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Surfaces à courbure moyenne constante dans les variétés homogènes

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Abstract

The goal of this thesis is to study constant mean curvature surfaces into homogeneous 3-manifolds with 4-dimensional isometry group.

In the first part of this thesis, we study constant mean curvature surfaces in the product manifolds $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. As a main result, we establish a local classification for constant mean curvature surfaces with constant intrinsic curvature in these spaces. In this classification, we present a new example of constant mean curvature surfaces with constant intrinsic curvature in $\mathbb{H}^2 \times \mathbb{R}$. As a consequence, we use the sister surface correspondence to classify the constant mean curvature surfaces with constant intrinsic curvature in the others homogeneous 3-manifolds with 4-dimensional isometry group, and then new examples with these conditions arise in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$.

We devote the second part of this thesis to study minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$. For this, we define a new Gauss map for surfaces in this space using the model of $\mathbb{S}^2 \times \mathbb{R}$ isometric to $\mathbb{R}^3 \setminus \{0\}$, endowed with a metric conformally equivalent to the Euclidean metric of $\mathbb{R}^3$. As a main result, we prove that any two minimal conformal immersions in $\mathbb{S}^2 \times \mathbb{R}$ with the same non-constant Gauss map differ by only two types of ambient isometries. Moreover, if the Gauss map is a singular, we show that it is necessarily constant and then the surface is a vertical cylinder over a geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$. We also study some particular cases, among them we also prove that there is no minimal conformal immersion into $\mathbb{S}^2 \times \mathbb{R}$ with anti-holomorphic non-constant Gauss map.

Keywords: Isometric immersions, Constant mean curvature surfaces, Homogeneous 3-manifolds.

Résumé

L'objectif de cette thèse est d'étudier les surfaces à courbure moyenne constante dans des variétés homogènes de dimension 3 avec un groupe d'isométries de dimension 4.

Dans la première partie de cette thèse, nous étudions les surfaces à courbure moyenne constante dans les variétés produites $\mathbb{S}^2 \times \mathbb{R}$ et $\mathbb{H}^2 \times \mathbb{R}$. Comme résultat principal, nous établissons une classification locale pour les surfaces à courbure moyenne constante et courbure intrinsèque constante dans ces espaces. Dans cette classification, nous présentons un nouvel exemple de surface à courbure moyenne constante et courbure intrinsèque constante dans $\mathbb{H}^2 \times \mathbb{R}$. En conséquence, nous utilisons la correspondence des surfaces soeurs pour classifier les surfaces à courbure moyenne constante et courbure intrinsèque constante dans les autres variétés homogènes de dimension 3 avec un groupe d'isométries de dimension 4, et donc sous ces conditions des nouveaux examples apparaissent dans $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$.

Nous consacrons la deuxième partie de cette thèse à l'étude des surfaces minimales dans $\mathbb{S}^2 \times \mathbb{R}$. À cet effet, nous définissons une nouvelle application de Gauss pour ces surfaces, en utilisant le modèle de $\mathbb{S}^2 \times \mathbb{R}$ qui est isométrique à $\mathbb{R}^3 \setminus \{0\}$, muni d'une métrique conformément équivalente à la métrique de l'espace euclidien $\mathbb{R}^3$. Comme résultat principal, nous montrons que deux immersions minimales conformes quelconques en $\mathbb{S}^2 \times \mathbb{R}$, avec la même application de Gauss non-constante, ne diffèrent que par des isométries de $\mathbb{S}^2 \times \mathbb{R}$ de deux types particuliers. De plus, si l'application de Gauss est singulière, nous montrons que cette application est forcément constante, et donc, la surface est un cylindre vertical sur une géodésique de $\mathbb{S}^2$ dans $\mathbb{S}^2 \times \mathbb{R}$. Nous étudions également quelques cas particuliers, et, parmi eux, nous prouvons qu'il n'existe pas d'immersion minimale conforme dans $\mathbb{S}^2 \times \mathbb{R}$ telle que l'application de Gauss soit non-constante et anti-holomorphe.

Mots-clés : Immersions isométriques, Surfaces à courbure moyenne constante, Variétés homogènes de dimension 3.

Resumo

O objetivo desta tese é estudar superfícies com curvatura média constante em Variedades homogêneas tridimensionais cujo grupo de isometrias tem dimensão 4.

Na primeira parte desta tese, nós estudamos superfícies com curvatura média contante nas variedades produtos $\mathbb{S}^2 \times \mathbb{R}$ e $\mathbb{H}^2 \times \mathbb{R}$. Como resultado principal, estabelecemos uma classificação local para superfícies com curvaturas média e intrínseca constantes nesses espaços. Nessa classificação, nós apresentamos um novo exemplo de superfície com curvaturas média e intrínseca constantes em $\mathbb{H}^2 \times \mathbb{R}$. Como consequência, utilizamos a correspondência das superfícies irmãs para classificar superfícies com curvaturas média e intrínseca constantes nas demais Variedades homogêneas tridimensionais, cujo grupo de isometrias tem dimensão 4, e então novos exemplos sob essas condições surgem em $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$.

Nós dedicamos a segunda parte desta tese para estudar superfícies mínimas em $\mathbb{S}^2 \times \mathbb{R}$. Com esse propósito, definimos uma nova aplicação de Gauss para essas superfícies, utilizando o modelo de $\mathbb{S}^2 \times \mathbb{R}$ isométrico à $\mathbb{R}^3 \setminus \{0\}$, munido com uma métrica conformemente equivalente a métrica do Espaço Euclidiano $\mathbb{R}^3$. Como principal resultado, provamos que quaisquer duas imersões mínimas conformes em $\mathbb{S}^2 \times \mathbb{R}$, com mesma aplicação de Gauss não-constante, diferem somente por isometrias ambientes, sendo essas dadas por apenas dois tipos. Mais ainda, se a aplicação de Gauss é singular, mostramos que a mesma é necessariamente constante, e, sendo assim, a superfície é um cilindro vertical sobre uma geodésica de $\mathbb{S}^2$ em $\mathbb{S}^2 \times \mathbb{R}$. Além disso, também estudamos alguns casos particulares, e, entre esses, provamos que não existe imersão mínima conforme em $\mathbb{S}^2 \times \mathbb{R}$ cuja a aplicação de Gauss é não-constante e anti-holomorfa.

Palavras-chave: Imersões isométricas, Superfícies com curvatura média constante, Variedades homogêneas tridimensionais.

Contents

Chapter 1

Introduction

1.1 Overview

Surfaces with constant mean curvature take an important place in Differential Geometry since 18th century. The first examples of minimal surfaces in $\mathbb{R}^3$ were given by Lagrange (1761), Meusnier (1776) and later, in the 19th century, Delaunay gave a systematic way to obtain non-zero constant mean curvature revolution surfaces in $\mathbb{R}^3$. In fact, the study of these surfaces was initially motivated by variational problems, such as the minimization of the area with or without a constraint on the volume enclosed by the surface, for instance the isoperimetric problem. However the connection of minimal surfaces theory and holomorphic functions developed in the 19th century by Riemann, Weierstrass, Schwarz and among others, introduced new techniques and the theory of constant mean curvature surfaces has been continuously expanded since then.

In this thesis, we concentrate in two kinds of problems related with constant mean curvature surfaces into Homogeneous 3-manifolds with 4-dimensional isometry group: The first one is a local classification problem for constant mean curvature surfaces with constant intrinsic curvature in these ambient manifolds. The second one is the definition of a new Gauss map for surfaces in $\mathbb{S}^2 \times \mathbb{R}$ along with the investigation of how a conformal minimal surface is determined by its conformal structure and its Gauss map. We notice that in the other simply connected Homogeneous 3-manifolds with 4-dimensional isometry group, more generally in 3-dimensional Lie group endowed with a left invariant Riemannian metric, the Lie group structure allows to define a *left invariant Gauss map* for surfaces in this spaces, and an extensive theory was developed by W. Meeks, P. Mira, J. Peréz, A. Ros, among others.

1.2 Constant mean curvature surfaces

An important geometric notion associated to a surface in the Euclidian space $\mathbb{R}^3$ is the notion of curvature. Given an oriented regular surface $\mathcal{S}$ in $\mathbb{R}^3$, there are two important functions defined on the surface: the Gauss curvature K and the mean curvature H.

The Gauss curvature K was formulated by Carl Friedrich Gauss (1777-1855) in 1827 when he proved the *Theorema Egregium*. The mean curvature H was defined by Marie-Sophie Germain (1771-1831). It was in the year of her death that she introduced for the first time the notion of mean curvature.

Both curvatures are given in terms of the two principal curvatures of the surface. Geometrically, at a point of $\mathcal{S}$, the principal curvatures can be obtained taking the maximum and the minimum values of the normal curvatures of all curves passing through this point on the surface; and they also can be seen as the eigenvalues of the shape operator of the surface. The Gauss curvature is the product of the principal curvatures while the mean curvature is the arithmetic average of the principal curvatures.

The theory of minimal surfaces in $\mathbb{R}^3$, that is the surfaces which the mean curvature H vanishes identically, started in the 18th century with Leonhard Euler, Joseph-Louis Lagrange and Jean Baptiste Meusnier. Initially motivated by variational problems, they considered the problem of finding a surface which minimize area within a given boundary configuration. In 1744, L. Euler gave the first example of a complete minimal surfaces that differs of the plane: the catenoid. Some years later in 1776, J. B. Meusnier proved that the helicoid, previously described by L. Euler, was another example of the complete minimal surface in $\mathbb{R}^3$.

In the 19th century many other examples of minimal surfaces were discovered. Besides, Schwarz, Riemann and mainly K. Weierstrass, developed the connection of minimal surfaces theory with the holomorphic functions. Weierstrass established the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ that is a fundamental contribution to the theory. Among others, important complete examples of the 19th century are: the Enneper surface, which is a non-embedded minimal surface described by A. Enneper in 1864; the Scharwz surface, which is a triply periodic minimal surface described by H. Schwarz in 1865; the Riemann surface, which is a singly periodic minimal surface described by B. Riemann (in a posthumous paper in 1867); the Henneberg surface, which is the first example of non-orientable minimal surface described by L. Henneberg in 1875, and more recently in the 20th century, the Costa surface, which is another important example of an embedded minimal surface of finite topology that was described by C. Costa in 1982. This surface is a counterexample of the conjecture that believed that the only complete embedded minimal surfaces with finite topology in $\mathbb{R}^3$ were the helicoid, the catenoid and the plane.

Apart from the right cylinder and the round sphere, the first examples of non-zero constant mean curvature surfaces in $\mathbb{R}^3$, which means that the mean curvature H is a constant function, were given in the 19th century. In 1841, Charles-Eugène Delaunay gave a systematic way to obtain constant mean curvature revolution surfaces which consists of rotating the roulettes of the conics about their own axis. In fact, he proved that the only revolution surfaces with constant mean curvature are the plane, the right cylinder, the sphere, the catenoid, the unduloid and the nodoid.

Different from the minimal surfaces, new examples of non-zero constant mean curvature surfaces in $\mathbb{R}^3$ appeared only in the 20th century. However several important results of characterization of these surfaces were established through this time. The first one was found by H. J. Jellet, in 1853. He proved that the only closed star-shaped constant mean curvature surfaces in $\mathbb{R}^3$ are the (round) spheres. More than one century later, the non-zero mean curvature surfaces gained notoriety when in 1955 Heinz Hopf proved the remarkable result known as Hopf's problem:

Theorem (H. Hopf [Hop83]). *Let $\mathcal{S}$ be a constant mean curvature topological sphere in $\mathbb{R}^3$. Then $\mathcal{S}$ is a round sphere.*

The fundamental key to prove this result was the definition of *Hopf differential*. Given $\mathcal{S}$ a surface on $\mathbb{R}^3$ the *Hopf differential* of $\mathcal{S}$ is the quadratic differential defined by

$$\mathcal{Q}(X,Y) = \langle X - iJX, S(Y - iJY)\rangle,$$

where S is the shape operator of $\mathcal{S}$ and J denotes the rotation by $\pi/2$ on $\mathrm{T}\mathcal{S}$. In fact, H. Hopf showed that for constant mean curvature surface $\mathcal{S}$ in $\mathbb{R}^3$, the *Hopf differential* $\mathcal{Q}$ is holomorphic on $\mathcal{S}$. Moreover, $|\mathcal{Q}|^2 = 4(H^2 - K)$, that is, $\mathcal{Q}$ vanishes at the umbilical points of $\mathcal{S}$.

At that time, H. Hopf conjectured that every constant mean curvature compact immersed surface in $\mathbb{R}^3$ was a round sphere, however only in 1986, H. C. Wente gave a family of counterexamples to this conjecture. He described constant mean curvature surfaces homeomorphic to the torus on $\mathbb{R}^3$: the Wente tori.

Another important contribution to the theory of constant mean curvature surfaces in $\mathbb{R}^3$ is due to H.B. Lawson. Indeed, in 1970, he established an isometric correspondence between constant mean curvature surfaces in 3-dimensional space-forms, known as *Lawson correspondence*. In particular, follows of his result that minimal surfaces in the unit 3-sphere $\mathbb{S}^3$ are locally isometric to constant mean curvature surfaces in $\mathbb{R}^3$ with $H = 1$.

1.3 Homogeneous 3-manifolds with 4-dimensional isometry group

In the last three centuries, the study of constant mean curvature (CMC) surfaces has been established on the 3-dimensional space forms: The Euclidian space $\mathbb{R}^3$, the unit 3-sphere $\mathbb{S}^3$ and the hyperbolic space $\mathbb{H}^3$. However in the last twenty years, the theory of CMC surfaces on simply connected homogeneous 3-manifolds has been developed after the definition of a holomorphic differential given by Abresch and H. Rosenberg. Namely *Abresch-Rosenberg differential*, this quadratic differential plays the same role for CMC surfaces in some homogeneous 3-manifolds as the Hopf differential plays to CMC surfaces on $\mathbb{R}^3$.

A simple connected Riemannian 3-manifold $\overline{M}$ is said to be homogeneous if its isometry group acts transitively on the manifold, that is, for all $p, q \in \overline{M}$ there is an isometry f such that $f(p) = q$. Naturally, these manifolds are completes.

It is known that simply connected Homogeneous 3-manifolds can be classified by the dimension of its isometry group. The Homogeneous 3-manifolds with 6-dimensional isometry group are space forms. When the isometry group has dimension 3, we have some Lie groups with a left invariant metric, as the space Sol_3 which is the only Thurston geometry in this case. In the case that the isometry group has dimension 4, there are five different kinds of 3-manifolds: The product manifolds $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$, the Heisenberg group endowed with a left invariant metric, denoted by Nil_3, the Berger spheres and the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ endowed with a left invariant metric, denoted by $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$.

It is important to remark that $\mathbb{S}^2\times\mathbb{R}$, $\mathbb{H}^2\times\mathbb{R}$, Nil_3 and $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ are four between the eight Thurston 3-dimensional geometries:

$$\mathbb{R}^3,\ \mathbb{S}^3,\ \mathbb{H}^3,\ \mathbb{S}^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ \mathrm{Nil}_3,\ \widetilde{\mathrm{PSL}}_2(\mathbb{R})\ \text{and}\ \mathrm{Sol}_3.$$

We denote by $\mathbb{E}(\kappa,\tau)$ a complete Homogeneous 3-manifolds with 4-dimensional isometry group. A such manifold is a Riemannian fibration of bundle curvature τ over a 2-dimensional space form $\mathbb{M}^2_\kappa$ of intrinsic curvature κ. Up to isometry, they are classified by the bundle curvature τ and the intrinsic curvature of the base κ, that satisfy $4\tau^2-\kappa\neq 0$. They are classified as follows:

- When $\tau=0$, $\mathbb{E}(\kappa,0)$ is the product manifold $\mathbb{M}^2_\kappa\times\mathbb{R}$,
- When $\tau\neq 0$ and $\kappa>0$, $\mathbb{E}(\kappa,\tau)$ is a Berger sphere,
- When $\tau\neq 0$ and $\kappa=0$, $\mathbb{E}(0,\tau)$ is the Heisenberg group Nil_3 endowed with a left invariant metric.

- When $\tau \neq 0$ and $\kappa < 0$, $\mathbb{E}(\kappa, \tau)$ is the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ endowed with a left invariant metric, denoted by $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$.

A comprehensive literature about constant mean curvature surfaces in these manifolds has been developed in the past few decades. For more details, we refer to [AR05, Dan07, FM10, Sco83].

1.3.1 Integrability equations and Sister correspondence

Let $(\Sigma, \mathrm{d}s^2)$ be an oriented Riemannian surface and $(\overline{M}, \mathrm{d}\mu^2)$ be a simply connected Riemannian 3-manifold and consider an isometric immersion $f : \Sigma \to \overline{M}$, i.e., f is an immersion satisfying $f^* \,\mathrm{d}\mu^2 = \mathrm{d}s^2$.

Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of Σ and $\overline{M}$, respectively, and by $N : \Sigma \to (\mathrm{T}\Sigma)^\perp$ the unit normal vector to Σ. If $X, Y \in \mathrm{T}\Sigma$ are local vector fields where $\overline{X}$ and $\overline{Y}$ are local extensions of X and Y on $\mathrm{T}M$, respectively, we have that $\nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^\top$. Moreover, the shape operator of f is the field of symmetric operators $S_p : \mathrm{T}_p\Sigma \to \mathrm{T}_p\Sigma$ given by $S(X) = -(\overline{\nabla}_X N)^\top$ at each point $p \in \Sigma$ and the mean curvature H is given by $H = (\mathrm{tr}\, S)/2$.

When $\overline{M} = \mathbb{R}^3$, a well-known result is that any surface on $\mathbb{R}^3$ satisfies the Gauss and Codazzi equations:

$$K = \det S, \qquad \text{(Gauss equation)}$$
$$\nabla_X(SY) - \nabla_Y(SX) - S[X, Y] = 0. \qquad \text{(Codazzi equation)}$$

Moreover, these equations are the integrability conditions in order to obtain an isometric immersion $f : \Sigma \to \mathbb{R}^3$ of a simply connected oriented Riemannian surface $(\Sigma, \mathrm{d}s^2)$ on $\mathbb{R}^3$, with a given field of symmetric operators $S_p : \mathrm{T}_p\Sigma \to \mathrm{T}_p\Sigma$. In this case, S will be the shape operator of the isometric immersion f. The same result also holds for the other two 3-space forms, when we consider the Gauss equation of the 3-space forms. In the literature, this result is called the Fundamental Theorem of surface theory:

Theorem (Fundamental theorem of surfaces theory). *Let $(\Sigma, \mathrm{d}s^2)$ be an oriented simply connected Riemannian surface, ∇ its Riemannian connection and K be the intrinsic curvature of $\mathrm{d}s^2$. Let S be a field of symmetric operators $S_p : \mathrm{T}_p\Sigma \to \mathrm{T}_p\Sigma$.*

Let $\mathbb{M}^3_c$ be the simply connected 3-space form of intrinsic curvature $c \in \mathbb{R}$. Then there is an isometric immersion $f : \Sigma \to \mathbb{M}^3_c$ such that the shape operator with respect to the normal N associated to f is

$$\mathrm{d}f \circ S \circ \mathrm{d}f^{-1}$$

if and only if the duple $(\mathrm{d}s^2, S)$ *satisfies the Gauss and Codazzi equations:*

$$K - c = \det S,$$
$$\nabla_X(SY) - \nabla_Y(SX) - S[X,Y] = 0.$$

In this case, the immersion f *is unique up to a global isometry of* $\mathbb{M}^3_c$*.*

As a consequence of the Fundamental Theorem of surface theory, H. B. Lawson established a relationship between constant mean curvature surfaces in 3-space forms. Denote by $\mathbb{M}^3_c$ denotes a 3-space form with sectional curvature c. The Lawson correspondence states that there exists an isometric correspondence between simply connected H_1-CMC surface in $\mathbb{M}^3_{c_1}$ and simply connected H_2-CMC surface in $\mathbb{M}^3_{c_2}$, when $c_1 - c_2 = H_2^2 - H_1^2$. Moreover, the shape operators of these surfaces are related by $S_2 - S_1 = (H_2 - H_1)\mathrm{I}$. In particular, minimal surfaces in $\mathbb{R}^3$ are locally isometric to $1-$CMC surfaces in $\mathbb{H}^3$.

It is remarkable that Gauss and Codazzi equations are necessary conditions to but not sufficient conditions to guarantee locally the immersion of a Riemannian surface isometrically into a $\mathbb{E}(\kappa,\tau)$. In this direction, in 2007 B. Daniel [Dan07] established necessary and sufficient condition for a $(\Sigma, \mathrm{d}s^2)$ Riemannian surface be locally isometrically immersed into a Homogeneous 3-manifolds with 4-dimensional isometry group.

Theorem 1.1 (Daniel [Dan07]). *Let* $(\Sigma, \mathrm{d}s^2)$ *be an oriented simply connected Riemannian surface,* ∇ *its Riemannian connection,* J *the rotation of angle* $\pi/2$ *on* $\mathrm{T}\Sigma$ *and* K *be the intrinsic curvature of* $\mathrm{d}s^2$*. Let* S *be a field of symmetric operators* $S_p : \mathrm{T}_p\Sigma \to \mathrm{T}_p\Sigma$*,* T *a vector field on* Σ *and* ν *a smooth function on* Σ *such that* $\|T\|^2 + \nu^2 = 1$*.*

Let $\mathbb{E}(\kappa,\tau)$ *be a 3-dimensional homogeneous manifolds with a 4-dimensional isometry group and* ξ *its vertical vector field. Let* κ *be its base curvature and* τ *its bundle curvature. Then there is an isometric immersion* $f : \Sigma \to \mathbb{M}^2_c \times \mathbb{R}$ *such that the shape operator with respect to the normal* N *associated to* f *is*

$$\mathrm{d}f \circ S \circ \mathrm{d}f^{-1}$$

and such that

$$\xi = \mathrm{d}f(T) + \nu N$$

if and only if the 4-uple $(\mathrm{d}s^2, S, T, \nu)$ *satisfies the following equations:*

$$K = \det S + \tau^2 + (\kappa - 4\tau^2)\nu^2,$$
$$\nabla_X SY - \nabla_Y SX - S[X,Y] = (\kappa - 4\tau^2)\nu\Big(\langle Y,T\rangle X - \langle X,T\rangle Y\Big),$$
$$\nabla_X T = \nu(SX - \tau JX),$$
$$\mathrm{d}\nu(X) + \langle SX - \tau JX, T\rangle = 0.$$

In this case, the immersion is unique up to a global isometry of $\mathbb{E}(\kappa,\tau)$ *preserving the orientations of both the fibers and the base of the fibration.*

The 4-uple $(\mathrm{d}s^2, S, T, \nu)$ is called Gauss-Codazzi data of the immersion f and that ν is its angle function.

