $H = \cup_{i=1} A'_i$. Note that $diam(A') \leq \frac{\delta}{2}$ and $\cup_{i=1}^k B_{\frac{r}{2}}(p_i) \subset H \subset \cup_{i=1}^k B_r(p_i)$.

Now, taking $t = \frac{r}{2} < \frac{\delta}{8}$, we apply the Proposition 3.4.3 to the ball $B_{\frac{r}{2}}(p_0)$ to find infinitely many points in $B_{\frac{r}{2}}(p_0)$ that have infinitely many zooming returns to $B_{\frac{r}{2}}(p_0)$. In particular, there are infinitely many points in A_0' that have infinitely many zooming returns to A_0'.

Let $\mathcal{E}_{\mathcal{Z}}$ the collection of zooming pre-balls. Given $x \in A_0'$, let $\Omega(x)$ be the collection of all $\mathcal{E}_{\mathcal{Z}}$-pre-images V of A_0' such that $x \in V$.

Let $h(x) = \{f^n(x) \mid n \text{ is a zooming time of } x\}$. The set $\Omega(x)$ is not empty for every $x \in A_0'$ that has a zooming return to A_0'. Indeed, if $x \in A_0'$ and $f^n(x) \in A_0' \cap h(x)$, then the ball $B_\delta(f^n(x)) = f^n(V_n(x)) \supset A_0'$ (because diameter$(A_0') < \frac{\delta}{2}$). Thus, for each h-return of a point $x \in A_0'$ we can associate the $\mathcal{E}_{\mathcal{Z}}$ - pre-image $P = f^{-n}_{|V_n(x)}(A_0')$ of A_0' with $x \in P$.

Definition 3.4.1 *The inducing time on A_0' associated to "the first $\mathcal{E}_{\mathcal{Z}}$-return to A_0'" is the function $R : B_r(p) \to \mathbb{N}$ given by*

$$R(x) = \begin{cases} \min\{ord(V) \mid V \in \Omega(x)\}, \text{ if } \Omega(x) \neq \emptyset \\ 0 \qquad\qquad\qquad\quad , \text{ if } \Omega(x) = \emptyset. \end{cases}$$

Definition 3.4.2 *The induced map F associated to "the first $\mathcal{E}_{\mathcal{Z}}$-return to A_0'" is the map given by $F(x) = f^{R(x)}(x),\ \forall x \in A_0'$.*

Since the collection $\Omega(x)$ is totally ordered by inclusion it follows from the Corollary 3.3.1 that there is a unique $I(x) \in \Omega(x)$ such that $ord(I(x)) = R(x)$ whenever $\Omega(x) \neq \emptyset$.

Lemma 3.4.4 *If $\Omega(x) \neq \emptyset \neq \Omega(y)$ then either $I(x) \cap I(y) = \emptyset$ or $I(x) = I(y)$*

Definition 3.4.3 *The Markov partition associated to "the first $\mathcal{E}_{\mathcal{Z}}$-return to A_0'" is the collection of open sets $\mathcal{P}$ given by $\mathcal{P} = \{I(x) \mid x \in A_0' \text{ and } \Omega(x) \neq \emptyset\}$.*

The following corollary guarantees that P is indeed a Markov partition of open sets.

Corollary 3.4.1 *Let $F, R, \mathcal{P}$ be as above. If $\mathcal{P} \neq \emptyset$, then $(F, \mathcal{P})$ is an induced full Markov map for f on A_0'.*

We have an induced full Markov map defined on A_0'. It remains to show that the induced full Markov map defined on A_0', respects the hole H. In fact, given an element $P \in \mathcal{P}$ and $0 \leq j < R(P)$ and suppose that we have $f^j(P) = f^{j-R(P)}(A_0') \cap H \neq \emptyset$.

There is $i \in \{1, 2, \ldots, k\}$ such that $f^j(P) = f^{j-R(P)}(A'_0) \cap A'_i \neq \emptyset$. As the collection $\mathcal{A}'$ is nested, we must have $f^j(P) \subset A'_i \subset H$. This holds for all $P \in \mathcal{P}$ and $0 \leq j < R(P)$. Then, the induced full Markov map respects the hole H. The Theorem is proved.

3.4.5 A Dense Partition

Now, we suppose that M is a differentiable manifold, the zooming set is dense and the zooming contractions are exponential. We will show that we can find p_0 such that the partition $\mathcal{P}$ can be constructed in such a way that it is dense in A'_0. In order to do it, we will follow ideas of Theorem 7 in [25] by using that the set of points in M that have infinitely many zooming times is dense. Now, by following the proof of the theorem, we obtain that there exists a dense set $\mathcal{D} \subset A'_0$ such that every point in $\mathcal{D}$ has a zooming return. As a consequence, the partition is dense.