In analogous way that happens on 3-space forms, the integrability conditions for Homogeneous 3-manifolds with 4-dimensional group make arises to a correspondence between CMC surfaces on theses spaces. The Lawson correspondence was generalised by B. Daniel [Dan07], to this context he states that:

Theorem 1.2 (Daniel [Dan07]). *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two 3-dimensional homogeneous manifolds with 4-dimensional groups, of base curvatures κ_1 and κ_2 and bundle curvatures τ_1 and τ_2, respectively, and such that*

$$\kappa_1^2 - 4\tau_1^2 = \kappa_2 - 4\tau_2^2.$$

Let H_1 and H_2 be two real numbers such that

$$\tau_1^2 + H_1^2 = \tau_2^2 + H_2^2.$$

Then there exists an isometric correspondence between simply connected H_1-CMC surfaces in $\mathbb{E}_1$ and simply connected H_2-CMC surfaces in $\mathbb{E}_2$.

In particular, minimal surfaces in Nil_3 are locally isometric to $1/2$-CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

1.3.2 The Abresch-Rosenberg differential on $\mathbb{M}_c^2 \times \mathbb{R}$

As mention before, the theory of CMC surfaces on simply connected Homogeneous 3-manifolds with 4-dimensional group has been developed principally after U. Abresch and H. Rosenberg definition of an holomorphic differential for any CMC surface on these spaces. With this definition, Abresch and Rosenberg proved the Hopf's problem to Homogeneous 3-manifolds with 4-dimensional group: any immersed CMC sphere in $\mathbb{E}(\kappa, \tau)$ is rotational.

Given a $H-$CMC surface Σ on $\mathbb{E}(\kappa, \tau)$, with Gauss-Codazzi data $(\mathrm{d}s^2, S, T, \nu)$, the quadratic differential Q_{AR} on Σ, defined by

$$Q_{AR}(X, Y) = 2(H + i\tau)\langle X - iJX, S(Y - iJY)\rangle - (\kappa - 4\tau^2)\langle T, X - iJX\rangle\langle T, Y - iJY\rangle,$$

is the *Abresch-Rosenberg differential* of the surface Σ.

Theorem (Abresch-Rosenberg [AR04, AR05]). *Let Σ be a $H-$CMC surface in $\mathbb{E}(\kappa, \tau)$, with Gauss-Codazzi data $(\mathrm{d}s^2, S, T, \nu)$. Then the Abresch-Rosenberg differential Q_{AR} of Σ is holomorphic, that is,*

$$(\nabla_Z Q_{AR} + i\nabla_{JZ} Q_{AR})(X, Y) = 0$$

for all $X, Y, Z \in \mathrm{T}\Sigma$.

When the curvature bundle τ vanishes, we have the product manifolds $\mathbb{M}^2_c \times \mathbb{R}$, where $\mathbb{M}^2_c$ denotes the 2-space form with curvature c. In this case, for $f : \Sigma \to \mathbb{M}^2_c \times \mathbb{R}$ an isometric immersion, with Gauss-Codazzi data $(\mathrm{d}s^2, S, T, \nu)$, $\xi = \partial_t$ is the unit upward pointing vertical field tangent to the factor $\mathbb{R}$. Moreover, $\mathrm{d}f(T)$ is nothing else that the gradient of height function h of f, defined as $h = \Pi_2 \circ f$, where $\Pi_2 : \mathbb{M}^2_c \times \mathbb{R} \to \mathbb{R}$ is the projection onto the factor $\mathbb{R}$.

Let Σ be a $H-$CMC surface on $\mathbb{M}^2_c \times \mathbb{R}$, with Gauss-Codazzi data $(\mathrm{d}s^2, S, T, \nu)$ and consider the quadratic real form given by

$$Q(X, Y) = 2H\langle SX, Y\rangle - c\langle \nabla h, X\rangle\langle \nabla h, Y\rangle,$$

where $h : \Sigma \to \mathbb{R}$ is the height function of Σ. Then, the holomorphic Abresch-Rosenberg differential Q_{AR} of the surface Σ is the $(2,0)-$part of the complexification of the quadratic real form $2Q$:

$$\begin{aligned} Q_{AR}(X, Y) &= Q(X, Y) - Q(JX, JY) - i(Q(JX, Y) + Q(X, JY)) \\ &= 2H\langle X - iJX, S(Y - iJY)\rangle - c\langle \nabla h, X - iJX\rangle\langle \nabla h, Y - iJY\rangle. \end{aligned}$$

Suppose that $\nu^2 < 1$ and consider the local orthonormal frame $\{e_1, e_2\} \subset \mathrm{T}\Sigma$ given by $e_1 = T/\|T\|$ and $Je_1 = e_2$, where J is the $\pi/2$-rotation on $\mathrm{T}\Sigma$. At a point $p \in \Sigma$, we have

$$Q_{AR}(e_1, e_1) = Q(e_1, e_1) - Q(e_2, e_2) - 2iQ(e_1, e_2).$$

Since $SJ + JS = 2HJ$ and, by Theorem 1.1, $ST = -\nabla\nu$ and $\|T\|^2 + \nu^2 = 1$, we get

$$\begin{aligned} (1-\nu^2)Q(e_1, e_1) &= -2H\langle \nabla\nu, \nabla h\rangle - c(1-\nu^2)^2 \\ (1-\nu^2)Q(e_2, e_2) &= 4H^2(1-\nu^2) + 2H\langle \nabla\nu, \nabla h\rangle \\ (1-\nu^2)Q(e_1, e_2) &= -2H\langle \nabla\nu, J\nabla h\rangle, \end{aligned}$$

and therefore

$$\begin{aligned} (1-\nu^2)^2|Q_{AR}(e_1, e_1)|^2 = {} & 16H^2(1-\nu^2)\|\nabla\nu\|^2 + 8Hc(1-\nu^2)^2\langle \nabla\nu, \nabla h\rangle \\ & + c^2(1-\nu^2)^4 + 16H^4(1-\nu^2)^2 + 8H^2c(1-\nu^2)^3 \\ & + 32H^3(1-\nu^2)\langle \nabla\nu, \nabla h\rangle. \end{aligned}$$

However, on the orthonormal frame $\{e_1, e_2\}$, for the shape operator S we have that

$$\det S = \frac{1}{1-\nu^2}(\|\nabla\nu\|^2 + 2H\langle \nabla\nu, \nabla h\rangle),$$

and, once more by Theorem 1.1, we get

$$\|\nabla\nu\|^2 = (-K + c\nu^2)(1-\nu^2) - 2H\langle \nabla\nu, \nabla h\rangle.$$

Hence,

$$(1-\nu^2)^2|Q_{AR}(e_1,e_1)|^2 = 16H^2(1-\nu^2)^2(-K+c\nu^2) + 8Hc(1-\nu^2)^2\langle\nabla\nu,\nabla h\rangle \\ + c^2(1-\nu^2)^4 + 16H^4(1-\nu^2)^2 + 8H^2c(1-\nu^2)^3,$$

i.e.,

$$|Q_{AR}(e_1,e_1)|^2 = 8Hc\langle\nabla\nu,\nabla h\rangle + 16H^2(H^2-K+c\nu^2) + 8H^2c(1-\nu^2) + c^2(1-\nu^2)^2,$$

that also holds at points which $\nu^2 = 1$, by continuity.

Since $|Q_{AR}(e_1,e_1)|^2$ does not depend of the choice of the unitary vector $e_1 \in \mathrm{T}_p\Sigma$, we consider the smooth function $q : \Sigma \to \mathbb{R}$ given by

$$q = 2Hc\langle\nabla\nu,\nabla h\rangle + 4H^2(H^2-K+c\nu^2) + 2H^2c(1-\nu^2) + \frac{c^2}{4}(1-\nu^2)^2.$$

The function q was introduced by J. Espinar and H. Rosenberg in [ER11], with a different approach: q is a normalization of the squared norm of the Abresch-Rosenberg differential, when they considered a conformal parameter on Σ. Since the Abresch-Rosenberg differential is holomorphic, then q either has isolated zeroes or vanishes identically. Moreover, in [ER11] the authors proved the following result that translates the holomorphic property of Q_{AR} to the function q:

Lemma (Espinar-Rosenberg [ER11]). *Let Σ be a $H-$CMC surface in $\mathbb{M}_c^2 \times \mathbb{R}$. Away from its isolated zeroes, the function q satisfies*

$$\Delta \log q = 4K.$$

1.4 Examples of CMC surfaces in $\mathbb{M}_c^2 \times \mathbb{R}$

The theory of surfaces in product manifolds $\mathbb{M}_c^2 \times \mathbb{R}$ started with H. Rosenberg and W. Meeks in [Ros02] and [MR05] and since then new examples of constant mean curvature surfaces in these ambient space has been studied. The first examples are horizontal surfaces and vertical cylinders over a curve with constant geodesic curvature in $\mathbb{M}_c^2$. In $\mathbb{S}_c^2 \times \mathbb{R}$, we also have minimal helicoids and minimal nodoids, while in $\mathbb{H}_c^2 \times \mathbb{R}$, we have minimal helicoids and minimal catenoids. We discuss some of these examples below among others based on works of Abresch and Rosenberg [AR05], Domínguez-Vázquez and Manzano [DVMM18], Leite [Lei07] and Sa Earp and Toubiana [SET05].

1. Horizontal surfaces

Given $a \in \mathbb{R}$, we call $\mathbb{M}_c^2 \times \{a\}$ of horizontal surface in $\mathbb{M}_c^2 \times \mathbb{R}$. The horizontal surfaces are totally geodesic surfaces in $\mathbb{M}_c^2 \times \mathbb{R}$, and therefore they are minimal. Moreover, they have intrinsic curvature $K = c$. For $\mathbb{M}_c^2 \times \mathbb{R}$, the height function is constant equal to a and, since the normal vector N of $\mathbb{M}_c^2 \times \{a\}$ is parallel to ∂_t and both are unitary vectors, the angle function ν satisfies $\nu^2 = 1$, in particular it is constant.

2. Vertical cylinders

Let $\gamma \subset \mathbb{M}_c^2$ be a curve with constant geodesic curvature k and $\pi : \mathbb{M}_c^2 \times \mathbb{R} \to \mathbb{M}_c^2$ is the Riemannian fibration. The vertical cylinder over γ is the surface $\Sigma = \pi^{-1}(\gamma)$ in $\mathbb{M}_c^2 \times \mathbb{R}$, denoted by $\gamma \times \mathbb{R}$. Since γ has constant geodesic curvature k then $\gamma \times \mathbb{R}$ is a $(k/2)$-CMC surface with intrinsic curvature $K = 0$. Moreover, the angle function ν vanishes since the normal vector N of $\gamma \times \mathbb{R}$ is orthogonal to ∂_t.

3. CMC surfaces in $\mathbb{M}_c^2 \times \mathbb{R}$ with vanishing Abresch-Rosenberg differential

A remarkable class of examples are the complete immersed constant mean curvature surfaces in $\mathbb{M}_c^2 \times \mathbb{R}$ with vanishing Abresch-Rosenberg differential. These surfaces were classified in the seminal paper [AR04] by Abresch and Rosenberg.

Constant mean curvature surfaces in $\mathbb{M}_c^2 \times \mathbb{R}$ whose Abresch-Rosenberg differential vanishes are divided in rotationally invariant surfaces and parabolic surfaces, this means surfaces invariant by parabolic isometries. On the Abresch-Rosenberg notation, between the rotationally invariant surfaces, we have: the horizontal surfaces $\mathbb{M}_c \times \{a\}$ with $a \in \mathbb{R}$, the spheres S_H^2 in $\mathbb{M}_c^2 \times \mathbb{R}$ satisfying $4H^2 + c > 0$, the surfaces D_H^2 in $\mathbb{H}_c^2 \times \mathbb{R}$ with a disk type satisfying $4H^2 + c \leq 0$, and the surfaces C_H^2 in $\mathbb{H}^2 \times \mathbb{R}$ with a catenoidal type satisfying $4H^2 + c < 0$. For the parabolic surfaces, we have: the vertical cylinders in $\mathbb{H}_c^2 \times \mathbb{R}$ satisfying $4H^2 + c = 0$ and the surfaces P_H^2 in $\mathbb{H}_c^2 \times \mathbb{R}$ satisfying $4H^2 + c < 0$, both foliated by horizontal horocycles.

We describe next the families of surfaces S_H^2, D_H^2, C_H^2 and P_H^2 and we use the parametrizations found by Domínguez-Vázquez and Manzano [DVMM18] and Leite [Lei07]. For this purpose, we use the common setting for $\mathbb{M}_c^2 \times \mathbb{R}$. Up to scaling, we suppose $c = \pm 1$. We consider the local model for $\mathbb{S}^2 \times \mathbb{R}$ given by $\mathbb{R}^3$ endowed with the Riemannian metric

$$\frac{1}{(1 + \frac{1}{4}(x^2 + y^2))^2}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}t^2.$$

This model above provides $\mathbb{S}^2 \times \mathbb{R}$ minus one fiber. For $c = -1$, we consider

$$\mathbb{H}_c^2 \times \mathbb{R} = \{(x, y, t) \in \mathbb{R}^3 : 1 - (x^2 + y^2)/4 > 0\}$$

endowed with the Riemannian metric

$$\frac{1}{(1 - \frac{1}{4}(x^2 + y^2))^2}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}t^2.$$

3.1 Rotationally invariant CMC spheres in $\mathbb{S}^2 \times \mathbb{R}$. Let $H > 0$. Consider the immersion $X : \mathbb{R}^2 \to \mathbb{S}^2 \times \mathbb{R}$ given by

$$X(u,v) = \left(\frac{1}{H}\cos(u)\cos(v), \frac{1}{H}\sin(u)\cos(v), \frac{4H}{\sqrt{4H^2+1}} \operatorname{arctanh}\left(\frac{\sin(v)}{\sqrt{4H^2+1}} \right) \right).$$

This surface is the analytic continuation of a surface that belongs to the rotationally invariant surfaces $S_{H,1,0}$ in $\mathbb{S}^2 \times \mathbb{R}$, whose the parametrization were described by Domínguez-Vázquez and Manzano [DVMM18], up to a vertical translation of $\mathbb{S}^2 \times \mathbb{R}$:

$$\widetilde{X}(u,t) = \left(t\cos(u), t\sin(u), \int_0^t \frac{-4H\sigma}{(4+\sigma^2)\sqrt{1-H^2\sigma^2}}\, d\sigma \right),$$

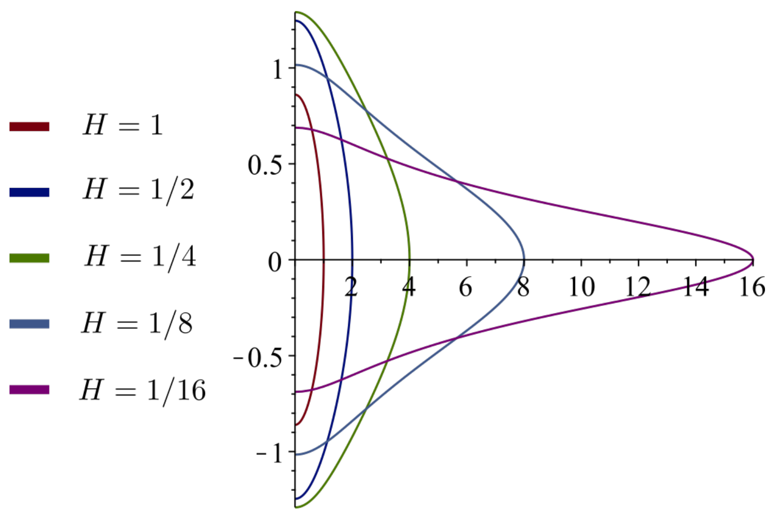
for $u \in \mathbb{R}$ and $t \in [0, 1/H)$. Indeed, by a direct computation, we get that

$$\int_0^t \frac{-4H\sigma}{(4+\sigma^2)\sqrt{1-H^2\sigma^2}}\, d\sigma = \\ = \frac{4H}{\sqrt{4H^2+1}} \left(\operatorname{arctanh}\left(\frac{\sqrt{1-H^2t^2}}{\sqrt{4H^2+1}} \right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{4H^2+1}} \right) \right).$$

With the change of coordinates given by $t = \cos(v)/H$, we get the parametrization X, up to a vertical translation of $\mathbb{S}^2 \times \mathbb{R}$.

For each $H \in \mathbb{R}^*$, $X(\mathbb{R}^2)$ is a complete surface with constant mean curvature H in $\mathbb{S}^2 \times \mathbb{R}$. Moreover, it is the immersed CMC sphere S_H^2 in $\mathbb{S}^2 \times \mathbb{R}$ studied by Hsiang and Hsiang in [HH89], Pedrosa and Ritoré in [PR99], and Abresch and Rosenberg in [AR05]. By the Abresch-Rosenberg results, the CMC sphere $S_H^2 \subset \mathbb{S}^2 \times \mathbb{R}$ is the solution of Hopf's problem in this ambient space.

Figure 1 – **Meridians of CMC spheres S_H^2 in $\mathbb{S}^2 \times \mathbb{R}$, with $H \neq 0$:**



3.2 Rotationally invariant CMC spheres in $\mathbb{H}^2 \times \mathbb{R}$. Let $H > 0$ such that $4H^2 - 1 > 0$. Consider the immersion $X : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$ given by

$$X(u,v) = \left(\frac{1}{H}\cos(u)\cos(v), \frac{1}{H}\sin(u)\cos(v), \frac{4H}{\sqrt{4H^2-1}} \arctan\left(\frac{\sin(v)}{\sqrt{4H^2-1}} \right) \right).$$

This surface is the analytic continuation of a surface that belongs to the rotationally invariant surfaces $S_{H,-1,0}$ in $\mathbb{H}^2 \times \mathbb{R}$ satisfying $4H^2 - 1 > 0$, whose the parametrization were described by Domínguez-Vázquez and Manzano [DVMM18], up to a vertical translation on $\mathbb{H}^2 \times \mathbb{R}$, up to a vertical translation of $\mathbb{H}^2 \times \mathbb{R}$:

$$\widetilde{X}(u,t) = \left(t\cos(u), t\sin(u), \int_0^t \frac{-4H\sigma}{(4-\sigma^2)\sqrt{1-H^2\sigma^2}}\, \mathrm{d}\sigma \right),$$

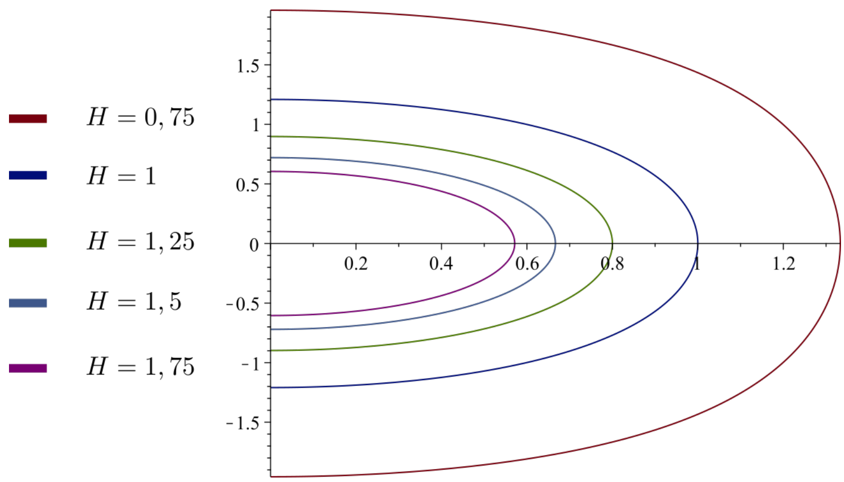
for $u \in \mathbb{R}$ and $t \in [0, 1/H)$. Indeed, by a direct computation, we get that

$$\int_0^t \frac{-4H\sigma}{(4-\sigma^2)\sqrt{1-H^2\sigma^2}}\, \mathrm{d}\sigma = \\ = \frac{4H}{\sqrt{4H^2-1}} \left(\arctan\left(\frac{\sqrt{1-H^2t^2}}{\sqrt{4H^2-1}} \right) - \arctan\left(\frac{1}{\sqrt{4H^2-1}} \right) \right).$$

With the change of coordinates given by $t = \cos(v)/H$, we get the parametrization X, up to a vertical translation of $\mathbb{H}^2 \times \mathbb{R}$.

For each $H \in \mathbb{R}$ such that $4H^2 - 1 > 0$, $X(\mathbb{R}^2)$ is a complete surface with constant mean curvature H in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, it is the immersed CMC sphere S^2_H in $\mathbb{H}^2 \times \mathbb{R}$ studied by Hsiang and Hsiang in [HH89], Pedrosa and Ritoré in [PR99], and Abresch and Rosenberg in [AR05]. By the Abresch-Rosenberg results, the CMC sphere $S^2_H \subset \mathbb{H}^2 \times \mathbb{R}$ is the solution of Hopf's problem in this ambient space.

Figure 2 – **Meridians of CMC spheres S^2_H in $\mathbb{H}^2 \times \mathbb{R}$ satisfying $4H^2 - 1 > 0$:**



3.3 Rotationally invariant CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with disk type. Let $H > 0$ such that $4H^2 - 1 < 0$. Let $v_0 = \arcsin(2H)$. Consider the immersion $X : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$ given by

$$X(u,v) = \Big(\frac{1}{H}\cos(u)\sin(v_0\tanh(v)), \frac{1}{H}\sin(u)\sin(v_0\tanh(v)), h(v)\Big),$$

where the funtion h is given by

$$h(v) = -\frac{4H}{\sqrt{1-4H^2}}\operatorname{arccoth}\left(\frac{\cos(v_0\tanh(v))}{\sqrt{1-4H^2}}\right).$$

This surface is the analytic continuation of a surface that belongs to the rotationally invariant surfaces $S_{H,-1,0}$ in $\mathbb{H}^2 \times \mathbb{R}$ satisfying $4H^2 - 1 < 0$, whose the parametrizations were described by Domínguez-Vázquez and Manzano [DVMM18], up to a vertical translation on $\mathbb{H}^2 \times \mathbb{R}$:

$$\widetilde{X}(u,t) = \left(t\cos(u), t\sin(u), \int_0^t \frac{-4H\sigma}{(4-\sigma^2)\sqrt{1-H^2\sigma^2}}\,\mathrm{d}\sigma\right)\Bigg),$$

for $u \in \mathbb{R}$ and $t \in [0,2)$. Indeed, by a direct computation, we get that

$$\int_0^t \frac{-4H\sigma}{(4-\sigma^2)\sqrt{1-H^2\sigma^2}}\,\mathrm{d}\sigma = \\ -\frac{4H}{\sqrt{1-4H^2}}\left(\operatorname{arccoth}\left(\frac{\sqrt{1-H^2t^2}}{\sqrt{1-4H^2}}\right) - \operatorname{arccoth}\left(\frac{1}{\sqrt{1-4H^2}}\right)\right).$$

With the change of coordinates given by $t = \sin(v_0\tanh(v))/H$, we get the parametrization X, up to a vertical translation of $\mathbb{H}^2 \times \mathbb{R}$.

For each $H \in \mathbb{R}^*$ such that $4H^2 - 1 < 0$, $X(\mathbb{R}^2)$ is a complete surface with constant mean curvature H in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, it is the immersed rotationally invariant CMC surface D^2_H in $\mathbb{H}^2 \times \mathbb{R}$ studied by Abresch and Rosenberg in [AR05].