Proposition 3.4.1 *For $r > 0$ sufficientely small, there exist $p_0 \in M$ and a set $\mathcal{D} \subset A'_0 = B^\star_r(p_0)$ of points with infinitely many zooming times dense in A'_0.*

Proof. Note that

$$W := \bigcap_{j=0}^{\infty} \bigcup_{j \geq n} \Big(\bigcup_{x \in Z_n(\alpha,\delta,f)} V_n(x) \Big)$$

is a residual set. Thus, the set of points $x \in W$ that are transitive $\omega(x) = M$ is also a residual set (because the set of transitive points is residual). Choose a transitive point $q \in W$. As $q \in W$, there are sequences $n_k \to \infty$ and $x_k \in H_{n_k}$ such that $q \in V_{n_k}(x_k), \forall k \in \mathbb{N}$ and $\lim_{k\to\infty} f^{n_k}(x_{n_k}) = p_0$, for some $p_0 \in M$. Of course that $x_k \to q$, for $d(x_k, q) \leq \sigma^{n_k}\delta, \forall k$.

Let $\alpha = \{\alpha_n\}_n$, where $\alpha_n(r) = \sigma^n r$. As f is backward separated (because $\#f^{-1}(x) < \infty \forall x \in M$) and as $\sup_{r>0} \sum_{n\geq 1} \alpha_n(r)/r < \infty$, we can choose any $r > 0$ sufficiently small and consider the zooming nested collection $\mathcal{A}' = \{B^\star_r(p_i), i = 0, 1, 2, \ldots, k\}$ as in Lemma 3.4.2.

We claim that there is $\mathcal{D} \subset B^\star_r(p_0)$ dense in $B^\star_r(p_0)$ and such that every $x \in \mathcal{D}$ has a zooming return to $B^\star_r(p_0)$. Indeed, for each $y \in B^\star_r(p_0)$ and $\gamma > 0$ one can find $m \in \mathbb{N}$ such that $d(f^m(q), y) < \gamma/2$. Taking $k > m$ big enough so that $d(f^m(x_k), f^m(q)) < \gamma/2$, it follows that $d(f^m(x_k, y) < \gamma, f^m(x_k) \in H_{n_k-m}$ and $f^{n_k-m}(x_k) \in B^\star_r(p_0)$. $\square$

As before, we can take any point in $\mathcal{D}$ to construct an element of the partition. Since the set is dense, we conclude that the partition is also dense.

3.5 Applications

In this section, we give examples of a zooming maps. We begin by defining a non-flat map.

3.5.1 Hyperbolic Times

The idea of hyperbolic times is a key notion on the study of non-uniformly hyperbolic dynamics and it was introduced by Alves et al. This is powerful to get expansion in the context of non-uniform expansion. Here, we recall the basic definitions and results on hyperbolic times that we will use later on. We will see that this notion is an example of a Zooming Time.

In the following, we give definitions taken from [1] and [25].

Definition 3.5.1 *Let M be a compact Riemannian manifold of dimension $d \geq 1$ and $f : M \to M$ a continuous map defined on M. The map f is called **non-flat** if it is a local $C^{1+\alpha}, (\alpha > 0)$ diffeomorphism in the whole manifold except in a non-degenerate set $\mathcal{C} \subset M$. We say that $\mathcal{C} \subset M$ is a **non-degenarate set** if there exist $\beta, B > 0$ such that the following two conditions hold.*

- $\frac{1}{B} d(x, \mathcal{C})^{\beta} \leq \frac{\|Df(x)v\|}{\|v\|} \leq B d(x, \mathcal{C})^{-\beta}$ *for all $v \in T_x M$.*

  For every $x, y \in M \backslash \mathcal{C}$ with $d(x, y) < d(x, \mathcal{C})/2$ we have

- $| \log \| Df(x)^{-1} \| - \log \| Df(y)^{-1} \| | \leq \frac{B}{d(x,\mathcal{C})^{\beta}} d(x, y)$.

In the following, we give the definition of a hyperbolic time [3], [25].