3.4 Rotationally invariant CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with disk type, satisfying $4H^2 = 1$. Let $H = 1/2$. Consider the immersion $X : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$ given by

$$X(u,v) = \Big(2\cos(u)\tanh(v), 2\sin(u)\tanh(v), -2(\cosh(v) - 1)\Big).$$

This surface is the analytic continuation of a surface that belongs to the rotationally invariant surfaces $S_{H,-1,0}$ in $\mathbb{H}^2 \times \mathbb{R}$ satisfying $4H^2 - 1 = 0$, whose the parametrizations were described by Domínguez-Vázquez and Manzano [DVMM18]:

$$\widetilde{X}(u,t) = \left(t\cos(u), t\sin(u), \int_0^t \frac{-4\sigma}{(4-\sigma^2)^{3/2}}\,\mathrm{d}\sigma\right),$$

Figure 3 – **Meridians of CMC disk type surfaces D^2_H in $\mathbb{H}^2 \times \mathbb{R}$ satisfying $4H^2 - 1 < 0$:**

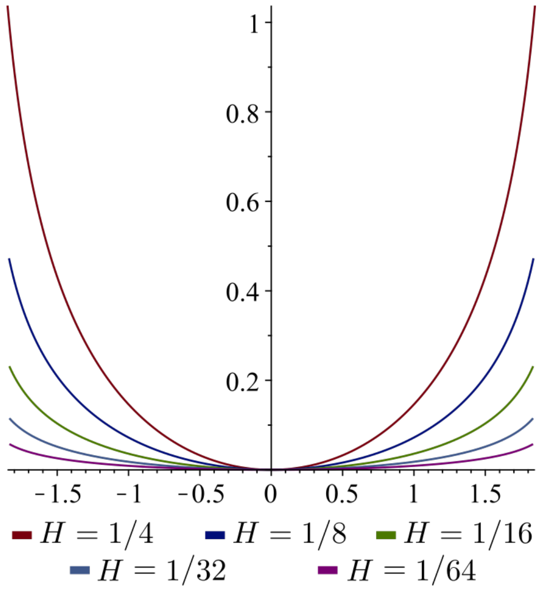
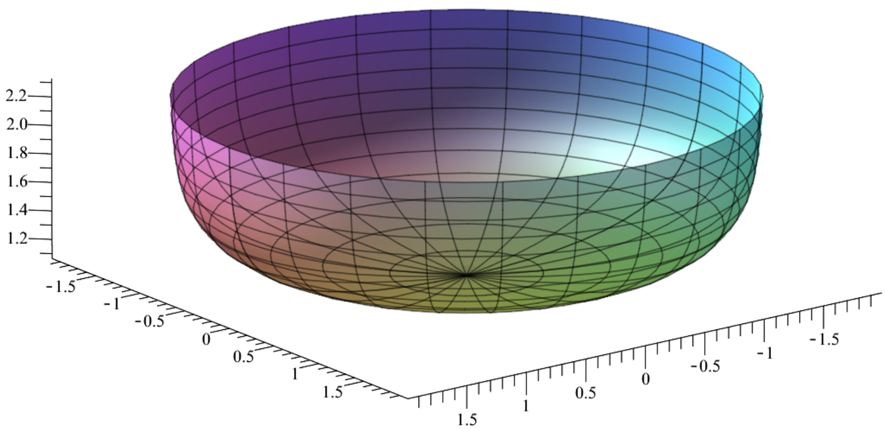


Figure 4 – **CMC disk type surface D^2_H in $\mathbb{H}^2 \times \mathbb{R}$ with $H = 1/8$:**



for $u \in \mathbb{R}$ and $t \in [0, 2)$. Indeed, by a direct computation, we get that

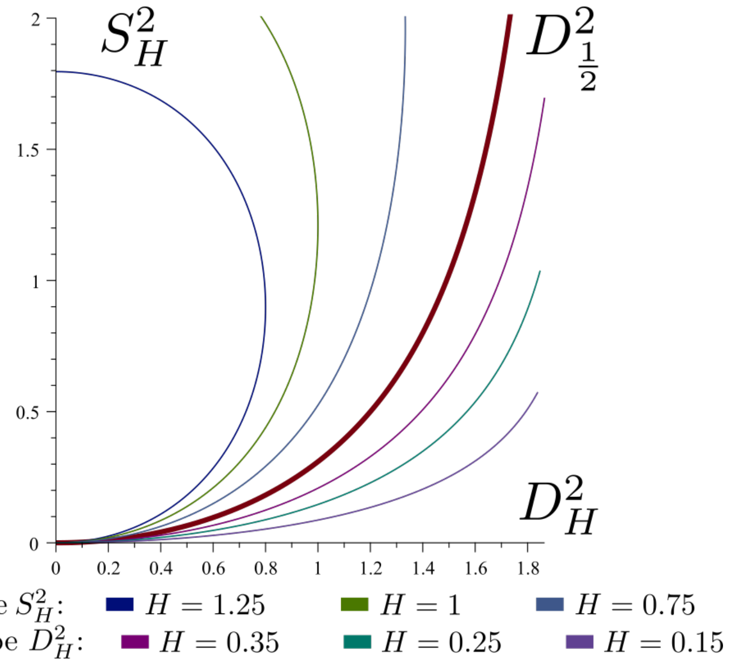
$$\int_0^t \frac{-4\sigma}{(4-\sigma^2)^{3/2}}\,\mathrm{d}\sigma = -2\Big(\frac{2}{\sqrt{4-t^2}} - 1\Big).$$

With the change of coordinates given by $t = 2\tanh(v)$, we get the parametrization X.

Therefore, the complete surface $X(\mathbb{R}^2)$ has constant mean curvature $1/2$ in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, it is the immersed rotationally invariant CMC surface D^2_H in $\mathbb{H}^2 \times \mathbb{R}$ with $4H^2 - 1 = 0$ studied by Abresch and Rosenberg in [AR05].

3.5 Rotationally invariant CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with catenoidal type. Let $H > 0$ be such that $4H^2 - 1 < 0$. Let $v_0 = \operatorname{arccosh}(1/2H)$. Consider the immersion

Figure 5 – **Meridian of CMC disk type surface $D^2_{\frac{1}{2}}$ in $\mathbb{H}^2 \times \mathbb{R}$:**



$X : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$ given by

$$X(u,v) = \Big(4H\cos(u)\cosh(v_0\tanh(v)), 4H\sin(u)\cosh(v_0\tanh(v)), h(v)\Big),$$

where the function h is given by

$$h(v) = -\frac{4H}{\sqrt{1-4H^2}}\operatorname{arctanh}\left(\frac{v_0\tanh(v)}{\sqrt{1-4H^2}}\right).$$

This surface is the analytic continuation of a surface that belongs to the rotationally invariant surfaces $C_{H,-1,0}$ in $\mathbb{H}^2 \times \mathbb{R}$ satisfying $4H^2 - 1 < 0$, whose the parametrizations were described by Domínguez-Vázquez and Manzano [DVMM18]:

$$\widetilde{X}(u,t) = \left(t\cos(u), t\sin(u), \int_{4H}^{t} \frac{-16H}{(4-\sigma^2)\sqrt{\sigma^2-16H^2}}\,\mathrm{d}\sigma\right),$$

for $u \in \mathbb{R}$ and $t \in [4H, 2)$. Indeed, by a direct computation, we get that

$$\int_{4H}^{t} \frac{-16H}{(4-\sigma^2)\sqrt{\sigma^2-16H^2}}\,\mathrm{d}\sigma = -\frac{4H}{\sqrt{1-4H^2}}\operatorname{arctanh}\left(\frac{\sqrt{t^2-16H^2}}{t\sqrt{1-4H^2}}\right).$$

With the change of coordinates given by $t = 4H\cosh(v_0\tanh(v))$, we get the parametrization X.

For each $H \in \mathbb{R}$ such that $4H^2 - 1 < 0$, $X(\mathbb{R}^2)$ is a complete surface with constant mean curvature H in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, it is the immersed CMC annulus rotationally invariant C^2_H in $\mathbb{H}^2 \times \mathbb{R}$ studied by Abresch and Rosenberg in [AR05].

Figure 6 – **Meridians of CMC catenoidal type surfaces C_H^2 in $\mathbb{H}^2 \times \mathbb{R}$:**

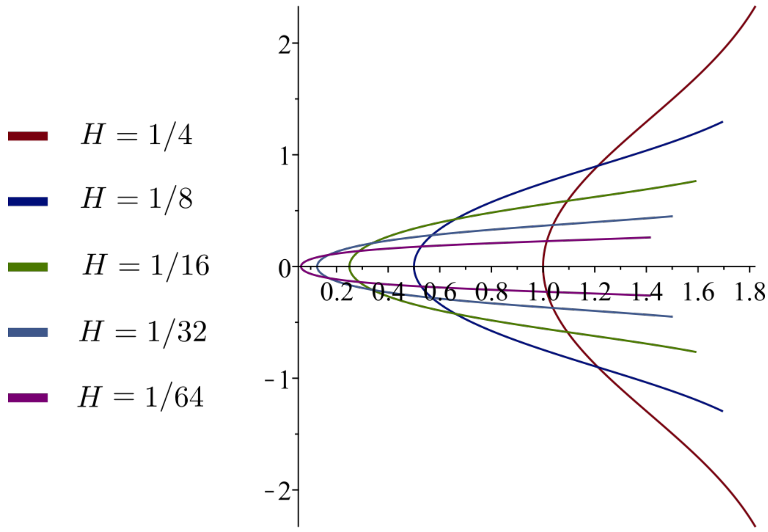
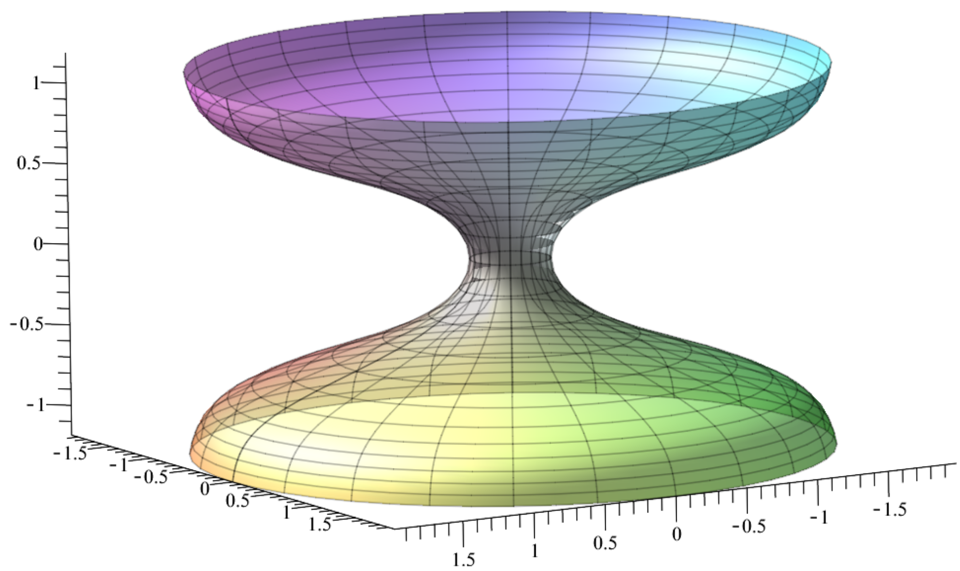


Figure 7 – **CMC catenoidal type surface C_H^2 in $\mathbb{H}^2 \times \mathbb{R}$ with $H = 1/16$:**



3.6 Abresch-Rosenberg-Leite surfaces. In the Abresch-Rosenberg classification, they proved that when $c < 0$ for each H such that $0 < 4H^2 < -c$, there exists a unique H-CMC surface P_H in $\mathbb{H}_c^2 \times \mathbb{R}$ invariant by parabolic isometries such that its Abresch-Rosenberg differential vanishes. Subsequently in [Lei07], M. Leite proved that these surfaces have constant intrinsic curvature $K = 4H^2 + c < 0$. Based on this, we call the surfaces $P_H \subset \mathbb{H}_c^2 \times \mathbb{R}$ of Abresch-Rosenberg-Leite surfaces, abbreviated ARL-surfaces.

Moreover, by the Abresch-Rosenberg classification, we can see that ARL-surfaces are the only ones that the angle function ν is constant with $0 < \nu^2 < 1$. More explicitly, we list the following properties for ARL-surfaces P_H in $\mathbb{H}^2 \times \mathbb{R}$:

- The constant mean curvature H satisfies $0 < 4H^2 < -c$.
- P_H has constant intrinsic curvature $K = 4H^2 + c$.
- The function q vanishes identically on P_H.
- The function ν is constant on P_H, satisfying $\nu^2 = \frac{4H^2+c}{c} \in (0,1)$.
- P_H is foliated by horizontal horocycles of principal curvature $2H$ orthogonally crossed by geodesics in $\mathbb{H}_c^2 \times \mathbb{R}$.

Up to scaling, suppose that $c = -1$ and let be $H > 0$ such that $0 < 4H^2 < 1$. Consider the half-plane model for $\mathbb{H}^2$, and let be

$$\mathbb{H}^2 \times \mathbb{R} = \{(x, y, t) \in \mathbb{R}^3 : y > 0\}$$

endowed with the metric

$$\frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}t^2.$$

We set $B = \sqrt{1 - 4H^2}/2H$ and consider the immersion $X : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$, given by

$$X(u, v) = (u, e^{-Bv}, v).$$

The explicit parametrization above was obtained by Leite in [Lei07]. By standard computations, we get that $P_H = X(\mathbb{R}^2)$ has constant mean curvature H with constant intrinsic curvature $K = 4H^2 - 1$. We note that, when $H \to 0$, P_H is a horizontal surface of $\mathbb{H}^2 \times \mathbb{R}$ and when $H \to 1/2$, P_H is a vertical cylinder of $\mathbb{H}^2 \times \mathbb{R}$.

Figure 8 – **Projection on yz-plane of ARL-surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H = 1/4$:**

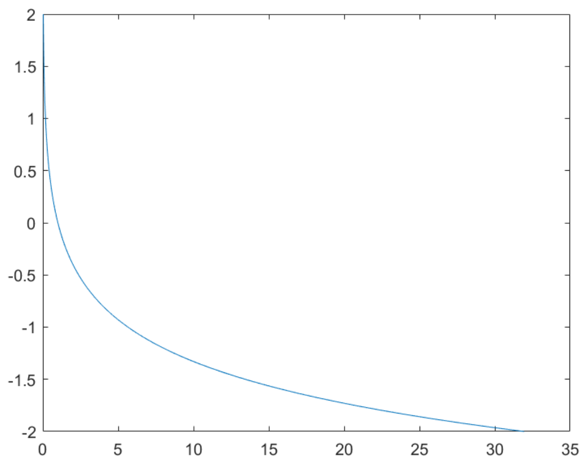
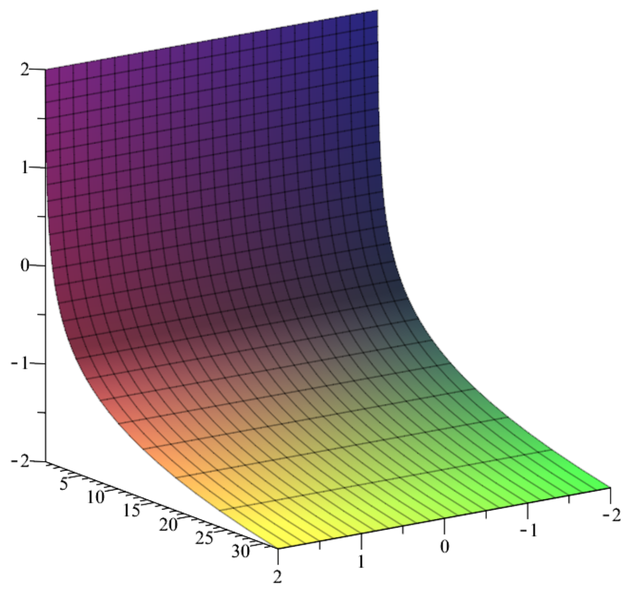


Figure 9 – **ARL-surface in** $\mathbb{H}^2 \times \mathbb{R}$ **with** $H = 1/4$**:**



We remark that between the examples mentioned above we have all examples of H-CMC surfaces in $\mathbb{M}_c^2 \times \mathbb{R}$ with constant angle function. This result was proved by J. Espinar and H. Rosenberg in [ER11].

Theorem 1.3 (Espinar-Rosenberg [ER11])**.** *Let* Σ *be an* H*-constant mean curvature surface in* $\mathbb{M}_c^2 \times \mathbb{R}$*. If the angle function* $\nu : \Sigma \to [-1, 1]$ *is constant then*

- *either* $\nu^2 = 1$*,* $K = c$*,* $H = 0$ *and* Σ *is part of a horizontal surface* $\mathbb{M}_c^2 \times \{a\}$*, for some* $a \in \mathbb{R}$*,*
- *or* $\nu = 0$*,* $K = 0$ *and* Σ *is part of a vertical cylinder* $\gamma \times \mathbb{R}$*, where* $\gamma \subset \mathbb{M}_c^2$ *is a curve of geodesic curvature* $2H$*,*
- *or* $0 < \nu^2 < 1$*,* $K = 4H^2 + c < 0$ *and* Σ *is part of an ARL-surface.*

4. Screw motion surfaces

The screw motion surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ were studied by R. Sa Earp and E. Toubiana in [SET05]. In their work, they proved that given $\ell \in \mathbb{R}$ there exists complete H-CMC screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with pitch ℓ whenever $0 < 4H^2 < 1$. More generally, without assumption about H, they established a *Bour's Lemma* for $\mathbb{H}^2 \times \mathbb{R}$ when $\ell \neq 0$, i.e., natural coordinates on the screw-motion surfaces depending on the arc length σ of a geodesic in the surface, where the induced metric is given by

$$\mathrm{d}\mu^2 = \mathrm{d}\sigma^2 + U(\sigma)\,\mathrm{d}\tau^2,$$

with $U(s)$ a positive smooth function. With this construction, for a given screw motion surface they obtained a (m, ℓ)-parameter family of screw motion surfaces, with $m, \ell \in \mathbb{R}^*$, isometric to the first one. Analogous results are proved also for $\mathbb{S}^2 \times \mathbb{R}$.

We consider below the Riemann sphere for $\mathbb{S}^2$ and the Poincaré disk for $\mathbb{H}^2$. Then $\mathbb{S}^2 \times \mathbb{R}$ is endowed with the metric

$$\frac{4}{(1 + x^2 + y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}t^2,$$

and $\mathbb{H}^2 \times \mathbb{R}$ is endowed with the metric

$$\frac{4}{(1 - (x^2 + y^2))^2}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}t^2.$$

4.1 Minimal screw motion surfaces in $\mathbb{S}^2 \times \mathbb{R}$. Let $\Omega \subset \mathbb{R}^2$ be a domain on $\mathbb{R}^2$ and consider the screw motion immersion $X : \Omega \to \mathbb{S}^2 \times \mathbb{R}$ given by

$$X(u, v) = (\tan(u/2)\cos(v), \tan(u/2)\sin(v), \lambda(u) + \ell v),$$

where the function $\lambda : (a, \pi - a) \to \mathbb{R}$ is given by

$$\lambda(u) = d \int_a^u \frac{\sqrt{\ell^2 + \sin^2(r)}}{\sin(r)\sqrt{\sin^2(r) - d^2}}\, \mathrm{d}r, \text{ with } d = \sin(a) \text{ and } 0 < a < \pi/2.$$

By Sa Earp-Toubiana results, $X(\Omega)$ is a minimal surface. When $\ell = 0$, $X(\Omega)$ is a rotational surface. If $\Omega = \mathbb{R}^2$ and λ vanishes identically, $X(\mathbb{R}^2)$ is a complete minimal helicoid in $\mathbb{S}^2 \times \mathbb{R}$, , that was studied by Rosenberg [Ros02].

4.2 Minimal screw motion in $\mathbb{H}^2 \times \mathbb{R}$. Let $\Omega \subset \mathbb{R}^2$ be a domain on $\mathbb{R}^2$ and consider the screw motion immersion $X : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ given by

$$X(u, v) = (\tanh(u/2)\cos(v), \tanh(u/2)\sin(v), \lambda(u) + \ell v),$$

where the smooth function λ is given by

$$\lambda(u) = d \int_a^u \frac{\sqrt{\ell^2 + \sinh^2(r)}}{\sinh(r)\sqrt{\sinh^2(r) - d^2}}\, \mathrm{d}r, \text{ with } d = \sinh(a).$$

By Sa Earp-Toubiana results, $X(\Omega)$ is a minimal surface. When $\ell = 0$, $X(\Omega)$ is a rotational surface. If $\Sigma = \mathbb{R}^2$ and λ vanishes identically, $X(\mathbb{R}^2)$ is the complete minimal helicoid in $\mathbb{H}^2 \times \mathbb{R}$, that was studied by Nelli and Rosenberg in [NR02].

Analogously, Sa Earp and Toubiana did the same study for H-CMC screw motions surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. To illustrate this case, we present a complete example of screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$.

4.3 Helicoidal surfaces in $\mathbb{H}^2_c \times \mathbb{R}$ satisfying $K = 4H^2 + c < 0$. Let $K, H \in \mathbb{R}$ be such that $H \neq 0$ and $K = 4H^2 - 1 < 0$. Consider the screw motion immersion $X : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$ given by

$$X(\sigma,\tau) = \left(\tanh\frac{\rho(\sigma)}{2}\cos\varphi(\sigma,\tau), \tanh\frac{\rho(\sigma)}{2}\sin\varphi(\sigma,\tau), \lambda(\sigma) + \varphi(\sigma,\tau)\right),$$

where the functions ρ, λ and φ are defined as

$$\rho(\sigma) = \operatorname{arccosh}\left(\frac{\cosh(\sqrt{-K}\sigma)}{\sqrt{-K}}\right),$$
$$\lambda(\sigma) = 2H\sigma + \arctan\left(\frac{e^{2\sqrt{-K}\sigma} + 2K + 1}{4H\sqrt{-K}}\right),$$
$$\varphi(\sigma,\tau) = \sqrt{-K}\tau - \arctan\left(\frac{e^{2\sqrt{-K}\sigma} + 2K + 1}{4H\sqrt{-K}}\right).$$

In Example 2.12, we show that the coordinates (σ, τ) are natural coordinates in the sense of Sa Earp-Toubiana and, by their results, $X(\mathbb{R}^2)$ has constant mean curvature H. We compute the induced metric $\mathrm{d}s^2$ on $X(\mathbb{R}^2)$ as

$$\mathrm{d}s^2 = \mathrm{d}\sigma^2 + \cosh^2(\sqrt{-K}\sigma)\,\mathrm{d}\tau^2,$$

which is complete and has intrinsic curvature $K = 4H^2 - 1$. Moreover, we show that the only H-CMC surfaces in $\mathbb{H}^2_c \times \mathbb{R}$ with constant intrinsic curvature $K \in \mathbb{R}^*$ are this helicoidal surface and the ARL-surface (Theorem 2.14). Hence, this is the only screw motion surface in $\mathbb{H}^2_c \times \mathbb{R}$ with constant mean curvature surface H and constant intrinsic curvature K.

Figure 10 – **Generating curve of Helicoidal surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H = 1/4$, satisfying $K = 4H^2 - 1 < 0$:**

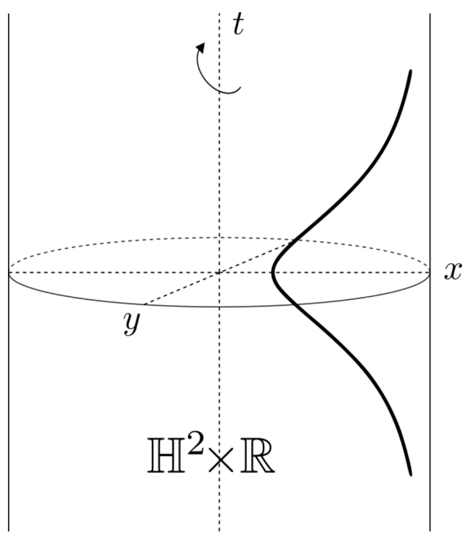
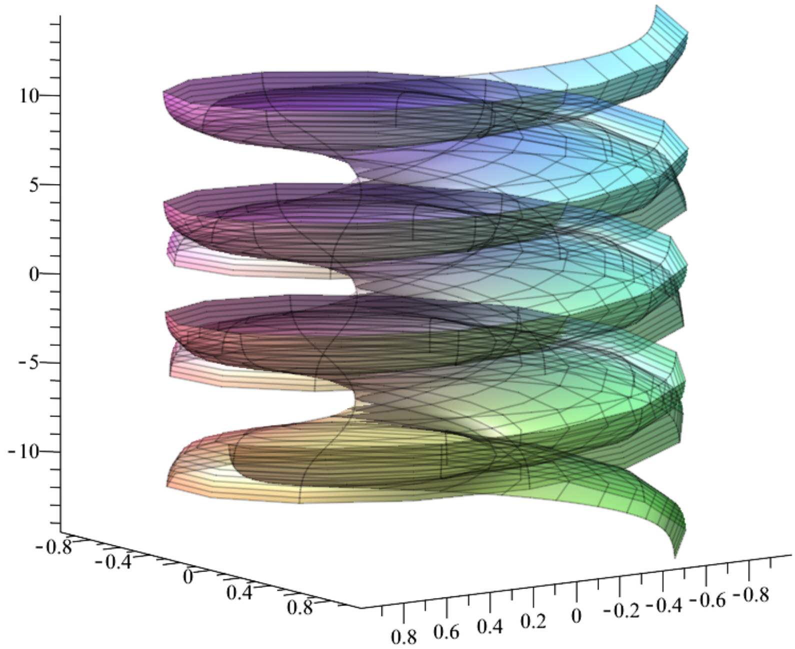


Figure 11 – **Helicoidal surface in** $\mathbb{H}^2 \times \mathbb{R}$ **with** $H = 1/4$**, satisfying** $K = 4H^2 - 1 < 0$**:**



It is important to point out that many others authors engaged themselves to construct constant mean curvatures surfaces in these product spaces using different techniques; and finding new examples of CMC surfaces in $\mathbb{M}^2_c \times \mathbb{R}$. More generally, the research of these kind of surfaces in simply connected Homogeneous 3-manifolds with 4-dimensional isometry group, is a very active subject nowadays.

1.5 Overview of Chapter 2: CMC surfaces with constant intrinsic curvature into $\mathbb{E}(\kappa, \tau)$

In the first part of this thesis, we concentrate on the local classification problem of surfaces with constant mean curvature surfaces with constant intrinsic curvature K in $\mathbb{E}(\kappa, \tau)$. To obtain this classification, we prove the following main result for $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ and later we use it with the Sister correspondence to get the same classification in $\mathbb{E}(\kappa, \tau)$.

Theorem (Theorem 2.14). *Let $H \neq 0$ and Σ be an H-constant mean curvature surface in $\mathbb{M}^2_c \times \mathbb{R}$ with constant intrinsic curvature K. Then one of the following holds:*

- *either $K = 0$ and Σ is part of a vertical cylinder $\gamma \times \mathbb{R}$, where $\gamma \subset \mathbb{M}^2_c$ is a curve of geodesic curvature $2H$,*
- *or $c < 0$, $K = 4H^2 + c < 0$ and Σ is part of either an ARL-surface (Example 3.6) or an helicoidal surface, described in the Example 4.3.*

We notice that for minimal surfaces with constant intrinsic curvature in $\mathbb{M}_c^2 \times \mathbb{R}$, the classification was obtained by F. Torralbo and F. Urbano in [TU15] when $c > 0$, and by B. Daniel [Dan15] when $c < 0$:

Theorem (Torralbo-Urbano [TU15] and Daniel [Dan15]). *Let Σ be a minimal surface in $\mathbb{M}_c^2 \times \mathbb{R}$ with constant intrinsic curvature K. Then one of the following holds:*

- *either $K = 0$ or $K = c$ and it is totally geodesic,*
- *$K = c < 0$ and it is part of an associate surface of the parabolic generalised catenoid in $\mathbb{H}_c^2 \times \mathbb{R}$.*

In the case $\tau \neq 0$, we show that apart from the cases with constant angle functions, we get either a minimal surface in $\mathbb{E}(\kappa, \tau)$ invariant by parabolic isometries with $K = \kappa$ or one of twin helicoidal surfaces in $\mathbb{E}(\kappa, \tau)$ with $K = 4H^2 + \kappa < 0$, both motivated by works of C. Peñafiel in [Pn12, Pn15]:

Theorem (Theorem 2.21). *Let κ and τ be real numbers such that $\tau \neq 0$ and $\kappa - 4\tau^2 \neq 0$, and Σ be an H-constant mean curvature surface in $\mathbb{E}(\kappa, \tau)$ with constant intrinsic curvature K. Then one of the following holds:*

- *either $K = 0$ and Σ is part of a vertical cylinder over a curve $\gamma \subset \mathbb{M}_\kappa^2$ with geodesic curvature $2H$,*
- *or $\kappa < 0$, $K = 4H^2 + \kappa < 0$ and Σ is part of a generalised ARL-surface,*
- *or $\kappa < 0$, $H = 0$, $K = \kappa$ and Σ is part of a minimal surface of Example 2.19,*
- *or $\kappa < 0$, $H \neq 0$, $K = 4H^2 + \kappa < 0$ and Σ is part of one of twin helicoidal surfaces of Example 2.20.*

1.6 Overview of Chapter 3: The Gauss map of minimal surfaces into $\mathbb{S}^2 \times \mathbb{R}$

In the second part of this thesis, we concentrate on the study of a Gauss map for minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$. More specifically, we are interested to know when a such minimal immersion is determined by its conformal structure and its Gauss map. As a main result, we show that any two minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ with the same non-constant Gauss map differ by an ambient isometries of only two types:

Theorem (Theorem 3.17). *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion and g be its non-constant Gauss map. If $\hat{X} : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ is another minimal conformal immersion with* *the same Gauss map of X, then $\hat{X} = f \circ X$, with $f \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ given by $f = (\mathrm{Id}, T)$ or $f = (\mathcal{A}, T)$, where $\mathcal{A}$ denotes the antipodal map on $\mathbb{S}^2$ and T is a translation on $\mathbb{R}$.*

We also get some characterizations of minimal conformal immersions into $\mathbb{S}^2 \times \mathbb{R}$ when we impose some conditions on the Gauss map g. For instance, we prove that there is no minimal conformal immersions into $\mathbb{S}^2 \times \mathbb{R}$ which anti-holomorphic non-constant Gauss map g and we show that only vertical cylinders over geodesics of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$ appears, and then the Gauss map is necessarily constant.

We remark that for the others simply connected Homogeneous 3-manifolds, an exhaustive literature about the relationship of constant mean curvature surfaces and its Gauss map is known. For instance in [MP12, MMPR13], W. Meeks, P. Mira, J. Peréz and A. Ros studied the surfaces with constant mean curvature in a 3-dimensional metric Lie groups and they considered a *left invariant Gauss map* related with the Lie group structure. However, since $\mathbb{S}^2 \times \mathbb{R}$ is the only simply connected homogeneous Riemannian 3-manifold that is not isometric to a Lie group endowed with a left invariant metric, this ambient manifold was excluded from this list.

1.7 Introduction aux résultats de la thèse en français

1.7.1 Chapitre 2 : Surfaces CMC à courbure intrinsèque constante dans $\mathbb{E}(\kappa, \tau)$

Dans le deuxième chapitre de cette thèse, qui est un travail en collaboration avec Benoît Daniel et Feliciano Vitório, on se concentre dans un problème de classification local des surfaces à courbures moyenne H et intrinsèque K constantes dans des variétés homogènes de dimension 3 avec un groupe d'isométries de dimension 4, dénotés par $\mathbb{E}(\kappa, \tau)$.

Pour obtenir la classification local des surfaces à courbures moyenne H et intrinsèque K constantes dans $\mathbb{E}(\kappa, \tau)$, on commence en étudiant le cas des variétés produites $\mathbb{S}^2_c \times \mathbb{R}$ et $\mathbb{H}^2_c \times \mathbb{R}$, et pour celui-là on a cinq étapes fondamentales pour la obtenir. On remarque que pour le cas des surfaces minimales, cette classification a été établie par [TU12, Torralbo et Urbano] et [Dan15, Daniel] dans $\mathbb{S}^2_c \times \mathbb{R}$ et $\mathbb{H}^2_c \times \mathbb{R}$: $K = 0$ ou $K = c$ et la surface est totalement géodésique, ou $c < 0$, $K = c$ et la surface est partie d'une surface associée à un caténoïde parabolique généralisé.

Donnée une surface Riemannienne $(\Sigma, \mathrm{d}s^2)$ orientée simplement connexe dans $\mathbb{M}^2_c \times \mathbb{R}$, où $\mathbb{M}^2_c$ dénote la 2-space forme simplement connexe à courbure constante c, avec $c \neq 0$. En Σ, on définit la fonction angle $\nu : \Sigma \to [-1, 1]$ donnée par

$$\nu = \langle N, \partial_t \rangle,$$

où N dénote le champ de vecteurs normal unitaire à Σ et ∂_t dénote le vecteur unitaire qui donne l'orientation de $\mathbb{R}$; et la fonction hauteur $h : \Sigma \to \mathbb{R}$ définie comme la restriction à Σ de la projection $\Pi_2 : \mathbb{M}_c^2 \times \mathbb{R} \to \mathbb{R}$ dans le factor $\mathbb{R}$, c'est-à-dire,

$$\Pi_2\Big|_\Sigma = h.$$

De plus, on a que $(\partial_t)^\top = \nabla h$.

Dès maintenant, on suppose que Σ est une surface à courbure moyenne constante H (on dénotera par H-CMC) dans $\mathbb{M}_c^2 \times \mathbb{R}$. La première étape pour la classification consiste en trouver un système d'équations aux dérivées partielles qui caractérise la surface donnée. Pour cela, on utilise le Théorème 1.1 pour montre qu'une immersion isométrique H-CMC dans $\mathbb{M}_c^2 \times \mathbb{R}$ est déterminée par la métrique et les fonctions angle et hauteur de la surface, si telles satisfont un système d'équations aux dérivées partielles.

Théorème (Théorème 2.2). *Soit Σ une surface H-CMC dans $\mathbb{M}_c^2 \times \mathbb{R}$. Alors, la fonction angle $\nu : \Sigma \to [-1, 1]$ et la fonction hauteur $h : \Sigma \to \mathbb{R}$ de Σ satisfont le système suivant*

$$\begin{aligned} \|\nabla\nu + H\nabla h\|^2 &= (H^2 - K + c\nu^2)(1 - \nu^2), \\ \Delta\nu &= (2K - c(1 + \nu^2) - 4H^2)\nu, \\ \|\nabla h\|^2 &= 1 - \nu^2, \\ \Delta h &= 2H\nu, \end{aligned}$$

où K dénote la courbure intrinsèque de Σ. Réciproquement, soit $(\Sigma, \mathrm{d}s^2)$ une surface Riemannienne orientée simplement connexe, à courbure K. Suppose que $\nu : \Sigma \to (-1, 1)$ et $h : \Sigma \to \mathbb{R}$ sont fonctions lisses en satisfaisant le système en haut composé par quatre équations aux dérivées partielles. Alors il existe une immersion isométrique $f : \Sigma \to \mathbb{M}_c^2 \times \mathbb{R}$ H-CMC telle que ν et h sont les fonctions angle et hauteur de Σ, respectivement. De plus, l'immersion est unique à moins de isométries horizontales de $\mathbb{M}_c^2 \times \mathbb{R}$ que préservent les orientations de $\mathbb{M}_c^2$ et $\mathbb{R}$.

On remarque que pour le cas des surfaces minimales dans $\mathbb{M}_c^2 \times \mathbb{R}$, Daniel a montré que les équations

$$\begin{aligned} \|\nabla\nu\|^2 &= (-K + c\nu^2)(1 - \nu^2) \\ \Delta\nu &= (2K - c(1 + \nu^2))\nu \end{aligned}$$

sont conditions nécessaires et suffisantes pour obtenir le même résultat en haut pour $H = 0$.

A partir du Théorème 2.2, on peut conclure la classification des surfaces H-CMC dans $\mathbb{M}_c^2 \times \mathbb{R}$ avec la fonction angle constante. Il est importante remarquer que ce résultat a été démontre avant par [ER11, Espinar et Rosenberg], où toutes les surfaces H-CMC dans $\mathbb{E}(\kappa, \tau)$ avec fonction angle constante sont classifiées.

Corollaire (Corollaire 2.8 [ER11]). *Soit Σ une surface H-CMC dans $\mathbb{M}_c^2 \times \mathbb{R}$. Si la fonction angle $\nu : \Sigma \to [-1, 1]$ est constante alors*

- *ou $\nu^2 = 1$, $K = c$, $H = 0$ et Σ est une partie d'une surface horizontale $\mathbb{M}_c^2 \times \{a\}$, pour $a \in \mathbb{R}$ fixé.*
- *ou $\nu = 0$, $K = 0$ et Σ est une partie d'un cylindre vertical $\gamma \times \mathbb{R}$, où $\gamma \subset \mathbb{M}_c^2$ a courbure géodésique $2H$.*
- *ou $0 < \nu^2 < 1$, $K = 4H^2 + c < 0$ et Σ est une partie d'une surface ARL.*

Dans la deuxième étape pour la classification, on considère la fonction $\phi : \Sigma \to \mathbb{R}$, donnée par $\phi = \nu + Hh$. En utilisant le Théorème 2.2, on voit que la fonction ϕ satisfait le système :

$$\begin{aligned} \|\nabla \phi\|^2 &= (1 - \nu^2)(H^2 - K + c\nu^2), \\ \Delta \phi &= \Big(2K - c(1 + \nu^2) - 2H^2\Big)\nu. \end{aligned}$$

Donc, par la formule de Weitzenböck-Bochner, on obtient une cinquième équation qui dépend de la métrique et des fonctions angle et hauteur de la surface.

Proposition (Proposition 2.11). *Soient ν et h les fonctions angle et auteur, respectivement, d'une surface Σ H-CMC dans Σ dans $\mathbb{M}_c^2 \times \mathbb{R}$. Alors ν et h satisfont*

$$\begin{aligned} 0 = &(H^2 - K + c\nu^2)\Delta K + \|\nabla K\|^2 - 6c\nu\langle \nabla K, \nabla \nu\rangle - 2Hc\nu\langle \nabla K, \nabla h\rangle \\ &+6Hc(H^2 - K - c\nu^2)\langle \nabla \nu, \nabla h\rangle + 4H^2c\nu^2(H^2 - K - 2c + 3c\nu^2) \\ &-4(H^2 - K + c\nu^2)(K - c - H^2)(K + 2c\nu^2). \end{aligned}$$

Dans la troisième étape pour la classification, on considère la fonction $q : \Sigma \to \mathbb{R}$, donnée par

$$q = 2Hc\langle \nabla \nu, \nabla h\rangle + 4H^2(H^2 - K + c\nu^2) + 2H^2c(1 - \nu^2) + \frac{c^2}{4}(1 - \nu^2)^2.$$

Cette fonction q a été introduit par [ER11, Espinar et Rosenberg]. En fait, q est une normalization de la norme carré de la différentielle de Abresch-Rosenberg. Comme la différentielle de Abresch-Rosenberg est holomorphe en Σ, alors les zéros de q sont tous isolés ou q s'annule en Σ. De plus, Espinar et Rosenberg ont montré que cette fonction satisfait

$$\Delta \log q = 4K$$

i.e.,

$$4Kq^2 = q\Delta q - \|\nabla q\|^2 \tag{1.1}$$

est la sixième équation satisfait par la métrique et les fonctions angle et hauteur de la surface.

Dans la quatrième étape, on suppose que Σ est une surface H-CMC dans $\mathbb{M}^2_c \times \mathbb{R}$ à courbure intrinsèque K constante avec $H \neq 0$. Alors, on transforme l'équation (1.1) dans un polynôme en ν, a partir des équations obtenus dans le Théorème 2.2 et Proposition 2.11.

Lemme (Lemme 2.13). *Soient $H \neq 0$, Σ une surface H-CMC dans $\mathbb{M}^2_c \times \mathbb{R}$ à courbure intrinsèque constante K et $U \subseteq \Sigma$ un ouvert tel que $(H^2 - K - c\nu^2) \neq 0$. Si $K \neq 4H^2 + c$, alors il existe un polynôme pair g de degré 18, tel que $g \circ \nu = 0$ en U.*

A partir du Lemme 2.13, on a que si $K \neq 4H^2 + c$ alors la fonction ν est constante en Σ et donc la classification est donné par le Corollaire 2.8. Alors, il suffit d'étudier le cas $K = 4H^2 + c$ et fonction angle non-constante.

La cinquième étape, avant de démontrer la classification, consiste en étudier le cas des surfaces H-CMC dans $\mathbb{M}^2_c \times \mathbb{R}$ à courbure intrinsèque $K = 4H^2 + c$ et fonction angle non-constante. Dans ce sens, nous présentons un nouvel exemple de surface à courbures moyenne et intrinsèque constantes dans $\mathbb{H}^2_c \times \mathbb{R}$. En effet, pour $K, H \in \mathbb{R}$ tels que $H \neq 0$ et $K = 4H^2 + c < 0$, on montre qu'il existe une immersion $X : \mathbb{R}^2 \to \mathbb{H}^2_c \times \mathbb{R}$, invariante par mouvement hélicoïdal telle que $X(\mathbb{R}^2)$ à courbure moyenne H et intrinsèque K constantes. Ce nouvel example est basé dans le travail de [SET05, Sa Earp et Toubiana] sur les surfaces hélicoïdales H-CMC dans $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. Sa fonction angle est non-constante.

Pour démontrer la classification (Théorème 2.14), on suppose que Σ est une surface H-CMC dans $\mathbb{M}^2 \times \mathbb{R}$ à courbure intrinsèque constante $K = 4H^2 + c$ et fonction angle non-constante, avec $H \neq 0$. Par le Théorème 2.2 et la Proposition 2.11, on a

$$\|\nabla\nu\|^2 = -\frac{1}{c}(4H^2 + c - c\nu^2)^2,$$
$$\Delta\nu = (4H^2 + c - c\nu^2)\nu.$$

Alors, $c < 0$ et par l'inégalité de Cauchy-Schwarz on a $4H^2 + c - c\nu^2 < 0$. Donc, $K = 4H^2 + c < 0$ et $|\nu| < \sqrt{\frac{K}{c}}$. Dès que ν satisfait le système en haut, alors ν est isoparamétrique. Donc, il existe un paramétrage local (x_1, x_2) en Σ, tel que $\nu(x_1, x_2) = x_1$, et

$$\mathrm{d}s^2 = \frac{1}{F(x_1)^2}\,\mathrm{d}x_1^2 + G(x_1)^2\,\mathrm{d}x_2^2,$$

avec

$$F(x_1) = \frac{1}{\sqrt{-c}}(cx_1^2 - c - 4H^2) \text{ et } G(x_1) = \frac{\sqrt{-c}}{(cx_1^2 - c - 4H^2)^{1/2}}.$$

Si $\{\partial_{x_1}, \partial_{x_2}\}$ sont les champs coordonnés de (x_1, x_2) en Σ, on a

$$\nabla\nu = F^2\partial_{x_1},$$
$$\nabla h = \frac{\partial h}{\partial x_1}F^2\partial_{x_1} + \frac{\partial h}{\partial x_2}\frac{1}{G^2}\partial_{x_2}.$$

Par le Théorème 2.2, on obtient que

$$h(x_1, x_2) = \frac{2H}{\sqrt{cK}}\,\mathrm{arctanh}\left(\sqrt{\frac{c}{K}}x_1\right) + \varepsilon x_2, \text{ pour } \varepsilon = \pm 1.$$

En utilisant un changement de coordonnés, on montre que

$$\begin{aligned} \mathrm{d}s^2 &= -c\,\mathrm{d}\sigma^2 - c\cosh^2(\sqrt{cK}\sigma)\,\mathrm{d}\tau^2 \\ \nu(\sigma) &= \sqrt{\frac{K}{c}}\tanh(\sqrt{cK}\sigma) \\ h(\sigma,\tau) &= 2H\sigma + \sqrt{-K}\tau, \end{aligned}$$

i.e., à échelle près, Σ est la surface hélicoïdale du Exemple 2.12. De cette façon, on peut énoncer la classification local des surfaces à courbures moyenne H et intrinsèque K constantes dans $\mathbb{M}^2_c \times \mathbb{R}$:

Théorème (Théorème 2.14). *Soient $H \neq 0$ et Σ une surface H-CMC dans $\mathbb{M}^2_c \times \mathbb{R}$ à courbure intrinsèque constante K. Alors :*

- *ou $K = 0$ et Σ est une partie d'un cylindre vertical $\gamma \times \mathbb{R}$, où $\gamma \subset \mathbb{M}^2_c$ est une courbe de courbure géodésique $2H$,*
- *ou $c < 0$, $K = 4H^2 + c < 0$ et Σ est une partie d'une surface ARL ou de la surface hélicoïdale du Exemple 2.12.*

Comme une application du Théorème 2.14 et de la Correspondence des surfaces soeurs (Théorème 1.2) établie par [Dan07, Daniel], on classifie les surfaces à courbures moyenne H et intrinsèque K constantes dans $\mathbb{E}(\kappa, \tau)$, pour $\kappa - 4\tau^2 \neq 0$.