Definition 3.5.2 *(Hyperbolic times). Let us fix $0 < b = \frac{1}{3} \min\{1, 1/\beta\} < \frac{1}{2} \min\{1, 1/\beta\}$. Given $0 < \sigma < 1$ and $\epsilon > 0$, we will say that n is a (σ, ϵ)**-hyperbolic time** for a point $x \in M$ (with respect to the non-flat map f with a β-non-degenerate critical/singular set $\mathcal{C}$) if for all $1 \leq k \leq n$ we have*

$$\prod_{j=n-k}^{n-1} \|(Df \circ f^j(x)^{-1}\| \leq \sigma^k \text{ and } dist_{\epsilon}(f^{n-k}(x), \mathcal{C}) \geq \sigma^{bk}.$$

We denote de set of points of M such that $n \in \mathbb{N}$ is a (σ, ϵ)-hyperbolic time by $H_n(\sigma, \epsilon, f)$.

Proposition 3.5.1 *(Positive frequence). Given $\lambda > 0$ there exist $\theta > 0$ and $\epsilon_0 > 0$ such that, for every $x \in U$ and $\epsilon \in (0, \epsilon_0]$,*

$$\#\{1 \leq j \leq n;\ x \in H_j(e^{-\lambda/4}, \epsilon, f)\} \geq \theta n,$$

whenever $\frac{1}{n}\sum_{i=0}^{n-1} \log \|(Df(f^i(x)))^{-1}\|^{-1} \geq \lambda$ and $\frac{1}{n}\sum_{i=0}^{n-1} -\log dist_\epsilon(x, \mathcal{C}) \leq \frac{\lambda}{16\beta}$.

If f is non-uniformly expanding, it follows from the proposition that the points of U have infinitely many moments with positive frequency of hyperbolic times. In particular, they have infinitely many hyperbolic times.

The following proposition shows that the hyperbolic times are indeed zooming times, where the zooming contraction is $\alpha_k(r) = \sigma^{k/2} r$.

Proposition 3.5.2 *Given $\sigma \in (0,1)$ and $\epsilon > 0$, there is $\delta, \rho > 0$, depending only on σ and ϵ and on the map f, such that if $x \in H_n(\sigma, \epsilon, f)$ then there exists a neighbourhood $V_n(x)$ of x with the following properties:*

(1) f^n maps $\overline{V_n(x)}$ diffeomorphically onto the ball $\overline{B_\delta(f^n(x))}$;

(2) $dist(f^{n-j}(y), f^{n-j}(z)) \leq \sigma^{j/2} dist(f^n(y), f^n(z)), \forall y, z \in V_n(x)$ and $1 \leq j < n$.

(3) $\log \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \leq \rho d(f^n(y), f^n(z))$.

for all $y, z \in V_n(x)$.

The sets $V_n(x)$ are called hyperbolic pre-balls and their images $f^n(V_n(x)) = B_\delta(f^n(x))$, hyperbolic balls.

In the following, we give an example of such a map.

3.5.2 Viana maps

Example 3.5.1 *(Viana maps) We recall the definition of the open class of maps with critical sets in dimension 2, introduced by M. Viana in [34]. We skip the technical points. It can be generalized for any dimension (See [1]).*

Let $a_0 \in (1,2)$ be such that the critical point $x=0$ is pre-periodic for the quadratic map $Q(x)=a_0-x^2$. Let $S^1=\mathbb{R}/\mathbb{Z}$ and $b: S^1 \to \mathbb{R}$ a Morse function, for instance $b(\theta)=\sin(2\pi\theta)$. For fixed small $\alpha>0$, consider the map

$$\begin{aligned} f_0 : S^1 \times \mathbb{R} &\longrightarrow S^1 \times \mathbb{R} \\ (\theta, x) &\longmapsto (g(\theta), q(\theta, x)) \end{aligned}$$

where g is the uniformly expanding map of the circle defined by $g(\theta)=d\theta(mod\mathbb{Z})$ for some $d \geq 16$, and $q(\theta,x)=a(\theta)-x^2$ with $a(\theta)=a_0+\alpha b(\theta)$. It is easy to check that for $\alpha>0$ small enough there is an interval $I \subset (-2,2)$ for which $f_0(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map f sufficiently close to f_0 in the C^0 topology has $S^1 \times I$ as a forward invariant region. We consider from here on these maps f close to f_0 restricted to $S^1 \times I$. Taking into account the expression of f_0 it is not difficult to check that for f_0 (and any map f close to f_0 in the C^2 topology) the critical set is non-degenerate.

The main properties of f in a C^3 neighbourhood of f that we will use here are summarized below (See [1], [6], [25]):

(1) f is ***non-uniformly expanding***, that is, there exist $\lambda>0$ and a Lebesgue full measure set $H \subset S^1 \times I$ such that for all point $p=(\theta,x) \in H$, the following holds
$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \| Df(f^i(p))^{-1} \|^{-1} < -\lambda.$$

(2) Its orbits have ***slow approximation to the critical set***, that is, for every $\epsilon>0$ the exists $\delta>0$ such that for every point $p=(\theta,x) \in H \subset S^1 \times I$, the following holds
$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} -\log \operatorname{dist}_\delta(p,\mathcal{C}) < \epsilon.$$
where
$$\operatorname{dist}_\delta(p,\mathcal{C}) = \begin{cases} dist(p,\mathcal{C}), & if \quad dist(p,\mathcal{C})<\delta \\ 1 & if \quad dist(p,\mathcal{C}) \geq \delta \end{cases}$$

(3) f is topologically mixing;

(4) f is strongly topologically transitive;

(5) it has a unique ergodic absolutely continuous invariant (thus SRB) measure;

(6) the density of the SRB measure varies continuously in the L^1 norm with f.

3.5.3 Benedicks-Carleson Maps

We study a class of non-hyperbolic maps of the interval with the condition of exponential growth of the derivative at critical values, called ***Collet-Eckmann Condition***. We also ask the map to be C^2 and topologically mixing and the critical points to have critical order $2 \le \alpha < \infty$.

Given a critical point $c \in I$, the ***critical order*** of c is a number $\alpha_c > 0$ such that $f(x) = f(c) \pm |g_c(x)|^{\alpha_c}$, $\forall x \in \mathcal{U}_c$ where g_c is a diffeomorphism $g_c : \mathcal{U}_c \to g(\mathcal{U}_c)$ and $\mathcal{U}_c$ is a neighbourhood of c.

Let $\delta > 0$ and denote $\mathcal{C}$ the set of critical points and $B_\delta = \cup_{c\in\mathcal{C}}(c-\delta, c+\delta)$. Given $x \in I$, we suppose that

- **(Expansion outside B_δ)**. There exists $\kappa > 1$ and $\beta > 0$ such that, if $x_k = f^k(x) \notin B_\delta$, $0 \le k \le n-1$ then $|Df^n(x)| \ge \kappa\delta^{(\alpha_{\max}-1)}e^{\beta n}$, where $\alpha_{\max} = \max\{\alpha_c, c \in \mathcal{C}\}$. Moreover, if $x_0 \in f(B_\delta)$ or $x_n \in B_\delta$ then $|Df^n(x)| \ge \kappa e^{\beta n}$.

- **(Collet-Eckmann Condition)**. There exists $\lambda > 0$ such that

$$|Df^n(f(c))| \ge e^{\lambda n}.$$

- **(Slow Recurrence to $\mathcal{C}$)**. There exists $\sigma \in (0, \lambda/5)$ such that

$$dist(f^k(x), \mathcal{C}) \ge e^{-\sigma k}.$$

3.5.4 Rovella Maps

There is a class of non-uniformly expanding maps known as ***Rovella Maps***. They are derived from the so-called *Rovella Attractor*, a variation of the *Lorenz Attractor*. We proceed with a brief presentation. See [5] for details.

Contracting Lorenz Attractor

The geometric Lorenz attractor is the first example of a robust attractor for a flow containing a hyperbolic singularity. The attractor is a transitive maximal invariant set for a flow in three-dimensional space induced by a vector field having a singularity at the origin for which the derivative of the vector field at the singularity has real eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$ with $\lambda_1 + \lambda_3 > 0$. The singularity is accumulated by regular orbits which prevent the attractor from being hyperbolic.

The geometric construction of the contracting Lorenz attractor (Rovella attractor) is the same as the geometric Lorenz attractor. The only difference is the condition (A1)(i) below that gives in particular $\lambda_1 + \lambda_3 < 0$. The initial smooth vector field X_0 in $\mathbb{R}^3$ has the following properties:

(A1) X_0 has a singularity at 0 for which the eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ of $DX_0(0)$ satisfy:

(i) $0 < \lambda_1 < -\lambda_3 < 0 < -\lambda_2$,

(ii) $r > s + 3$, where $r = -\lambda_2/\lambda_1, s = -\lambda_3/\lambda_1$;