Théorème (Théorème 2.21). *Soient κ et τ nombres réels tels que $\tau \neq 0$ et $\kappa - 4\tau^2 \neq 0$, et Σ une surface H-CMC dans $\mathbb{E}(\kappa, \tau)$ à courbure intrinsèque K constante. Alors :*

- *ou $K = 0$ et Σ est une partie d'un cylindre vertical sur une courbe $\gamma \subset \mathbb{M}^2_\kappa$ à courbure géodésique $2H$;*
- *ou $\kappa < 0$, $K = 4H^2 + \kappa < 0$ et Σ est une partie d'une surface ARL généralisée ;*
- *ou $\kappa < 0$, $H = 0$, $K = \kappa$ et Σ est une partie de la surface minimale du Exemple 2.19 ;*
- *ou $\kappa < 0$, $H \neq 0$, $K = 4H^2 + \kappa < 0$ et Σ est une partie d'une des surfaces hélicoïdales jumelles du Exemple 2.20.*

1.7.2 Chapitre 3 : L'application de Gauss des surfaces minimales dans $\mathbb{S}^2 \times \mathbb{R}$

Dans le troisième chapitre de cette thèse, on se concentre à étudier les surfaces minimales dans $\mathbb{S}^2 \times \mathbb{R}$ a partir d'une nouvelle application de Gauss qu'on définit pour les surfaces dans $\mathbb{S}^2 \times \mathbb{R}$. On utilise un modèle de $\mathbb{S}^2 \times \mathbb{R}$ isométrique à $\mathbb{R}^3 \setminus \{0\}$ muni d'une métrique conformément equivalente à la métrique Euclidienne de $\mathbb{R}^3$, et donc on définit l'application de Gauss pour surfaces dans ce modèle comme elle est définie dans $\mathbb{R}^3$. Alors, on s'intéresse pour la question de quand une immersion minimale est déterminée par leur structure conforme et leur application de Gauss.

Soient $\mathbb{S}^2 \subset \mathbb{R}^3$ la sphère de rayon 1 munie de la métrique induite par la métrique Euclidienne, et $\mathbb{S}^2 \times \mathbb{R}$ la variété produite munie de la métrique produite (nommé modèle standard de $\mathbb{S}^2 \times \mathbb{R}$). Soient $\Pi_1 : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2$ et $\Pi_2 : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ les projetions dans $\mathbb{S}^2$ et $\mathbb{R}$, respectivement.

On considère $\mathbb{R}^3 \setminus \{0\}$ muni de la métrique

$$\mathrm{d}\mu^2 = \frac{1}{x_1^2 + x_2^2 + x_3^2}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2).$$

Alors, l'application $\phi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}$, donnée par $\phi(y,t) = e^t y$ est une isométrie globale. En plus, on considère les projections $\pi_j = \Pi_j \circ \phi^{-1}$, pour $j = 1, 2$, c'est-à-dire,

$$z \mapsto \pi_1(x) = \frac{x}{|x|} \text{ et } z \mapsto \pi_2(x) = \log|x|.$$

Dès maintenant, on considère

$$\mathbb{S}^2 \times \mathbb{R} \text{ comme } (\mathbb{R}^3 \setminus \{0\}, \mathrm{d}\mu^2).$$

Le groupe d'isométries de $\mathbb{S}^2 \times \mathbb{R}$ est un groupe de dimension 4 isomorphe à $\mathrm{Iso}(\mathbb{S}^2) \times \mathrm{Iso}(\mathbb{R})$. Considère $f = (M, T) \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R}) \simeq \mathrm{Iso}(\mathbb{S}^2) \times \mathrm{Iso}(\mathbb{R})$. Si f préserve l'orientation de $\mathbb{R}$, alors $T(t) = s + t$, et dont $\phi^{-1} \circ f \circ \phi$ correspond à isométrie f dans notre modèle, donnée par

$$x \in \mathbb{R}^3 \setminus \{0\} \mapsto e^s M(x),$$

avec $s \in \mathbb{R}$ et $M \in O(3)$, où $O(3)$ dénote le groupe Orthogonal de dimension 3. Si f inverse l'orientation de $\mathbb{R}$, alors $T(t) = s - t$, alors $\phi^{-1} \circ f \circ \phi$ correspond à isométrie f dans notre modèle, donnée par

$$x \in \mathbb{R}^3 \setminus \{0\} \mapsto \frac{e^s}{|x|^2} M(x),$$

avec $s \in \mathbb{R}$ et $M \in O(3)$. On dit que $f = (M, T) \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ est une translation vertical de $\mathbb{S}^2 \times \mathbb{R}$ quand $M = \mathrm{Id}$ est l'identité en $\mathbb{S}^2$ et f préserve l'orientation de $\mathbb{R}$, c'est-à-dire, $f = (\mathrm{Id}, T)$ où $T(t) = s + t$ est une translation en $\mathbb{R}$, pour $s \in \mathbb{R}$ fixé.

Dans notre modèle de $\mathbb{S}^2 \times \mathbb{R}$, on considère $\Pi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{C}$ comme la projection dans $\mathbb{R}^2$ (identifié avec $\mathbb{C}$) et $\pi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ la projection dans $\mathbb{R}$, respectivement, données par

$$\Pi(x_1, x_2, x_3) = x_1 + ix_2 \ \text{ et } \ \pi(x_1, x_2, x_3) = x_3.$$

Soit Σ une surface Riemannienne orientée, $z = u + iv$ une coordonné conforme en Σ. Considère $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ une immersion conforme. On dénote par $N : \Sigma \to \mathrm{U}(\mathbb{S}^2 \times \mathbb{R})$ le champ de vecteurs normal unitaire à Σ, où $\mathrm{U}(\mathbb{S}^2 \times \mathbb{R})$ est le fibré tangent unitaire de $\mathbb{S}^2 \times \mathbb{R}$. On définit $F = \Pi \circ X : \Sigma \to \mathbb{C}$ et $h = \pi \circ X : \Sigma \to \mathbb{R}$, alors l'immersion conforme X est écrit comme $X = (F, h)$.

Si $\{\partial_{x_j}\}_{j=1,2,3}$ est la base orthonormée de $\mathbb{R}^3$, on considère $E_j(x) = |x|\partial_{x_j}$, pour $x \in \mathbb{S}^2 \times \mathbb{R}$ et $j = 1, 2, 3$. Alors $\{E_j(x)\}_{j=1,2,3}$ est une base orthonormée de $\mathbb{S}^2 \times \mathbb{R}$. Donc, pour chaque $x \in \mathbb{S}^2 \times \mathbb{R}$, on identifie la 2-sphère unitaire de $T_x(\mathbb{S}^2 \times \mathbb{R})$ avec la 2-sphère unitaire de $\mathbb{R}^3$, i.e., on identifie $v_1E_1 + v_2E_2 + v_3E_3 \in T_x(\mathbb{S}^2 \times \mathbb{R})$ unitaire avec $v_1\partial_{x_1} + v_2\partial_{x_2} + v_3\partial_{x_3} \in \mathbb{S}^2 \subset \mathbb{R}^3$. Ainsi, on considère $N(z) \in \mathbb{S}^2 \subset \mathbb{R}^3$ à cet identification près.

Définition (Définition 3.4)**.** L'application de Gauss de X est définie par $g = \varphi \circ N : \Sigma \to \bar{\mathbb{C}}$, où $\varphi : \mathbb{S}^2 \to \bar{\mathbb{C}}$ est la projection stéréographique par rapport au pôle sud, c'est-à-dire, si $N = N_1E_1 + N_2E_2 + N_3E_3$ alors

$$g = \frac{N_1 + iN_2}{1 + N_3} : \Sigma \to \bar{\mathbb{C}}.$$

On remarque que pour l'autre choix de N, on change g par $\tilde{g} = -1/\bar{g}$.

On suppose maintenant que $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ est une immersion conforme minimale. Alors, les premiers examples de surfaces minimales dans $\mathbb{S}^2 \times \mathbb{R}$ sont les surfaces horizontales et les cylindres verticaux sur des géodésiques de $\mathbb{S}^2$. Les 2-sphères totalement géodésiques $\mathbb{S}^2 \times \{a\}$ dans le modèle standard de $\mathbb{S}^2 \times \mathbb{R}$, pour $a \in \mathbb{R}$, sont des 2-sphères de rayon e^a centrées à l'origine de $\mathbb{R}^3$ dans notre modèle de $\mathbb{S}^2 \times \mathbb{R}$. Les cylindres verticaux $\gamma \times \mathbb{R}$ sur des géodésiques $\mathbb{S}^2$ dans le modèle standard de $\mathbb{S}^2 \times \mathbb{R}$ sont des plans en passant par l'origine de $\mathbb{R}^3$ dans notre modèle de $\mathbb{S}^2 \times \mathbb{R}$. Dans ce dernier cas, g est constante.

Un premier résultat dans l'étude des surfaces minimales dans $\mathbb{S}^2 \times \mathbb{R}$ a partir de cet application de Gauss, c'est une caractérisation des cylindres verticaux $\gamma \times \mathbb{R}$ sur des géodésiques $\mathbb{S}^2$ dans $\mathbb{S}^2 \times \mathbb{R}$.

Proposition (Proposition 3.5)**.** *Soit $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ une immersion conforme minimale avec l'application de Gauss g. Alors g est constante si, et seulement si, $X(\Sigma)$ est une partie d'un cylindre vertical sur une géodésique de $\mathbb{S}^2$ dans $\mathbb{S}^2 \times \mathbb{R}$.*

Pour une immersion conforme minimale $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$, on considère les applications $p : \Sigma \to \bar{\mathbb{C}}$ et $r : \Sigma \to \mathbb{R}$ définies, respectivement, par $p(z) = \varphi \circ \pi_1 \circ X(z)$ et $r(z) = \pi_2 \circ X(z)$, i.e.,

$$z \mapsto p(z) = \frac{F(z)}{|X(z)| + h(z)} \quad \text{et} \quad z \mapsto r(z) = \log |X(z)|.$$

On note que p est une application harmonique dans $(\bar{\mathbb{C}}, 4/(1+|w|^2)^2 |\,\mathrm{d}w|^2)$ et r est une fonction harmonique dans $\mathbb{R}$, une fois que X est minimale. Alors, a partir des définitions et propriétés de g, p et r, on montre un ensemble de quatre équations dans les deux résultats suivants :

Proposition (Proposition 3.6)**.** *Soit $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ une immersion conforme minimale. Soit $g : \Sigma \to \bar{\mathbb{C}}$ l'application de Gauss de X et suppose que $g \neq \infty$. Alors*

$$\begin{aligned} &(1+|p|^2)p_{z\bar{z}} - 2\bar{p}p_z p_{\bar{z}} = 0, \\ &(\bar{g}-\bar{p})^2 p_z + (1+\bar{g}p)^2 \bar{p}_z = 0, \\ &(1+|p|^2)^2 g_z = (1+g\bar{p})^2 p_z + (g-p)^2 \bar{p}_z, \end{aligned}$$

quand $p \neq \infty$.

Proposition (Proposition 3.8)**.** *L'application de Gauss g et l'application p satisfont*

$$\begin{aligned} \Big(|g-p|^2 - |1+\bar{g}p|^2\Big)(1+|g|^2)g_{z\bar{z}} + 2(g-p)(1+g\bar{p})|g_z|^2 \\ + 2\Big(\bar{g}|1+\bar{g}p|^2 - (\bar{g}-\bar{p})(1+|g|^2)\Big)g_z g_{\bar{z}} = 0 \end{aligned}$$

quand $g \neq \infty$.

En vertu des équations établies dans les propositions précédentes, on cherche classifier les immersions conformes minimales dans $\mathbb{S}^2 \times \mathbb{R}$ a partir de leurs applications de Gauss. Dans cette direction, on montre que sur hypothèse d'une application de Gauss non-constante et anti-holomorphe, cette condition est une obstruction naturelle pour l'existence d'une immersion conforme minimale.

Proposition (Proposition 3.9)**.** *Il n'existe pas immersion conforme minimale $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ telle que l'application de Gauss g soit non-constante et anti-holomorphe.*

De plus, on montre que l'application de Gauss non-constante holomorphe est une caractérisation des 2-sphères totalement géodésiques dans $\mathbb{S}^2 \times \mathbb{R}$.

Proposition (Proposition 3.12)**.** *Soit $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ une immersion conforme minimale avec l'application de Gauss g non constante et holomorphe. Alors, $X(\Sigma)$ est une partie d'une sphère totalement géodésique dans $\mathbb{S}^2 \times \mathbb{R}$.*

Comme avec en $\mathbb{R}^3$, on montre, par le lemme suivant, que les zéros de la courbure K de la métrique induite par X sont les points singuliers de l'application de Gauss, quand X est une immersion conforme minimale.

Lemme (Lemme 3.13). *La courbure K de la métrique induite par X est donnée par*

$$K = \frac{(|g-p|^2 - |1+\bar{g}p|^2)^2}{(1+|g|^2)^2(1+|p|^2)^2|g_z|^2}(|g_z|^2 - |\bar{g}_z|^2),$$

quand $g \neq \infty$. En particulier, $K = 0$ dans les points singuliers de g.

Avec cette lemme, on caractérise les immersions conformes minimales telles que leurs applications de Gauss sont singulières. En plus, cette condition implique g constante.

Proposition (Proposition 3.14). *Soient $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ une immersion conforme minimale et g l'application de Gauss de X. Si g est singulière alors $X(\Sigma)$ est une partie d'un cylindre vertical sur une géodésique de $\mathbb{S}^2$ dans $\mathbb{S}^2 \times \mathbb{R}$. En particulier, g est constante.*

Une dernière caractérisation sont pour les immersions conformes minimales telles que leurs applications de Gauss sont non-constantes et harmoniques.

Corollaire (Corollaire 3.15). *Soient $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ une immersion conforme minimale et g l'application de Gauss de X. Si g est une application non-constante harmonique, alors $X(\Sigma)$ est une partie d'une sphère totalement géodésique dans $\mathbb{S}^2 \times \mathbb{R}$.*

Comme principal résultat du chapitre 3, on montre qu'une surface minimale dans $\mathbb{S}^2 \times \mathbb{R}$ est déterminée par leur structure conforme et leur application de Gauss, à deux types d'isométries près :

Théorème (Théorème 3.17). *Soient $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ une immersion conforme minimale et g l'application de Gauss non-constante de X. Si $\hat{X} : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ est une autre immersion conforme minimale avec la même application de Gauss g de X, alors $\hat{X} = f \circ X$, avec $f \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ donnée par $f = (\mathrm{Id}, T)$ ou $f = (\mathcal{A}, T)$, où $\mathcal{A}$ dénote l'application antipodal en $\mathbb{S}^2$ et T est une translation en $\mathbb{R}$.*

L'idée de démonstration de ce résultat commence par montrer que, donnée une application g non-constante, il existe au maximum deux applications p telles que ces satisfont la Proposition 3.8. En effet, note que g est forcement régulière, sinon g est constante. Donc, soit $U \subset \Sigma$ ensemble ouvert tel que $|g| \neq 1$. On réécrit l'équation de la Proposition 3.8 comme

$$A(1-|p|^2) + Bp + C\bar{p} = 0, \tag{1.2}$$

où

$$\begin{aligned}
A &= -(1-|g|^4)g_{z\bar{z}} + 2g|g_z|^2 - 2|g|^2\bar{g}g_z g_{\bar{z}},\\
B &= -2\bar{g}(1+|g|^2)g_{z\bar{z}} - 2|g_z|^2 + 2\bar{g}^2 g_z g_{\bar{z}},\\
C &= -2g(1+|g|^2)g_{z\bar{z}} + 2g^2|g_z|^2 + 2(1+2|g|^2)g_z g_{\bar{z}}.
\end{aligned}$$

On observe que si $A = 0$ en $z_0 \in U$, alors $p(z_0) = 0$, quand $g_z(z_0) \neq 0$. Par contre, si $A \neq 0$ en $z_0 \in U$, alors en z_0, on a que

$$(\bar{A}B - A\bar{C})p^2 + (|C|^2 - |B|^2)p - (A\bar{B} - \bar{A}C) = 0$$

a deux solutions distinctes pour $p(z_0)$, quand $g_z(z_0) \neq 0$. Ainsi, l'équation (1.2) a au maximum deux solutions distinctes pour chaque $p(z_0)$ en U, quand g est non-constante.

Soit $\hat{X} : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ une immersion conforme minimale avec la même application de Gauss g de X. On utilise le symbole ˆ pour se référer aux objets associés à $\hat{X}$.

Si $p = \hat{p}$, alors par la définition de p on a $X/|X| = \hat{X}/|\hat{X}|$, i.e., $\hat{X} = q(z)X$, où $q : \Sigma \to \mathbb{R}^+$ est donnée par $q(z) = |\hat{X}(z)|/|X(z)|$. Par contre, on montre que $(\log q)_{\bar{z}} = 0$. Alors, q est holomorphe et donc $\hat{X} = f \circ X$, où $f = (\mathrm{Id}, T) \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ est une translation vertical de $\mathbb{S}^2 \times \mathbb{R}$ avec $T(t) = t + \log q_0$.

Si $p \neq \hat{p}$, on affirme que $\hat{p} = -1/\bar{p}$. En effet, on voit que $-1/\bar{p}$ satisfait l'équation de la Proposition 3.8, et donc $\hat{p} = -1/\bar{p}$. De plus, $\hat{p}\bar{p} = -1$ implique que $\varphi^{-1}(\hat{p})$ et $\varphi^{-1}(p)$ sont des points antipodaux en $\mathbb{S}^2$. Alors, on a $\hat{X} = q(z)\mathcal{A} \circ X$, où $\mathcal{A}$ est l'application antipodal en $\mathbb{S}^2$ et $q(z) = |\hat{X}(z)|/|X(z)|$. Par contre, on montre que $(\log q)_{\bar{z}} = 0$. Alors, q est holomorphe et donc $\hat{X} = f \circ X$, où $f = (\mathcal{A}, T) \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ avec $T(t) = t + \log q_0$ une translation en $\mathbb{R}$.

Chapter 2

Constant mean curvature surfaces with constant intrinsic curvature into $\mathbb{E}(\kappa, \tau)$

This chapter is contained in the joint work with Benoît Daniel and Feliciano Vitório:

Constant mean curvature isometric immersions into $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$ *and related results*

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Abstract

In this chapter, we study constant mean curvature isometric immersions into $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ and we classify these isometric immersions when the surface has constant intrinsic curvature. As a application, we use the first classification with the sister surface correspondence to classify the constant mean curvature surfaces with constant intrinsic curvature in the 3-dimensional homogenous manifolds $\mathbb{E}(\kappa, \tau)$. It is worthwhile to point out that these classifications provide new examples.

2.1 Introduction

This chapter deals with the classification of constant mean curvature surfaces and parallel mean curvature surfaces with constant intrinsic curvature in some ambient manifolds. In $\mathbb{R}^3$, it is a classical result that an H-constant mean curvature surface with constant intrinsic curvature K is either part of a plane, when $H = 0$, or part of a right circular cylinder with $K = 0$ or part of a round sphere of radius $1/H$ with $K = H^2$, when $H \neq 0$ (Levi-Civita [LC37]).

In the 3-space forms $\mathbb{M}^3_c$, surfaces with H and K constants are isoparametric surfaces, that is, the two principal curvatures are constant. In this direction, the first results of local classification in $\mathbb{S}^3_c$ and $\mathbb{H}^3_c$ started with E. Cartan [Car38] and since then many related results have been obtained. For example, minimal surfaces with constant intrinsic curvature in 3-dimensional space forms $\mathbb{M}^3_c$ are either totally geodesic with $K = c$ or a part of the Clifford torus with $K = 0$ in $\mathbb{S}^3_c$ (Chen [Che72], Lawson [Law69]).

In codimension 2, for 4-dimensional space forms $\mathbb{M}^4_c$, minimal surfaces with constant intrinsic curvature are either totally geodesics with $K = c$, or a part of Clifford torus with $K = 0$ in a totally geodesics $\mathbb{S}^3_c$ of $\mathbb{S}^4_c$, or a part of Veronese surfaces with $K = c/3$ in $\mathbb{S}^4_c$ (Kenmotsu [Ken83]). For parallel mean curvature surfaces with constant intrinsic curvature in the same ambient manifold either $K = 0$ or $K = H^2 + c$. If $c \geq 0$, then the surface is either a part of product of circles with $K = 0$ or a part of 2-sphere with $K = H^2 + c$, where H denotes the norm of the mean curvature vector (Hoffman [Hof73]). When the ambient manifold is a complex 2-dimensional space form ($\mathbb{C}^2$, $\mathbb{CH}^2(c)$, $\mathbb{CP}^2(c)$), parallel mean curvature surfaces with constant intrinsic curvature are either a product of circles, or a cylinder, or a round sphere, or a slant surface or one of the Hirakawa examples (Hirakawa [Hir06]).

The aim of the work is to generalise these results to some other ambient homogeneous manifolds.

First, we classify the constant mean curvature surfaces with constant intrinsic curvature in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ (Theorem 2.14). The minimal case has been treated by the first author in [Dan15]: either $K = 0$ or $K = c$ and the surface is totally geodesic, or $K = c < 0$ and the surface is part of an associate surface of the parabolic generalised catenoid in $\mathbb{H}^2_c \times \mathbb{R}$. In this chapter we consider the non-minimal case. Our study is based on a system of partial differential equations satisfied by the metric, the *angle* and *height functions* of the surface (which holds, more generally, without the condition of constant intrinsic curvature).

We show that apart from the vertical cylinders with $K = 0$, there are exactly two examples of H-constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with constant intrinsic curvature. Both satisfy the relation $K = 4H^2 + c < 0$. The first one is a surface found in the classification of surfaces whose Abresch-Rosenberg differential vanishes [AR04] and the second one is a helicoidal surface that appears in the study of screw motion surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, due to Sa Earp and Toubiana [SET05]; however, it was not explicit there that this surface has both constant intrinsic and mean curvatures.

In particular, this classification provides an example of two CMC isometric immersions of the same Riemannian surface with a same mean curvature and which are not congruent up to an isometric reparametrization; this answers a question by Torralbo and Urbano [TU12, Remark 1].

As a corollary (Theorem 2.21), using the sister surface correspondence [Dan07], we obtain a classification of CMC surfaces with constant intrinsic curvature in the homogeneous 3-manifolds $\mathbb{E}(\kappa, \tau)$, for $\tau \neq 0$ and $\kappa - 4\tau^2 \neq 0$. We show that apart from the cases with constant angle functions, we get either a minimal surface in $\mathbb{E}(\kappa, \tau)$ invariant by parabolic isometries with $K = \kappa$ or one of twin helicoidal surfaces in $\mathbb{E}(\kappa, \tau)$ with $K = 4H^2 + \kappa < 0$. The first one is a surface found by Peñafiel in [Pn12] and the second ones are motivated by the study of screw motion surfaces in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ in [Pn15] of the same author.

This work is organized as follows: In Section 2.2, we fix some notations and we recall previous results about surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. Moreover, from the study the angle and height functions and compatibility equations obtained in [Dan07, Theorem 3.3], we establish necessary and sufficient conditions for a Riemannian surface to be isometrically immersed as a constant mean curvature surface in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ (Theorem 2.2). After, we derive an additional order-1 equation using the classical Weitzenböck-Bochner formula.

In Section 2.3, we suppose that the metric of the surface has constant intrinsic curvature and we consider a function associated to the Abresch-Rosenberg differential. This function satisfies a new partial differential equation that can be reduced to an order-0 equation by the previous results. We get the classification for the H-constant mean curvature surfaces in $\mathbb{M}^2_c \times \mathbb{R}$ with constant intrinsic curvature (Theorem 2.14).

Finally, in Section 2.4, we use the sister surface correspondence [Dan07, Theorem 5.2] to get the classification of H-constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$ with constant intrinsic curvature, for $\tau \neq 0$.

2.2 The angle and height functions

We fix real numbers c and H such that $c \neq 0$. Let $\mathbb{M}^2_c$ be the simply connected Riemannian manifold of constant intrinsic curvature c, that is, $\mathbb{M}^2_c = \mathbb{S}^2_c$ is the 2-sphere for $c > 0$ and $\mathbb{M}^2_c = \mathbb{H}^2_c$ is the hyperbolic plane for $c < 0$. For simplicity, we sometimes use the normalization $c = \pm 1$. In this case, $\mathbb{S}^2_1 = \mathbb{S}^2$ and $\mathbb{H}^2_{-1} = \mathbb{H}^2$.

Theorem 2.1 (Daniel, [Dan09]). *Let $(\Sigma, \mathrm{d}s^2)$ be an oriented simply connected Riemannian surface, ∇ its Riemannian connection and K be the intrinsic curvature of $\mathrm{d}s^2$. Let $S : \mathrm{T}\Sigma \to \mathrm{T}\Sigma$ be a field of symmetric operators, $T \in \mathcal{X}(\Sigma)$ and $\nu : \Sigma \to [-1, 1]$ be a smooth function. Then* *there is an isometric immersion $f:\Sigma\to\mathbb{M}^2_c\times\mathbb{R}$ such that the shape operator with respect to the normal N associated to f is*

$$\mathrm{d}f\circ S\circ\mathrm{d}f^{-1}$$

and such that

$$\partial_t=\mathrm{d}f(T)+\nu N$$

if and only if the 4-uple $(\mathrm{d}s^2,S,T,\nu)$ satisfies the following equations on Σ:

$$K=\det S+c\nu^2, \tag{C1}$$
$$\nabla_X SY-\nabla_Y SX-S[X,Y]=c\nu\Big(\langle Y,T\rangle X-\langle X,T\rangle Y\Big), \tag{C2}$$
$$\nabla_X T=\nu SX, \tag{C3}$$
$$\mathrm{d}\nu(X)+\langle SX,T\rangle=0, \tag{C4}$$
$$\|T\|^2+\nu^2=1. \tag{C5}$$

Moreover the immersion is unique up to an isometry of $\mathbb{M}^2_c\times\mathbb{R}$ that preserve both orientations of $\mathbb{M}^2_c$ and $\mathbb{R}$.

We will say that $(\mathrm{d}s^2,S,T,\nu)$ are Gauss-Codazzi data of the immersion f and that ν is its angle function. The height function of f is the map $h=p\circ f$ where $p:\mathbb{M}^2_c\times\mathbb{R}\to\mathbb{R}$ is the projection onto the factor $\mathbb{R}$.

As a first result, we derive from Theorem 2.1 a system of compatibility equations for H-constant mean curvature surfaces in $\mathbb{M}^2_c\times\mathbb{R}$ in terms of the angle and height functions. We will say that an isometry of $\mathbb{M}^2_c\times\mathbb{R}$ is horizontal if it is the identity on the factor $\mathbb{R}$.

Theorem 2.2. *Let Σ be an H-constant mean curvature surface in $\mathbb{M}^2_c\times\mathbb{R}$. Then the angle function $\nu:\Sigma\to[-1,1]$ and the height function $h:\Sigma\to\mathbb{R}$ of Σ satisfy the following system*

$$\|\nabla\nu+H\nabla h\|^2=(H^2-K+c\nu^2)(1-\nu^2), \tag{2.1}$$
$$\Delta\nu=(2K-c(1+\nu^2)-4H^2)\nu, \tag{2.2}$$
$$\|\nabla h\|^2=1-\nu^2, \tag{2.3}$$
$$\Delta h=2H\nu, \tag{2.4}$$

where K denotes the intrinsic curvature of Σ.

Conversely, let $(\Sigma,\mathrm{d}s^2)$ be an oriented simply connected Riemannian surface, with curvature K. Assume that $\nu:\Sigma\to(-1,1)$ and $h:\Sigma\to\mathbb{R}$ are smooth functions satisfying the system of four partial differential equations above. Then there is a H-constant mean curvature isometric immersion $f:\Sigma\to\mathbb{M}^2_c\times\mathbb{R}$ such that ν and h are the angle function and the height function of Σ, respectively. Moreover the immersion is unique up to horizontal isometries of $\mathbb{M}^2_c\times\mathbb{R}$ that preserves the orientation of $\mathbb{M}^2_c$.

Proof. Let Σ be an H-constant mean curvature surface in $\mathbb{M}_c^2 \times \mathbb{R}$. Let $(\mathrm{d}s^2, S, T, \nu)$ be the Gauss-Codazzi data on Σ and J the rotation of angle $\pi/2$ on $\mathrm{T}\Sigma$. By the symmetry of S and $\operatorname{tr} S = 2H$, a straightforward computation shows that $SJ + JS = 2HJ$, and the compatibility equation (C1) implies that $\nabla\nu = -ST$. Away from the points where $T = 0$, we consider the orthonormal frame $\{T/\|T\|, JT/\|T\|\}$. Then S has the form

$$S = \frac{1}{\|T\|^2}\begin{pmatrix} -\,\mathrm{d}\nu(T) & -\,\mathrm{d}\nu(JT) \\ -\,\mathrm{d}\nu(JT) & 2H\|T\|^2 + \mathrm{d}\nu(T) \end{pmatrix}.$$

From the height function definition, we have that

$$\mathrm{d}h(X) = \langle X, \partial_t\rangle, \text{ for every } X \in \mathrm{T}\Sigma,$$

that is, $T = \nabla h$ and then

$$\det S = -\frac{1}{\|T\|^2}\Big(\|\nabla\nu\|^2 + 2H\langle\nabla\nu, \nabla h\rangle\Big).$$

Hence, the equations (C1) and (C5) imply that

$$(K - c\nu^2)(1 - \nu^2) = -\|\nabla\nu\|^2 - 2H\langle\nabla\nu, \nabla h\rangle,$$

i.e., (2.1) holds. When $\nu^2 = 1$, the right and left sides of (2.1) equals to zero, then for this case (2.1) also holds.

Let L be the Jacobi operator of Σ, given by $L = \Delta + \|S\|^2 + \overline{\mathrm{Ric}}(N)$, where $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\mathbb{M}_c^2 \times \mathbb{R}$. Since $\|S\|^2 = 4H^2 - 2\det S$ and $\overline{\mathrm{Ric}}(N) = c(1 - \nu^2)$, the equation (C1) implies that

$$L = \Delta - 2K + c(1 + \nu^2) + 4H^2.$$

On the other hand, as ∂_t is a Killing field and $\nu = \langle\partial_t, N\rangle$ then $L\nu = 0$, so (2.2) holds.

Since $\nabla h = T$, equation (2.3) follows directly from equation (C5). Moreover, equation (C3) gives that $\nabla_X\nabla h = \nu SX$, which concludes (2.4) by the definition of divergence, finishing the first assertion. Note that also by equation (C3), if $\nu^2 \neq 1$, T satisfies the following equation for every $X \in \mathrm{T}\Sigma$:

$$\nabla_X T = -\frac{\nu}{1-\nu^2}\,\mathrm{d}\nu(X)T - \frac{\nu}{1-\nu^2}\Big(\,\mathrm{d}\nu(JX) + 2H\langle T, JX\rangle\Big)JT. \tag{2.5}$$

We now prove the second part of the theorem. Let $(\Sigma, \mathrm{d}s^2)$ be a simply connected Riemannian surface, $\nu : \Sigma \to (-1, 1)$ and $h : \Sigma \to \mathbb{R}$ smooth functions on Σ satisfying equations (2.1), (2.2), (2.3) and (2.4).

Claim: The vector field $T = \nabla h \in \mathrm{T}\Sigma$ satisfies equation (2.5).

By equation (2.3) and symmetry of the Hessian of h, that is $\langle\nabla_X\nabla h, Y\rangle = \langle\nabla_Y\nabla h, X\rangle$ for every $X, Y \in \mathrm{T}\Sigma$, we have

$$\nabla_{\nabla h}\nabla h = -\nu\nabla\nu.$$

Since $\nu^2 < 1$, considering the orthonormal frame $\{\nabla h/\|\nabla h\|, J\nabla h/\|\nabla h\|\}$, by the symmetry of Hessian of h we have

$$\nabla_X\nabla h = -\frac{\nu}{1-\nu^2}\,\mathrm{d}\nu(X)\nabla h + \frac{1}{1-\nu^2}\langle\nabla_{J\nabla h}\nabla h, X\rangle J\nabla h$$

for every $X \in \mathrm{T}\Sigma$. On the other hand, again by the symmetry of Hessian of h we have

$$\nabla_{JX}\nabla h + J\nabla_X\nabla h = \Delta h JX, \text{ for every } X \in \mathrm{T}\Sigma.$$

Since equation (2.4) holds, we obtain that

$$\nabla_X\nabla h = -\frac{\nu}{1-\nu^2}\,\mathrm{d}\nu(X)\nabla h - \frac{\nu}{1-\nu^2}\Big(\mathrm{d}\nu(JX) + 2H\langle\nabla h, JX\rangle\Big)J\nabla h,$$

that is, the vector field ∇h satisfies equation (2.5).

Since $T = \nabla h$ does not vanish on Σ, because $\nu^2 < 1$, there is a unique symmetric operator $S : \mathrm{T}\Sigma \to \mathrm{T}\Sigma$ with constant trace $2H$, such that $ST = -\nabla\nu$.

We affirm that the 4-uple $(\mathrm{d}s^2, S, T, \nu)$ is Gauss-Codazzi data on Σ. To prove this, it is sufficient to show the equations (C1) and (C2). This is because (C4) is follows of definition of S, (C5) is the same of (2.3) and the (C3) follows of (2.5), when we write any $X \in \mathrm{T}\Sigma$ in the basis $\{\nabla h/\|\nabla h\|, J\nabla h/\|\nabla h\|\}$ and use $\nabla\nu = -ST$.

In a previous calculation, from the fact that $\nabla\nu = -ST$ and S is a symmetric operator with $\operatorname{tr} S = 2H$, we have shown that

$$\det S = -\frac{1}{\|T\|^2}\Big(\|\nabla\nu\|^2 + 2H\,\mathrm{d}\nu(T)\Big).$$

Using equations (2.1) and (2.3) we obtain the Gauss equation (C1), that is,

$$K = \det S + c\nu^2.$$

To show equation (C2), it is sufficient to verify for $X = T$ and $Y = JT$. Since the rotation J of angle $\pi/2$ on $\mathrm{T}\Sigma$ commutes with ∇_X, for every $X \in \mathrm{T}\Sigma$, and $SJ + JS = 2HJ$, by the equation (C3) we get

$$[T, JT] = 2\nu(JS - HJ)T.$$

By the symmetry of S and $ST = -\nabla\nu$ we have

$$\langle S[T, JT], T\rangle = 2H\nu\,\mathrm{d}\nu(JT).$$

Furthermore, again since the rotation J of angle $\pi/2$ on $\mathrm{T}\Sigma$ commutes with ∇_X, $SJ + JS = 2HJ$, $\nabla_T T = -\nu\nabla\nu$ and $ST = -\nabla\nu$, we get

$$\nabla_T SJT - \nabla_{JT} ST = -2H\nu J\nabla\nu + J\nabla_T\nabla\nu + \nabla_{JT}\nabla\nu,$$

and, by the symmetry of Hessian of ν and since $J^2 = -\mathrm{I}$, we have

$$\langle \nabla_T SJT - \nabla_{JT} ST, T\rangle = 2H\nu\,\mathrm{d}\nu(JT),$$

then

$$\langle \nabla_T SJT - \nabla_{JT} ST - S[T, JT], T\rangle = 0.$$

On the other hand, again by the symmetry of S, $SJ + JS = 2HJ$, $ST = -\nabla\nu$ and the fact that J is an isometry, we have

$$\langle S[T, JT], JT\rangle = -2\nu\Big(\|\nabla\nu\|^2 + 3H\,\mathrm{d}\nu(T) + 2H^2\|T\|^2\Big).$$

Moreover, since J is an isometry

$$\langle \nabla_T SJT - \nabla_{JT} ST, JT\rangle = \|T\|^2\Delta\nu - 2H\nu\,\mathrm{d}\nu(T).$$

Then

$$\langle \nabla_T SJT - \nabla_{JT} ST - S[T, JT], JT\rangle = \|T\|^2\Big(\Delta\nu + 4H^2\nu\Big) + 2\nu\Big(\|\nabla\nu\|^2 + 2H\,\mathrm{d}\nu(T)\Big)$$

and by equations (2.1), (2.2) and (2.3), we obtain

$$\langle \nabla_T SJT - \nabla_{JT} ST - S[T, JT], JT\rangle = -c\nu(1-\nu^2)^2.$$

On the other hand, computing $c\nu\Big(\langle Y, T\rangle X - \langle X, T\rangle Y\Big)$ for $X = T$ and $Y = JT$, respectively, we get

$$c\nu\Big\langle \langle T, JT\rangle T - \langle T, T\rangle JT, T\Big\rangle = 0$$

and

$$c\nu\Big\langle \langle T, JT\rangle T - \langle T, T\rangle JT, JT\Big\rangle = -c\nu(1-\nu^2)^2.$$

Thus we showed the Codazzi equation (C2) and then $(\mathrm{d}s^2, S, T, \nu)$ are Gauss-Codazzi data on U. Since the operator S and the vector field T are determined in a unique way, the uniqueness follows from Theorem 2.1 and the fact that the height function is prescribed. □

Remark 2.3. Assuming $\nu^2 < 1$ for the converse, it is possible to rewrite Theorem 2.2 changing the gradient ∇h by vector field T if the function ν and the a vector field T

satisfy

$$\begin{aligned} \|\nabla\nu + HT\|^2 &= (H^2 - K + c\nu^2)(1-\nu^2), \\ \Delta\nu &= (2K - c(1+\nu^2) - 4H^2)\nu, \\ \|T\|^2 &= 1-\nu^2, \\ \operatorname{div} T &= 2H\nu, \\ \langle \nabla_T T, JT\rangle &= \langle \nabla_{JT} T, T\rangle. \end{aligned}$$

This is because the last equation is equivalent to the fact that the $1-$form $\langle T, \cdot\rangle$ is closed and this is a necessary condition to show that T satisfies equation (2.5).

Remark 2.4. The minimal case has been studied by the first author in [Dan15]. He obtained a slightly different result, involving only ν and equations (2.1) and (2.2).

Remark 2.5. For a fixed H, the pair $(\nu, \nabla h)$ satisfies (2.1), (2.2), (2.3) and (2.4) for H if and only if the pair $(-\nu, -\nabla h)$ does for H and the pairs $(\nu, -\nabla h)$ and $(-\nu, \nabla h)$ do for $-H$. The isometric immersions corresponding to the pairs $(\nu, \nabla h)$ and $(-\nu, -\nabla h)$ are the same up to a π-rotation around a horizontal geodesic of $\mathbb{M}^2_c \times \mathbb{R}$. For the pair $(\nu, -\nabla h)$ (respectively, $(-\nu, \nabla h)$), its correspondent isometric immersion is the same of the isometric immersion correspond to $(\nu, \nabla h)$ up to an isometry of $\mathbb{M}^2_c \times \mathbb{R}$ that preserves the orientation of $\mathbb{M}^2_c$ and reverses the orientation of $\mathbb{R}$ (respectively, reverses the orientation of $\mathbb{M}^2_c$ and preserves the orientation of $\mathbb{R}$), see also [Dan09, Proposition 3.8].

From Theorem 2.2 follows the next result that characterizes H-constant mean surfaces in $\mathbb{M}^2_c \times \mathbb{R}$ with constant angle function. This result was already proved by Espinar and Rosenberg in [ER11].

Before that, we consider the smooth function $q : \Sigma \to \mathbb{R}$ introduced by Espinar and Rosenberg, in [ER11]. This function q is a normalization of the squared norm of the Abresch-Rosenberg differential; it will play an important role in the case of H-constant mean curvature surface Σ in $\mathbb{M}^2_c \times \mathbb{R}$ with constant intrinsic curvature in the next section.

In [ER11, Lemma 2.2], the authors show that

$$\|\nabla\nu\|^2 = \frac{4H^2 + c - c\nu^2}{4c}(4(H^2 - K + c\nu^2) + c(1-\nu^2)) - \frac{q}{c}.$$

Combining this relation and Theorem 2.2, we can see that q satisfies

$$q = 2Hc\langle \nabla\nu, \nabla h\rangle + 4H^2(H^2 - K + c\nu^2) + 2H^2c(1-\nu^2) + \frac{c^2}{4}(1-\nu^2)^2. \tag{2.6}$$

Example 2.6 (Horizontal surfaces and vertical cylinders)**.** The simplest examples of H-CMC surfaces in $\mathbb{M}^2_c \times \mathbb{R}$ are the horizontal surfaces and the vertical cylinders over curves of constant geodesic curvature. Given $a \in \mathbb{R}$, then $\mathbb{M}^2_c \times \{a\}$ is a totally geodesic surface in $\mathbb{M}^2_c \times \mathbb{R}$ (then $H = 0$) with intrinsic curvature $K = c$. The height function is constant and, since the normal vector N of $\mathbb{M}^2_c \times \{a\}$ is parallel to ∂_t and both are unitary vectors, the angle function ν satisfies $\nu^2 = 1$, in particular it is constant. Let $\gamma \subset \mathbb{M}^2_c$ be a curve with constant geodesic curvature k. Then $\gamma \times \mathbb{R}$ is a $(k/2)$-CMC surface in $\mathbb{M}^2_c \times \mathbb{R}$ with intrinsic curvature $K = 0$. Since the normal vector N of $\gamma \times \mathbb{R}$ is orthogonal to ∂_t, then the angle function vanishes and the gradient of the height function is a principal direction of Σ by equation (C4) and $2H = k$.

Example 2.7 (ARL-surfaces). Abresch and Rosenberg [AR04] classified CMC surfaces with vanishing Abresch-Rosenberg differential. In particular, when $c < 0$ they proved that, for each H such that $0 < 4H^2 < -c$, there exists a unique H-CMC surface P_H in $\mathbb{H}^2_c \times \mathbb{R}$ invariant by parabolic isometries such that its Abresch-Rosenberg differential vanishes. Moreover, Leite [Lei07] proved that P_H has constant intrinsic curvature $K = 4H^2 + c$. We note that, in the limit, P_H is a horizontal surface of $\mathbb{H}^2 \times \mathbb{R}$ when $H \to 0$ and a vertical cylinder of $\mathbb{H}^2 \times \mathbb{R}$ when $4H^2 \to -c$.

In this work, we say that P_H is an Abresch-Rosenberg-Leite surface, abbreviated ARL-surface. In the Abresch-Rosenberg classification of surfaces with vanishing Abresch-Rosenberg differential, we can also see that ARL-surfaces are the only ones that the angle function ν is constant with $0 < \nu^2 < 1$. More explicitly, the ARL-surfaces P_H have the following properties:

- The constant mean curvature H satisfies $0 < 4H^2 < -c$.
- P_H has constant intrinsic curvature $K = 4H^2 + c$.
- The function q vanishes identically on P_H.
- The function ν is constant on P_H, satisfying $\nu^2 = \frac{4H^2+c}{c} \in (0, 1)$.
- P_H is foliated by horizontal horocycles of principal curvature $2H$ orthogonally crossed by geodesics in $\mathbb{H}^2_c \times \mathbb{R}$.

Corollary 2.8 ([ER11]). *Let Σ be an H-constant mean curvature surface in $\mathbb{M}^2_c \times \mathbb{R}$. If the angle function $\nu : \Sigma \to [-1, 1]$ is constant then*

- *either $\nu^2 = 1$, $K = c$, $H = 0$ and Σ is part of a horizontal surface $\mathbb{M}^2_c \times \{a\}$, for some $a \in \mathbb{R}$,*
- *or $\nu = 0$, $K = 0$ and Σ is part of a vertical cylinder $\gamma \times \mathbb{R}$, where $\gamma \subset \mathbb{M}^2_c$ is a curve of geodesic curvature $2H$,*
- *or $0 < \nu^2 < 1$, $K = 4H^2 + c < 0$ and Σ is part of an ARL-surface.*

Proof. Assume that $\nu^2 = 1$. Then (2.2) implies $K = c + 2H^2$ and (2.3) implies that $T = 0$, that is, the height function h is constant on Σ, and so $H = 0$, by (2.4). Hence Σ is part of a horizontal surface $\mathbb{M}^2_c \times \{a\}$ of curvature $K = c$, for some $a \in \mathbb{R}$.

Assume that $\nu = 0$. Then equation (2.1) implies that $K = 0$. On the other hand, $\partial_t = T$, that is, the vertical vector is tangent to Σ and it is a principal direction of Σ, by the equation (C4). Then the other principal direction has eigenvalue $2H$, that is, Σ is part of a vertical cylinder $\gamma \times \mathbb{R}$, where $\gamma \subset \mathbb{M}^2_c$ has geodesic curvature $2H$.

Assume that $0 < \nu^2 < 1$. Then equations (2.1) and (2.3) imply $K = c\nu^2$. By (2.2), we obtain the following equation

$$c(1-\nu^2) + 4H^2 = 0. \tag{2.7}$$

Note that there is no solution to (2.7) if $H = 0$ or $c > 0$. Then we have $c < 0$ and $K = 4H^2 + c$. Moreover $c < K < 0$ because $0 < \nu^2 < 1$. Since ν constant and $K = c\nu^2$, again by equations (2.1) and (2.7) we get that the function q vanishes on Σ; therefore Σ is part of an ARL-surface. □

The following result will be useful for future computations.

Corollary 2.9. *Let Σ be an H-constant mean curvature surface in $\mathbb{M}^2_c \times \mathbb{R}$. The function $\phi = \nu + Hh$ is constant if and only if Σ is either part of a minimal horizontal surface $\mathbb{M}^2_c \times \{a\}$, for some $a \in \mathbb{R}$, or part of a minimal vertical cylinder $\gamma \times \mathbb{R}$, where γ is a geodesic of $\mathbb{M}^2_c$.*

Proof. If $\phi = \nu + Hh$ is a constant function on Σ then $(H^2 - K + c\nu^2)(1-\nu^2) = 0$ and $\nu\Big(2K - c(1+\nu^2) - 2H^2\Big) = 0$. Then from (2.1), (2.2) and (2.4) we get a system of two algebraic equations on ν:

$$\begin{aligned} &(H^2 - K + c\nu^2)(1-\nu^2) = 0,\\ &\nu\Big(-2(H^2 - K + c\nu^2) - c(1-\nu^2)\Big) = 0. \end{aligned}$$

At the points where $0 < \nu^2 < 1$, the first equation of the system above implies $H^2 - K + c\nu^2 = 0$; then replacing this in the second one we have $-c(1-\nu^2) = 0$, which cannot occur because $c \neq 0$. Then $|\nu|$ does not take values in the interval $(0,1)$. Then, by connectedness and continuity, either $\nu^2 = 1$ or $\nu = 0$ and then Σ is part of a horizontal surface in $\mathbb{M}^2_c \times \mathbb{R}$ or Σ is part of a vertical cylinder $\gamma \times \mathbb{R}$, where $\gamma \subset \mathbb{M}^2_c$, respectively.