(A2) there is an open set $U \subset \mathbb{R}^3$, which is positively invariant under the flow, containing the cube $\{(x, y, z) :| x |\leq 1, | y |\leq 1, | x |\leq 1\}$ and supporting the *Rovella attractor*

$$\Lambda_0 = \bigcap_{t\geq 0} X_0^t(U).$$

The top of the cube is a Poincaré section foliated by stable lines $\{x = \text{const}\} \cap \Sigma$ which are invariant under Poincaré first return map P_0. The invariance of this foliation uniquely defines a one-dimensional map $f_0 : I\backslash\{0\} \to I$ for which

$$f_0 \circ \pi = \pi \circ P_0,$$

where I is the interval $[-1, 1]$ and π is the canonical projection $(x, y, z) \mapsto x$;

(A3) there is a small number $\rho > 0$ such that the contraction along the invariant foliation of lines x =const in U is stronger than ρ.

See [5] for properties of the map f_0.

Rovella Parameters

The Rovella attractor is not robust. However, the chaotic attractor persists in a measure theoretical sense: there exists a one-parameter family of positive Lebesgue measure of C^3 close vector fields to X_0 which have a transitive non-hyperbolic attractor. In the proof of that result, Rovella showed that there is a set of parameters $E \subset (0, a_0)$ (that we call *Rovella parameters*) with a_0 close to 0 and 0 a full density point of E, i.e.

$$\lim_{a \to 0} \frac{\mid E \cap (0, a) \mid}{a} = 1,$$

such that:

(C1) there is $K_1, K_2 > 0$ such that for all $a \in E$ and $x \in I$

$$K_2 \mid x \mid^{s-1} \leq f_a'(x) \leq K_1 \mid x \mid^{s-1},$$

where $s = s(a)$. To simplify, we shall assume s fixed.

(C2) there is $\lambda_c > 1$ such that for all $a \in E$, the points 1 and -1 have *Lyapunov exponents* greater than λ_c:

$$(f_a^n)'(\pm 1) > \lambda_c^n, \ \forall n \geq 0;$$

(C3) there is $\alpha > 0$ such that for all $a \in E$ the *basic assumption* holds:

$$\mid f_a^{n-1}(\pm 1) \mid > e^{-alphan}, \ \forall n \geq 1;$$

(C4) the forward orbits of the points ± 1 under f_a are dense in $[-1, 1]$ for all $a \in E$.

Definition 3.5.3 *We say that a map f_a with $a \in E$ is a **Rovella Map**.*

Theorem 3.5.1 *(Alves-Soufi [5]) Every Rovella map is non-uniformly expanding.*

Bibliography

[1] J. F. Alves, *Statistical Analysis of Non-Uniformly Expanding Dynamical Systems*, 24° Colóquio Brasileiro de Matemática, 2003.

[2] J. F. Alves, S. Luzzatto, V. Pinheiro, *Markov Structures for Non-Uniformly Expanding Maps on Compact Manifolds in Arbitrary Dimension*, Eletronic Research Annoucements of the American Mathematical Society, 9, 2003, 26-31.

[3] J. F. Alves, S. Luzzatto, V. Pinheiro, *Markov Structures and Decay of Correlations for Non-Uniformly Expanding Dynamical Systems*, Annales de l'Institute Henri Poincaré, 22, 2005, 817-839.

[4] J. F. Alves, V. Ramos, J. Siqueira, *Equilibrium Stability for Non-Uniformly Hyperbolic Systems*, Ergodic Theory and Dynamical Systems, Published Online in 2018.

[5] J. F. Alves, M. Soufi, *Statistical Stability and Limit Laws for Rovella Maps*, Nonlinearity, 25, 2012, 3527-3552.

[6] J. F. Alves, M. Viana, *Statistical Stability for Robust Classes of Maps with Non-uniform Expansion*, Ergodic Theory and Dynamical Systems, 22, 2002, 1-32.

[7] A. Arbieto, C. Matheus, K. Oliveira, *Equilibrium States for Random Non-Uniformly Expanding Maps*, Nonlinearity, 17, 2004, 581-593.

[8] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Springer-Verlag, 2008.

[9] H. Bruin, M. Demers, I. Melbourne, *Existence and Convergence Properties of Physical Measures for Certain Dynamical Systems with Holes*, Ergodic Theory and Dynamical Systems, 30, 2010, 687-728.

[10] H. Bruin, G. Keller, *Equilibrium States for S-Unimodal Maps*, Ergodic Theory and Dynamical Systems, 18, 1998, 765-789.

[11] H. Bruin, M. Todd, *Equilibrium States for Interval Maps: The Potential* $-t \log \mid df \mid$, Annales Scientifiques de l'École Normale Superieure, 42, 4, 2009, 559-600.