For the minimality of Σ, note that if $\nu^2 = 1$, Corollary 2.8 implies that $K = c$ and $H = 0$. If $\nu = 0$, Corollary 2.8 implies that $K = 0$, and by the first equation of the system above we get $H = 0$.

Conversely, if Σ is a minimal horizontal surface, then the unit normal N is in the same direction of ∂_t. If Σ is a minimal vertical cylinder over a geodesic of $\mathbb{M}^2_c$, then the unit normal N is orthogonal to ∂_t. Then both cases imply that ν is constant on Σ. Since Σ is minimal surface, ϕ is a constant function on Σ. □

Let Σ be an H-constant mean curvature surface in $\mathbb{M}_c^2 \times \mathbb{R}$ with Gauss-Codazzi data $(\mathrm{d}s^2, S, T, \nu)$. Consider a local orthonormal frame $\{e_1, e_2\}$ on Σ such that $Je_1 = e_2$ where J is the rotation of angle $\pi/2$ on $\mathrm{T}\Sigma$.

Given a smooth function $f : \Sigma \to \mathbb{R}$ on Σ, we will set

$$f_i = \langle \nabla f, e_i \rangle \ \text{ and } \ f_{ij} = (\nabla^2 f)(e_i, e_j),$$

where $\nabla^2 f$ is the symmetric Hessian 2-tensor of f. Then if ν and h are the angle function and height function of Σ, respectively, setting $\phi = \nu + Hh$ we found the following system of two partial differential equations from Theorem 2.2:

$$\phi_1^2 + \phi_2^2 = -(1-\nu^2)(K - c\nu^2 - H^2), \tag{2.8}$$

$$\phi_{11} + \phi_{22} = \Big(2K - c(1+\nu^2) - 2H^2\Big)\nu. \tag{2.9}$$

Lemma 2.10. *The function ϕ satisfies*

$$\begin{aligned} 2\phi_{12}\|\nabla\phi\|^2 = {} & (1-\nu^2)(6c\nu\phi_1\phi_2 - K_1\phi_2 - K_2\phi_1) \\ & + 2H\nu(H^2 - K - c + 2c\nu^2)(h_1\phi_2 + h_2\phi_1) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} (\phi_{11} - \phi_{22})\|\nabla\phi\|^2 = {} & (1-\nu^2)\Big(3c\nu(\phi_1^2 - \phi_2^2) - K_1\phi_1 + K_2\phi_2\Big) \\ & + 2H\nu(H^2 - K - c + 2c\nu^2)(h_1\phi_1 - h_2\phi_2). \end{aligned} \tag{2.11}$$

Proof. Differentiating (2.8) with respect to e_1 and e_2, we have

$$2(\phi_1\phi_{11} + \phi_2\phi_{12}) = 2\Big(K - H^2 + c(1-2\nu^2)\Big)\nu\nu_1 - (1-\nu^2)K_1, \tag{2.12}$$

$$2(\phi_1\phi_{12} + \phi_2\phi_{22}) = 2\Big(K - H^2 + c(1-2\nu^2)\Big)\nu\nu_2 - (1-\nu^2)K_2. \tag{2.13}$$

Making ϕ_2(2.12) $+$ ϕ_1(2.13) and using (2.9) we obtain (2.10). In an analogous way, making ϕ_1(2.12) $-$ ϕ_2(2.13) we find

$$\begin{aligned} 2(\phi_1^2\phi_{11} - \phi_2^2\phi_{22}) = {} & 2\Big(K - H^2 + c(1-2\nu^2)\Big)\nu(\phi_1^2 - \phi_2^2) - (1-\nu^2)(K_1\phi_1 - K_2\phi_2) \\ & + 2H\nu\Big(K - H^2 + c(1-2\nu^2)\Big)(\phi_1 h_1 - \phi_2 h_2). \end{aligned}$$

Since

$$2(\phi_1^2\phi_{11} - \phi_2^2\phi_{22}) = (\Delta\phi)(\phi_1^2 - \phi_2^2) + (\phi_{11} - \phi_{22})\|\nabla\phi\|^2,$$

we get (2.11), using (2.8). □

Proposition 2.11. *The functions* ν *and* h *satisfy*

$$\begin{aligned} 0 = (H^2 - K + c\nu^2)\Delta K + \|\nabla K\|^2 - 6c\nu\langle\nabla K, \nabla\nu\rangle - 2Hc\nu\langle\nabla K, \nabla h\rangle \\ +6Hc(H^2 - K - c\nu^2)\langle\nabla\nu, \nabla h\rangle + 4H^2c\nu^2(H^2 - K - 2c + 3c\nu^2) \\ -4(H^2 - K + c\nu^2)(K - c - H^2)(K + 2c\nu^2). \end{aligned} \tag{2.14}$$

Proof. If ϕ is constant on a non empty open set of Σ, then by analyticity ϕ is constant on Σ. By Corollary 2.9, ν and K are constant functions on Σ, $H = 0$ and $H^2 - K + c\nu^2 = 0$, so (2.14) holds.

From now on, we assume that ϕ is not a constant function on Σ. By analyticity, it is sufficient to prove equation (2.14) on a non empty set $U \subset \Sigma$ on which $\nabla\phi$ does not vanish. Restricting U if necessary, we can consider an orthonormal frame $\{e_1, e_2\}$ on U and assume all the previous notations. The classical Weitzenböck-Bochner formula reads as

$$\frac{1}{2}\Delta\|\nabla\phi\|^2 = \langle\nabla\phi, \nabla\Delta\phi\rangle + \|\nabla^2\phi\|^2 + K\|\nabla\phi\|^2.$$

Note that the Hessian term can be written

$$\|\nabla^2\phi\|^2 = \frac{1}{2}(\Delta\phi)^2 + \frac{1}{2}(\phi_{11} - \phi_{22})^2 + 2\phi_{12}^2.$$

Then by Lemma 2.10 we have

$$\begin{aligned} 2\|\nabla\phi\|^4\|\nabla^2\phi\|^2 = {} & \|\nabla\phi\|^4(\Delta\phi)^2 \\ & + (1 - \nu^2)^2\|\nabla\phi\|^2\Big\{9c^2\nu^2\|\nabla\phi\|^2 + \|\nabla K\|^2 - 6c\nu\langle\nabla K, \nabla\phi\rangle\Big\} \\ & - 4H\nu(1 - \nu^2)\|\nabla\phi\|^2(H^2 - K - c + 2c\nu^2)\langle\nabla K, \nabla h\rangle \\ & + 12cH\nu^2(1 - \nu^2)\|\nabla\phi\|^2(H^2 - K - c + 2c\nu^2)\langle\nabla\phi, \nabla h\rangle \\ & + 4H^2\nu^2(1 - \nu^2)\|\nabla\phi\|^2(H^2 - K - c + 2c\nu^2)^2. \end{aligned}$$

Dividing the expression above by $(1 - \nu^2)\|\nabla\phi\|^2$ and using (2.4) we get

$$\begin{aligned} 2(H^2 - K + c\nu^2)\|\nabla^2\phi\|^2 = {} & (H^2 - K + c\nu^2)(\Delta\phi)^2 \\ & + (1 - \nu^2)\Big\{9c^2\nu^2\|\nabla\phi\|^2 + \|\nabla K\|^2 - 6c\nu\langle\nabla K, \nabla\phi\rangle\Big\} \\ & - 4H\nu(H^2 - K - c + 2c\nu^2)\langle\nabla K, \nabla h\rangle \\ & + 12cH\nu^2(H^2 - K - c + 2c\nu^2)\langle\nabla\nu, \nabla h\rangle \\ & + 4H^2\nu^2(H^2 - K - c + 2c\nu^2)(H^2 - K - c\nu^2 + 2c). \end{aligned} \tag{2.15}$$

By equation (2.8), and since $\Delta\nu^2 = 2(\nu\Delta\nu + \|\nabla\nu\|^2)$, we have

$$\begin{aligned} -\frac{1}{2}\Delta\|\nabla\phi\|^2 = {} & \frac{1}{2}(1 - \nu^2)\Delta K - 2\nu\langle\nabla K, \nabla\nu\rangle \\ & + (H^2 - K - c + 2c\nu^2)\nu\Delta\nu + (H^2 - K - c + 6c\nu^2)\|\nabla\nu\|^2, \end{aligned}$$

and by equation (2.9)

$$\begin{aligned}\langle \nabla\phi, \nabla\Delta\phi\rangle &= 2\nu\langle\nabla K, \nabla\nu\rangle + 2H\nu\langle\nabla K, \nabla h\rangle + (-2H^2 + 2K - c - 3c\nu^2)\|\nabla\nu\|^2 \\ &\quad + H(-2H^2 + 2K - c - 3c\nu^2)\langle\nabla\nu, \nabla h\rangle.\end{aligned}$$

Since equation (2.8) implies $K\|\nabla\phi\|^2 = K(H^2 - K + c\nu^2)(1-\nu^2)$ and equation (2.1) implies $\|\nabla\nu\|^2 = (-K + c\nu^2)(1-\nu^2) - 2H\langle\nabla\nu, \nabla h\rangle$, we get

$$\begin{aligned}-\frac{1}{2}\Delta\|\nabla\phi\|^2 + \langle\nabla\phi, \nabla\Delta\phi\rangle + K\|\nabla\phi\|^2 &= \frac{1}{2}(1-\nu^2)\Delta K + (H^2 - K - c + 2c\nu^2)\nu\Delta\nu \\ &\quad + (-H^2 + K - 2c + 3c\nu^2)(-K + c\nu^2)(1-\nu^2) \\ &\quad + 3Hc(1-3\nu^2)\langle\nabla\nu, \nabla h\rangle + 2H\nu\langle\nabla K, \nabla h\rangle \\ &\quad + K(H^2 - K + c\nu^2)(1-\nu^2).\end{aligned}$$

Multiplying the expression above by $2(H^2 - K + c\nu^2)$ and using (2.8), the Weitzenböck-Bochner formula implies that

$$\begin{aligned}0 = (1-\nu^2)&\Big\{(H^2 - K + c\nu^2)\Delta K + \|\nabla K\|^2 - 6c\nu\langle\nabla K, \nabla\phi\rangle + 4Hc\nu\langle\nabla K, \nabla h\rangle\Big\} \\ &+6Hc(1-\nu^2)(H^2 - K - c\nu^2)\langle\nabla\nu, \nabla h\rangle + \|\nabla\phi\|^2\Big\{9c^2\nu^2(1-\nu^2) \\ &+2(-K + c\nu^2)(-H^2 + K - 2c + 3c\nu^2) + 2K(H^2 - K + c\nu^2)\Big\} \\ &+(H^2 - K + c\nu^2)(\Delta\phi)^2 + 2(H^2 - K + c\nu^2)(H^2 - K - c + 2c\nu^2)\nu\Delta\nu \\ &+4H^2\nu^2(H^2 - K - c + 2c\nu^2)(H^2 - K - c\nu^2 + 2c).\end{aligned}$$

Since $\nabla\phi = \nabla\nu + H\nabla h$, observing that $(H^2 - K - c\nu^2 + 2c) = (H^2 - K + c\nu^2) + 2c(1-\nu^2)$, we get

$$\begin{aligned}0 = (1-\nu^2)&\Big\{(H^2 - K + c\nu^2)\Delta K + \|\nabla K\|^2 - 6c\nu\langle\nabla K, \nabla\nu\rangle - 2Hc\nu\langle\nabla K, \nabla h\rangle\Big\} \\ &+6Hc(1-\nu^2)(H^2 - K - c\nu^2)\langle\nabla\nu, \nabla h\rangle + \|\nabla\phi\|^2\Big\{9c^2\nu^2(1-\nu^2) \\ &+2(-K + c\nu^2)(-H^2 + K - 2c + 3c\nu^2) + 2K(H^2 - K + c\nu^2) \\ &+3c\nu^2(2H^2 - 2K + c + 2c\nu^2)\Big\} + 8H^2c\nu^2(1-\nu^2)(H^2 - K - c + 2c\nu^2).\end{aligned}$$

Dividing by $(1-\nu^2)$ and using (2.8) we get equation (2.14). □

2.3 CMC surfaces with constant intrinsic curvature into $\mathbb{M}_c^2 \times \mathbb{R}$

Minimal surfaces in $\mathbb{M}_c^2 \times \mathbb{R}$ with constant intrinsic curvature were classified in [Dan15, Theorem 4.2]: such a surface is either totally geodesic or part of an associate surface of the parabolic generalised catenoid (a certain limit of catenoids). Regarding the non minimal case, Corollary 2.8 provides some examples of H-constant mean curvature surfaces in $\mathbb{M}^2_c \times \mathbb{R}$ with constant intrinsic curvature, with $H \neq 0$. We will see next that these are not the only ones in $\mathbb{H}^2_c \times \mathbb{R}$.

This new example is based on the work [SET05] by Sa Earp and Toubiana, where they study H-constant mean curvature screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$.

Example 2.12 (Helicoidal surfaces in $\mathbb{H}^2_c \times \mathbb{R}$ satisfying $K = 4H^2 + c < 0$)**.** Up to scaling, suppose that $c = -1$. Consider the Poincaré disk model for $\mathbb{H}^2$. Let K and H be real numbers such that $H \neq 0$ and $K = 4H^2 - 1 < 0$. Let $X : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$ be the screw motion immersion given by

$$X(\sigma,\tau) = \Big(\tanh\frac{\rho(\sigma)}{2}\cos\varphi(\sigma,\tau), \tanh\frac{\rho(\sigma)}{2}\sin\varphi(\sigma,\tau), \lambda(\sigma) + \varphi(\sigma,\tau)\Big), \tag{2.16}$$

where the functions ρ, λ and φ are defined as

$$\begin{aligned}
\rho(\sigma) &= \operatorname{arccosh}\Big(\frac{\cosh(\sqrt{-K}\sigma)}{\sqrt{-K}}\Big),\\
\lambda(\sigma) &= 2H\sigma + \arctan\Big(\frac{e^{2\sqrt{-K}\sigma} + 2K + 1}{4H\sqrt{-K}}\Big),\\
\varphi(\sigma,\tau) &= \sqrt{-K}\tau - \arctan\Big(\frac{e^{2\sqrt{-K}\sigma} + 2K + 1}{4H\sqrt{-K}}\Big).
\end{aligned}$$

The height function is

$$\begin{aligned}
h(\sigma,\tau) &= \lambda(\sigma) + \varphi(\sigma,\tau)\\
&= 2H\sigma + \sqrt{-K}\tau.
\end{aligned} \tag{2.17}$$

We compute

$$\rho'(\sigma) = \frac{\sqrt{-K}\sinh(\sqrt{-K}\sigma)}{\sqrt{\cosh^2(\sqrt{-K}\sigma) + K}}$$

and

$$\lambda'(\sigma) = 2H\frac{\cosh^2(\sqrt{-K\sigma})}{\cosh^2(\sqrt{-K\sigma}) + K}.$$

By [SET05, Proposition 9] with $l = 1$ we have that, choosing the appropriate orientation, the angle function is

$$\begin{aligned}
\nu(\sigma) &= \frac{\sinh\rho(\sigma)}{\sqrt{1 + \sinh^2\rho(\sigma) + (\lambda'(\sigma)/\rho'(\sigma))^2\sinh^2\rho(\sigma)}}\\
&= \sqrt{-K}\tanh(\sqrt{-K}\sigma),
\end{aligned} \tag{2.18}$$

and we also compute

$$\frac{(\lambda'(\sigma)/\rho'(\sigma))\sinh^2\rho(\sigma)}{\sqrt{1 + \sinh^2\rho(\sigma) + (\lambda'(\sigma)/\rho'(\sigma))^2\sinh^2\rho(\sigma)}} = \frac{2H}{\sqrt{-K}}\cosh(\sqrt{-K}\sigma).$$

Hence the first formula of [SET05, Lemma 11] implies that $X(\mathbb{R}^2)$ has constant mean curvature H. Moreover, we see that equation $(*)$ in that lemma is satisfied for $d = 0$ (see Figures 11 and 12 in [SET05, Theorem 17] for pictures of this surface).

Next, a straightforward computation shows that the induced metric $\mathrm{d}s^2$ on $X(\mathbb{R}^2)$ is

$$\mathrm{d}s^2 = \mathrm{d}\sigma^2 + \cosh^2(\sqrt{-K}\sigma)\,\mathrm{d}\tau^2. \tag{2.19}$$

By standard arguments, we obtain that this metric is complete and has intrinsic curvature K. Note that this surface when $H > 0$ is the surface obtained in Theorem 19 in [SET05] for $a = \sqrt{-K}$, $m = 1/a$, $l = 1$ and $U(\sigma) = \cosh(a\sigma)$.

Therefore, given $H \in \mathbb{R}$ satisfying $0 < 4H^2 < 1$, there is an H-constant mean curvature isometric immersion of $\mathbb{R}^2$ endowed with the metric (2.19) into $\mathbb{H}^2 \times \mathbb{R}$, with constant intrinsic curvature $K = 4H^2 - 1$, such that the angle and height functions are given by (2.18) and (2.17), respectively.

Let Σ be an H-constant mean curvature surface in $\mathbb{M}^2_c \times \mathbb{R}$; we recall the smooth function $q : \Sigma \to \mathbb{R}_+$ defined on Σ as before by

$$q = 2Hc\langle\nabla\nu, \nabla h\rangle + 4H^2(H^2 - K + c\nu^2) + 2H^2c(1-\nu^2) + \frac{c^2}{4}(1-\nu^2)^2.$$

Since the Abresch-Rosenberg differential is holomorphic, q either has isolated zeroes or vanishes identically. Moreover, away from its zeroes, it is proved in [ER11] that the function q satisfies the following equation

$$\Delta \log q = 4K$$

i.e.,

$$4Kq^2 = q\Delta q - \|\nabla q\|^2. \tag{2.20}$$

Therefore, this equation holds by continuity on the isolated zeroes of q, and also when q vanishes identically.

If $H \neq 0$, when Σ has constant mean curvature H and also constant intrinsic curvature K, Proposition 2.11 reads as

$$2Hcr\langle\nabla\nu, \nabla h\rangle + W = 0$$

where

$$W = 4H^2c\nu^2(H^2 - K - 2c + 3c\nu^2) - 4(H^2 - K + c\nu^2)(K - c - H^2)(K + 2c\nu^2)$$

and

$$r = 3(H^2 - K - c\nu^2).$$

Note that the condition $r = 0$ implies that ν is a constant function on Σ and then Σ is characterized by Corollary 2.8. Then restricting in an open set of Σ where $r \neq 0$, by Proposition 2.11 we can written the function q as

$$q = \frac{p}{r}$$

where p is defined by

$$p = -W + r\Big(4H^2(H^2 - K + c\nu^2) + 2H^2c(1-\nu^2) + \frac{c^2}{4}(1-\nu^2)^2\Big).$$

Since K, H and c are constants, the functions W, p and r are polynomials of ν on Σ. In the next lemma, we transform equation (2.20).

Lemma 2.13. *Let $H \neq 0$, Σ be an H-constant mean curvature surface in $\mathbb{M}^2_c \times \mathbb{R}$ with constant intrinsic curvature K and $U \subseteq \Sigma$ be an open set on which $r \neq 0$. If $K \neq 4H^2 + c$, there is an even polynomial g with degree 18, such that $g \circ \nu = 0$ on U.*

Proof. In the open set $U \subseteq \Sigma$, since $rq = p$ we have that

$$r^4 q\Delta q = r^2 p\Delta p - p^2 r\Delta r - 2p\langle r\nabla p - p\nabla r, \nabla r\rangle,$$
$$r^4\|\nabla q\|^2 = r^2\|\nabla p\|^2 - 2p\langle r\nabla p - p\nabla r, \nabla r\rangle - p^2\|\nabla r\|^2.$$

Subtracting the equations above, multiplying by r and using equation (2.20) we get

$$4Kp^2r^3 - pr^2(r\Delta p) + r^2(r\|\nabla p\|^2) - p^2(r\|\nabla r\|^2) + p^2r(r\Delta r) = 0. \tag{2.21}$$

Note that the quantities between parentheses in (2.21) are all polynomial of ν. In fact, if $f \in \mathcal{C}^\infty(\Sigma)$ is an even polynomial of ν of degree at most 6, given by

$$f = f_0 + f_2\nu^2 + f_4\nu^4 + f_6\nu^6,$$

where f_i are constants, we have that

$$\begin{aligned}\|\nabla f\|^2 &= \nu^2(2f_2 + 4f_4\nu^2 + 6f_6\nu^4)^2\|\nabla\nu\|^2\\ \Delta f &= f_2\Delta\nu^2 + f_4\Delta\nu^4 + f_6\Delta\nu^6\\ &= (2f_2 + 4f_4\nu^2 + 6f_6\nu^4)\nu\Delta\nu + (2f_2 + 12f_4\nu^2 + 30f_6\nu^4)\|\nabla\nu\|^2.\end{aligned}$$

On the other hand, since $2Hcr\langle\nabla\nu, \nabla h\rangle + W = 0$, by equation (2.1) we have

$$\begin{aligned}r\|\nabla\nu\|^2 &= r(-K + c\nu^2)(1-\nu^2) + \frac{1}{c}W\\ &= \Big[-Kr_0 + \frac{1}{c}W_0\Big] + \Big[(K+c)r_0 - Kr_2 + \frac{1}{c}W_2\Big]\nu^2\\ &\quad + \Big[-cr_0 + (K+c)r_2 + \frac{1}{c}W_4\Big]\nu^4 - cr_2\nu^6\end{aligned}$$

and

$$\nu\Delta\nu = (2K - 4H^2 - c)\nu^2 - c\nu^4.$$

With these expressions we can see that equation (2.21) has degree at most 20 in ν, and now we proceeds to compute its coefficients of degrees 20 and 18.