[12] J. Buzzi, O. Sarig, *Uniqueness of Equilibrium Measures for Countable Markov Shifts and Multidimensional Piecewise Expanding Maps*, Ergodic Theory and Dynamamical Systems, 23, 2003, 1383-1400.

[13] M. Demers, M. Todd, *Equilibrium States, Pressure and Escape for Multimodal Maps with Holes*, Israel Journal of Mathematics, 221, 1, 2017, 367-424.

[14] M. Denker, M. Urbanski, *Ergodic Theory of Equilibrium States for Rational Maps*, Nonlinearity, 4, 1991, 103-134.

[15] K. Díaz-Ordaz, M. P. Holland, S. Luzzatto, *Statistical Properties of One-Dimensional Maps with Critical Points and Singularities*, Stochastics and Dynamics, 6, 4, 2006, 423-458.

[16] G. Iommi, T. Jordan, M. Todd, *Recurrence and Transience for Suspension Flows*, Israel Journal of Mathematics, 209, 2, 2015, 547-592.

[17] G. Iommi, M. Todd, *Thermodynamic Formalism for Interval Maps: Inducing Schemes*, Dynamical Systems, 28, 3, 2013, 354-380.

[18] G. Iommi, M. Todd, *Natural Equilibrium States for Multimodal Maps*, Communications in Mathematical Physics, 300, 2010, 65-94.

[19] H. Li, J. Rivera-Letelier, *Equilibrium States of Weakly Hyperbolic One-Dimensional Maps for Hölder Potentials*, Communications in Mathematical Physics, 328, 2014, 397-419.

[20] R. Leplaideur, K. Oliveira, I. Rios, *Invariant Manifolds and Equilibrium States for Non-Uniformly Hyperbolic Horseshoes*, Nonlinearity, 19, 2006, 2667-2694.

[21] K. Oliveira, *Equilibrium States for Non-Uniformly Expanding Maps*, Ergodic Theory and Dynamical Systems, 23, 06, 2003, 1891-1905.

[22] K. Oliveira, M. Viana, *Foundations of Ergodic Theory*, Cambridge University Press, 2016.

[23] Ya. Pesin, *On the Work of Omri Sarig on Countable Markov Chains and Thermodynamic Formalism*, Journal of Modern Dynamics, 8, 2014, 1-14.

[24] Ya. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, The University of Chicago Press, 2016.

[25] V. Pinheiro, *Expanding Measures*, Annales de l'Institute Henri Poincaré, 28, 2011, 889-939.

[26] V. Pinheiro, *Sinai Ruelle Bowen Measures for Weakly Expanding Maps*, Nonlinearity, 19, 2006, 1185-1200.

[27] V. Ramos, M. Viana, *Equilibrium States for Hyperbolic Potentials*, Nonlinearity, 30, 2017, 825-847.

[28] O. Sarig, *Thermodynamic Formalism for Countable Markov Shifts*, Ergodic Theory and Dynamical Systems, 19, 1999, 1565-1593.

[29] O. Sarig, *Existence of Gibbs Measures for Countable Markov Shifts*, Proceedings of the American Mathematical Society, 131, 6, 2003, 1751-1758.

[30] O. Sarig, *Thermodynamic Formalism for Null Recurrent Potentials*, Israel Journal of Mathematics, 121, 2001, 285-311.

[31] O. Sarig, *Lecture Notes on Thermodynamic Formalism for Topological Markov Shifts*, Penn State, 2009.

[32] O. Sarig, *Phase transitions for countable markov shifts* Communications of Mathematics Physics, 217, 2001, 555-577.

[33] P. Varandas, M. Viana, *Existence, Uniqueness and Stability of Equilibrium States for Non-Uniformly Expanding Maps*, Annales de l'Institute Henri Poincaré, 27, 2010, 555-593.

[34] M. Viana, *Multidimensional NonHyperbolic Attractors*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 85, 1, 1997, 63-96.

[35] P. Walters, *A Variational Principle for the Pressure of Continuous Transformations*, American Journal of Mathematics, 97 1975 937â997.

[36] M. Yuri, *Thermodynamical Formalism for Countable to One Markov Systems*, Transactions of the American Mathematical Society, 335, 2003, 2949-2971.

[37] R. Zweimüller, *Invariant Measure for General(ized) Induced Transformations*, Proceedings of the American Mathematical Society, 133, 8, 2005, 2283-2295.