For the term $4Kp^2r^3$, we have

$$\begin{aligned}(4Kp^2r^3)_{20} &= 0,\\ (4Kp^2r^3)_{18} &= 4Kp_6^2r_2^3.\end{aligned}$$

For the term $-pr^2(r\Delta p)$, we have

$$\begin{aligned}(-pr^2(r\Delta p))_{20} =& -6p_6^2r_2^3(\nu\Delta\nu)_4 - 30p_6^2r_2^2(r\|\nabla\nu\|^2)_6,\\ (-pr^2(r\Delta p))_{18} =& -p_6r_2^2\Big(6p_6r_0(\nu\Delta\nu)_4 + 4p_4r_2(\nu\Delta\nu)_4 + 6p_6r_2(\nu\Delta\nu)_2\\ &+ 12p_4(r\|\nabla\nu\|^2)_6 + 30p_6(r\|\nabla\nu\|^2)_4\Big)\\ &- 2p_6r_0r_2\Big(6p_6r_2(\nu\Delta\nu)_4 + 30p_6(r\|\nabla\nu\|^2)_6\Big)\\ &- p_4r_2^2\Big(6p_6r_2(\nu\Delta\nu)_4 + 30p_6(r\|\nabla\nu\|^2)_6\Big).\end{aligned}$$

For the term $r^2(r\|\nabla p\|^2)$, we have

$$\begin{aligned}(r^2(r\|\nabla p\|^2))_{20} &= 36p_6^2r_2^2(r\|\nabla\nu\|^2)_6,\\ (r^2(r\|\nabla p\|^2))_{18} &= 72p_6^2r_0r_2(r\|\nabla\nu\|^2)_6 + r_2^2\Big(36p_6^2(r\|\nabla\nu\|^2)_4 + 48p_4p_6r_2^2(r\|\nabla\nu\|^2)_6\Big).\end{aligned}$$

For the term $-p^2(r\|\nabla r\|^2)$, we have

$$\begin{aligned}(-p^2(r\|\nabla r\|^2))_{20} =& -4p_6^2r_2^2(r\|\nabla\nu\|^2)_6,\\ (-p^2(r\|\nabla r\|^2))_{18} =& -4p_6^2r_2^2(r\|\nabla\nu\|^2)_4 - 8p_4p_6r_2^2(r\|\nabla\nu\|^2)_6.\end{aligned}$$

And finally, for the term $p^2r(r\Delta r)$, we have

$$\begin{aligned}(p^2r(r\Delta r))_{20} =& p_6^2r_2\Big(2r_2^2(\nu\Delta\nu)_4 + 2r_2(r\|\nabla\nu\|^2)_6\Big)\\ (p^2r(r\Delta r))_{18} =& p_6^2r_2\Big(2r_0r_2(\nu\Delta\nu)_4 + 2r_2^2(\nu\Delta\nu)_2 + 2r_2(r\|\nabla\nu\|^2)_4\Big)\\ &+ p_6^2r_0\Big(2r_2^2(\nu\Delta\nu)_4 + 2r_2(r\|\nabla\nu\|^2)_6\Big)\\ &+ 2p_4p_6r_2\Big(2r_2^2(\nu\Delta\nu)_4 + 2r_2(r\|\nabla\nu\|^2)_6\Big).\end{aligned}$$

Summing all these terms of order 20 and 18, we get respectively

$$(2.21)_{20} = 4p_6^2r_2^2\Big((r\|\nabla\nu\|^2)_6 - r_2(\nu\Delta\nu)_4\Big),$$

and

$$\begin{aligned}(2.21)_{18} = 4Kp_6^2r_2^3 - 14p_6^2r_0r_2^2(\nu\Delta\nu)_4 - 6p_4p_6r_2^3(\nu\Delta\nu)_4 - 4p_6^2r_2^3(\nu\Delta\nu)_2\\ + 14p_6^2r_0r_2(r\|\nabla\nu\|^2)_6 + 2p_4p_6r_2^2(r\|\nabla\nu\|^2)_6 + 4p_6^2r_2^2(r\|\nabla\nu\|^2)_4.\end{aligned}$$

Since $p_4 = -c^2(101H^2 - 29K + 26c)/4$, $p_6 = -3c^3/4$, $r_0 = 3(H^2 - K)$, $r_2 = -3c$, $(r\|\nabla\nu\|^2)_4 = c(17H^2 - 8K + 5c)$, $(r\|\nabla\nu\|^2)_6 = -cr_2$, $(\nu\Delta\nu)_2 = -4H^2 + 2K - c$ and $(\nu\Delta\nu)_4 = -c$, we have that $(2.21)_{20} = 0$ and

$$(2.21)_{18} = -486c^9(4H^2 + c - K).$$

If $K \neq 4H^2 + c$ then (2.21) implies that exists an even polynomial g of degree 18, such that $g \circ \nu = 0$ on U. □

Theorem 2.14. *Let $H \neq 0$ and Σ be an H-constant mean curvature surface in $\mathbb{M}_c^2 \times \mathbb{R}$ with constant intrinsic curvature K. Then one of the following holds:*

- *either $K = 0$ and Σ is part of a vertical cylinder $\gamma \times \mathbb{R}$, where $\gamma \subset \mathbb{M}_c^2$ is a curve of geodesic curvature $2H$,*
- *or $c < 0$, $K = 4H^2 + c < 0$ and Σ is part of either an ARL-surface or a surface of Example 2.12.*

Proof. Let $\nu : \Sigma \to [-1, 1]$ be the angle function of Σ. If $r = 0$ then ν is a constant function on Σ and so Σ is characterized by Corollary 2.8. On an open set where $r \neq 0$, suppose that $K \neq 4H^2 + c$. Then by Lemma 2.13 there is an even polynomial g such that $g \circ \nu = 0$; then ν is a constant function on Σ and again Σ is characterized by Corollary 2.8.

Suppose that $K = 4H^2 + c$. If ν is constant on Σ then the result follows from Corollary 2.8 and Σ is part of an ARL-surface. If ν is not a constant function on Σ, Proposition 2.11 implies that

$$c\langle\nabla\nu, \nabla h\rangle = 2H(4H^2 + c - c\nu^2) \tag{2.22}$$

and by Theorem 2.2 we obtain the following system

$$\|\nabla\nu\|^2 = -\frac{1}{c}(4H^2 + c - c\nu^2)^2, \tag{2.23}$$

$$\Delta\nu = (4H^2 + c - c\nu^2)\nu. \tag{2.24}$$

The equation (2.23) implies that $c < 0$. Using the Cauchy-Schwarz inequality for $\langle \nabla\nu, \nabla h\rangle$ in (2.22), since $r \neq 0$, by equations (2.1) and (2.3) we have that $4H^2 + c - c\nu^2 < 0$. Consequently, $K = 4H^2 + c < 0$ and $|\nu| < \sqrt{\frac{K}{c}}$.

The system (2.24) and (2.23) implies that the function ν is isoparametric, that is, $\|\nabla\nu\|^2$ and $\Delta\nu$ are functions of ν. Then there is a local parametrization of Σ such that ν is one of coordinates (see, e.g., [KM00] and [Eis40, page 163]), i.e., there is local coordinates (x_1, x_2) on Σ such that $\nu(x_1, x_2) = x_1$, for $x_1 \in I$, where $I \subseteq \Big(-\sqrt{\frac{K}{c}}, \sqrt{\frac{K}{c}}\Big)$ is an open interval, and

$$\mathrm{d}s^2 = \frac{1}{F(x_1)^2}\,\mathrm{d}x_1^2 + G(x_1)^2\,\mathrm{d}x_2^2,$$

for $F(x_1) = \|\nabla\nu\|$ and $G : I \to \mathbb{R}$ defined by

$$F(x_1)G(x_1) = \exp\Big(\int \frac{\Delta\nu}{\|\nabla\nu\|^2}\,\mathrm{d}x_1\Big).$$

By (2.22) we have

$$F(x_1) = \frac{1}{\sqrt{-c}}(cx_1^2 - c - 4H^2)$$

and we compute that, up to multiplication by a positive constant,

$$G(x_1) = \frac{\sqrt{-c}}{(cx_1^2 - c - 4H^2)^{1/2}}.$$

Let $\{\partial_{x_1}, \partial_{x_2}\}$ be the coordinate fields of the local parametrization (x_1, x_2) on Σ. In the basis $\{\partial_{x_1}, \partial_{x_2}\}$ of $\mathrm{T}\Sigma$, the gradient of height function h of Σ is written as

$$\nabla h = \frac{\partial h}{\partial x_1} F^2 \partial_{x_1} + \frac{\partial h}{\partial x_2}\frac{1}{G^2}\partial_{x_2},$$

and the following system holds:

$$\|\nabla h\|^2 = 1 - \nu^2, \tag{2.25}$$

$$c\langle \nabla\nu, \nabla h\rangle = 2H(4H^2 + c - cx_1^2). \tag{2.26}$$

Since $\nabla\nu = F^2\partial_{x_1}$, equations (2.25) and (2.26) imply that

$$\frac{\partial h}{\partial x_1} = \frac{2H}{cx_1^2 - c - 4H^2} \quad \text{and} \quad \frac{\partial h}{\partial x_2} = \varepsilon,$$

for $\varepsilon = \pm 1$ and then, up to the addition of a constant, we obtain

$$h(x_1, x_2) = \frac{2H}{\sqrt{cK}}\,\mathrm{arctanh}\Big(\sqrt{\frac{c}{K}}x_1\Big) + \varepsilon x_2.$$

Considering the following change of coordinates

$$(x_1,x_2)\mapsto(\sigma,\tau)=\left(\frac{1}{\sqrt{cK}}\operatorname{arctanh}\left(\sqrt{\frac{c}{K}}x_1\right),\frac{\varepsilon x_2}{\sqrt{-K}}\right),$$

we can see that the metric $\mathrm{d}s^2$, the angle and height functions of Σ are also given, respectively, by

$$\begin{aligned}\mathrm{d}s^2&=-c\,\mathrm{d}\sigma^2-c\cosh^2(\sqrt{cK}\sigma)\,\mathrm{d}\tau^2,\\ \nu(\sigma)&=\sqrt{\frac{K}{c}}\tanh(\sqrt{cK}\sigma),\\ h(\sigma,\tau)&=2H\sigma+\sqrt{-K}\tau,\end{aligned}$$

that is, up to scaling, Σ is the helicoidal surface of Example 2.12. □

Remark 2.15. Note that Theorem 2.14 together with [Dan15, Theorem 4.2] give a complete classification of constant mean curvature surfaces in $\mathbb{M}^2_c\times\mathbb{R}$ with constant intrinsic curvature.

Remark 2.16. For a given $H\neq 0$, the ARL-surface and the helicoidal surface of Example 2.12 are complete surfaces in $\mathbb{H}^2\times\mathbb{R}$ with the same non-zero constant mean curvature, which are intrinsically isometric but not congruent; up to our knowledge this is the first example of such a pair (see [TU12, Remark 1]).

2.4 CMC surfaces with constant intrinsic curvature in $\mathbb{E}(\kappa,\tau)$

2.4.1 Preliminaries and first examples

In this section, as an application of Theorem 2.14, we classify constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$, for $\kappa-4\tau^2\neq 0$, with constant intrinsic curvature. From now on, we will assume that $\tau\neq 0$. As already mentioned in the product case, some examples of H-constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$ with constant intrinsic curvature K appears when angle function is constant [ER11]. Note that, by the recent classification by Domínguez-Vázquez and Manzano in [DVMM18], these constant angle CMC surfaces turn out to be the only isoparametric surfaces in $\mathbb{E}(\kappa,\tau)$.

Example 2.17 (Vertical cylinders in $\mathbb{E}(\kappa,\tau)$)**.** Let $\gamma\subset\mathbb{M}^2_\kappa$ be a curve with constant geodesic curvature k. If $\pi:\mathbb{E}(\kappa,\tau)\to\mathbb{M}^2_\kappa$ is the Riemannian fibration then $\pi^{-1}(\gamma)$ is a $(k/2)$-CMC surface in $\mathbb{E}(\kappa,\tau)$ with intrinsic curvature $K=0$. Since the normal vector N of $\pi^{-1}(\gamma)$ is orthogonal to the unit Killing vector field ξ, the angle function vanishes.

Example 2.18 (Generalised ARL-surfaces in $\mathbb{E}(\kappa,\tau)$ with $\kappa<0$)**.** In this work, we call generalised ARL-surface the H-constant mean curvature surfaces $P_{H,\kappa,\tau}$ in $\mathbb{E}(\kappa,\tau)$ such that:

- $P_{H,\kappa,\tau}$ has constant intrinsic curvature $K = 4H^2 + \kappa < 0$.
- The Abresch-Rosenberg differential vanishes identically on $P_{H,\kappa,\tau}$.
- The function ν is constant on $P_{H,\kappa,\tau}$ satisfying $\nu^2 = \frac{4H^2+\kappa}{\kappa-4\tau^2} \in (0,1)$.

These surfaces generalises the ARL-surfaces and have also been studied by Verpoort [Ver14]. Also in [DVMM18], Domínguez-Vázquez and Manzano gave an explicit parametrization of $P_{H,\kappa,\tau}$ has an entire graph.

Moreover, in [ER11] it is shown that generalised ARL-surfaces are the only H-constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$ such that the angle function ν is constant and satisfies $0 < \nu^2 < 1$.

2.4.2 New examples in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$

Example 2.19 (Minimal surfaces in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ with $K < 0$ satisfying $K = \kappa$)**.** Up to scaling, we suppose that $\kappa = -1$. Consider $\widetilde{\mathrm{PSL}}_2(\mathbb{R}) = \{(x,y,t) \in \mathbb{R}^3 : y > 0\}$ endowed with the metric

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} + \Big(-\frac{2\tau}{y}\,\mathrm{d}x + \mathrm{d}t\Big)^2, \text{ with } \tau \neq 0.$$

Let $\Omega \subset \mathbb{R}^2$ be the open set given by $\Omega = \mathbb{R} \times (0,1)$. Consider the immersion $X : \Omega \to \widetilde{\mathrm{PSL}}_2(\mathbb{R})$ given by

$$X(x,y) = \Big(x, y, \sqrt{4\tau^2+1}\arcsin y\Big). \tag{2.27}$$

This surface is the one studied by Peñafiel in [Pn12, Lemma 4.2] with $d = 1$ (the other examples, with $d \in \mathbb{R}^*$, in that lemma can be reduced to this one if we consider the isometries of $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ given by $F_1(x,y,t) = (x/d, y/d, t)$, for $d > 0$, and $F_2(x,y,t) = (x/d, -y/d, -t)$, for $d < 0$, with a change of coordinates; for $d = 0$, the surface is a generalised ARL-surface). Moreover, this surface is invariant by 1-parameter group of parabolic isometries of $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ and by Peñafiel's results, $X(\Omega)$ is a minimal surface.

The induced metric on $X(\Omega)$ is

$$\mathrm{d}s^2 = \frac{4\tau^2+1}{y^2}\,\mathrm{d}x^2 - \frac{4\tau\sqrt{4\tau^2+1}}{y\sqrt{1-y^2}}\,\mathrm{d}x\,\mathrm{d}y + \frac{4\tau^2y^2+1}{y^2(1-y^2)}\,\mathrm{d}y^2. \tag{2.28}$$

Choosing the appropriate orientation, the angle function ν is

$$\nu(y) = \frac{\sqrt{1-y^2}}{\sqrt{4\tau^2+1}}. \tag{2.29}$$

Since the coefficients of the first fundamental form of $X(\Omega)$ depend only of y, standard computations using the Christoffel symbols show that the intrinsic curvature K of $\mathrm{d}s^2$ is given by

$$K(y) = \frac{1}{4(EG-F^2)^2}\Big\{E_y(EG)_y - 2(EG-F^2)E_{yy} - 2FE_yF_y\Big\}. \tag{2.30}$$

We compute all terms involved in (2.30):

$$EG-F^2 = \frac{4\tau^2+1}{y^4(1-y^2)},\ E_y = -\frac{2(4\tau^2+1)}{y^3},\ E_{yy} = \frac{6(4\tau^2+1)}{y^4},$$

$$F_y = -\frac{2\tau\sqrt{4\tau^2+1}(2y^2-1)\sqrt{1-y^2}}{y^2(1-y^2)^2}$$

and

$$(EG)_y = \frac{16\tau^2y^4(4\tau^2+1) - 2y^2(16\tau^4-8\tau^2-3) - 4(4\tau^2+1)}{y^5(1-y^2)^2}.$$

A straightforward computation using these expressions and (2.30) shows that $X(\Omega)$ has constant intrinsic curvature $K=-1$.

Therefore, X is a minimal isometric immersion of Ω endowed with the metric (2.28) into $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$, with constant intrinsic curvature $K=-1$, such that the angle function is given by (2.29). Considering the π-rotation around the geodesic $\{x=0, t=\frac{\pi}{2}\sqrt{4\tau^2+1}\}$, we get a complete embedded minimal surface invariant by parabolic isometries of $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ (see Figures 10 and 11 in [Pn12, Example 4.3] for pictures of this surface).

Example 2.20 (Helicoidal surfaces in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ satisfying $K=4H^2+\kappa<0$)**.** Since the relation $K=4H^2+\kappa<0$ is invariant by scaling the metric of $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$, we may multiply this metric by $1/\sqrt{-\kappa}$ and so assume that $\kappa=-1$. Now we consider $\widetilde{\mathrm{PSL}}_2(\mathbb{R}) = \{(x,y,t)\in\mathbb{R}^3 : x^2+y^2<1\}$ endowed with the metric

$$\lambda^2(\mathrm{d}x^2+\mathrm{d}y^2) + \Big(2\tau\frac{\lambda_y}{\lambda}\,\mathrm{d}x - 2\tau\frac{\lambda_x}{\lambda}\,\mathrm{d}y + \mathrm{d}t\Big)^2,$$

with $\lambda = 2/(1-(x^2+y^2))$.

Let K and H be real numbers, such that $H>0$ and $K=4H^2-1<0$. Let $\varepsilon=\pm1$. We set

$$A = \frac{H\sqrt{1-4H^2}}{\sqrt{H^2+\tau^2}}\Big(\sqrt{4\tau^2+1}-2\varepsilon\tau\Big),$$

$$B = -\frac{H\sqrt{1-4H^2}}{\sqrt{H^2+\tau^2}}\Big(\sqrt{4\tau^2+1}+2\varepsilon\tau\Big),$$

$$C = -\frac{\varepsilon\tau\sqrt{1-4H^2}}{2H\sqrt{H^2+\tau^2}}.$$

We note that

$$1-A^2=\frac{\Big(\varepsilon\tau(1-4H^2)+2H^2\sqrt{4\tau^2+1}\Big)^2}{H^2+\tau^2};$$

hence $0<A\leq 1$, and $A=1$ if and only if $\varepsilon\tau<0$ and $\tau^2(1-8H^2)=4H^4$. Similarly,

$$1-B^2=\frac{\Big(\varepsilon\tau(1-4H^2)-2H^2\sqrt{4\tau^2+1}\Big)^2}{H^2+\tau^2};$$

hence $-1\leq B<0$, and $B=-1$ if and only if $\varepsilon\tau>0$ and $\tau^2(1-8H^2)=4H^4$.

Consider the screw motion invariant immersion $X_\varepsilon:\mathbb{R}^2\to\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ given by

$$X_\varepsilon(\sigma,\theta)=\Big(f(\sigma)\cos(\theta),f(\sigma)\sin(\theta),u_\varepsilon(\sigma)+\Big(-2\tau+\varepsilon\sqrt{4\tau^2+1}\Big)\theta\Big),$$

where $f:\mathbb{R}\to(-1,1)$ and $u_\varepsilon:\mathbb{R}\to\mathbb{R}$ are defined as follows.

If $\varepsilon\tau>0$ or if $\varepsilon\tau<0$ and $\tau^2(1-8H^2)\neq 4H^4$, the function f is defined as

$$f(\sigma)=\frac{\sqrt{\cosh(\sigma)-A}}{\sqrt{\cosh(\sigma)-B}}.$$

If $\varepsilon\tau<0$ and $\tau^2(1-8H^2)=4H^4$, the function f is defined as

$$f(\sigma)=\frac{\tanh(\sigma/2)}{\sqrt{4H^2\tanh^2(\sigma/2)+1-4H^2}}.$$

If $\tau^2(1-8H^2)\neq 4H^4$, the function u_ε is defined as

$$\begin{aligned}u_\varepsilon(\sigma)=&\frac{2H\sqrt{4\tau^2+1}}{\sqrt{1-4H^2}}\sigma\\&+\frac{2\sqrt{H^2+\tau^2}}{1-4H^2}\left\{\frac{A(A-C)}{\sqrt{1-A^2}}\arctan\left(\frac{\sqrt{1+A}}{\sqrt{1-A}}\tanh(\sigma/2)\right)\right.\\&\left.-\frac{B(B-C)}{\sqrt{1-B^2}}\arctan\left(\frac{\sqrt{1+B}}{\sqrt{1-B}}\tanh(\sigma/2)\right)\right\}.\end{aligned}$$

If $\varepsilon\tau>0$ and $\tau^2(1-8H^2)=4H^4$, the function u_ε is defined as

$$u_\varepsilon(\sigma)=\frac{2H\sqrt{1-4H^2}}{\sqrt{1-8H^2}}\sigma+\sqrt{1-8H^2}\arctan\left(\frac{\sqrt{1-4H^2}}{2H}\tanh(\sigma/2)\right).$$

If $\varepsilon\tau<0$ and $\tau^2(1-8H^2)=4H^4$, the function u_ε is defined as

$$u_\varepsilon(\sigma)=\frac{2H\sqrt{1-4H^2}}{\sqrt{1-8H^2}}\sigma-\sqrt{1-8H^2}\arctan\left(\frac{2H}{\sqrt{1-4H^2}}\tanh(\sigma/2)\right).$$

We note that f and u_ε remain unchanged if we multiply both ε and τ by -1. In all cases, X_ε is analytic and

$$u_\varepsilon'(\sigma) = \frac{2H\sqrt{4\tau^2+1}}{\sqrt{1-4H^2}}\frac{\Big(\cosh(\sigma)-C\Big)\cosh(\sigma)}{\Big(\cosh(\sigma)-A\Big)\Big(\cosh(\sigma)-B\Big)}.$$

This surface is the analytic continuation of a surface that belongs to the screw motion invariant family in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ studied by Peñafiel in [Pn15]. In fact, if we consider the change of coordinates given by

$$\sigma(\rho) = \operatorname{arccosh}\left(\frac{H\sqrt{1-4H^2}}{\sqrt{H^2+\tau^2}}\Big(\sqrt{4\tau^2+1}\cosh(\rho)-2\varepsilon\tau\Big)\right)$$

for $\rho > \operatorname{arccosh}\frac{2\tau\varepsilon H\sqrt{1-4H^2}+\sqrt{H^2+\tau^2}}{H\sqrt{1-4H^2}\sqrt{4\tau^2+1}}$, we obtain the screw motion invariant surface of [Pn15, Section 3.3] with

$$l = -2\tau + \varepsilon\sqrt{4\tau^2+1} \quad \text{and} \quad d = \frac{\varepsilon\tau(1-4H^2)}{H\sqrt{4\tau^2+1}}$$

(l is the pitch if the screw motion). Indeed, we have

$$(u_\varepsilon\circ\sigma)'(\rho) = \frac{\Big(2H^2\sqrt{4\tau^2+1}\cosh(\rho)+\varepsilon(1-4H^2)\tau\Big)\Big(\sqrt{4\tau^2+1}\cosh(\rho)-2\tau\varepsilon\Big)}{\sinh(\rho)\sqrt{(1-4H^2)H^2\Big(\sqrt{4\tau^2+1}\cosh(\rho)-2\tau\varepsilon\Big)^2-H^2-\tau^2}}.$$

and $f(\sigma(\rho)) = \tanh(\rho/2)$. By Peñafiel's results, $X_\varepsilon(\mathbb{R}^2)$ has constant mean curvature H.

The induced metric on $X_\varepsilon(\mathbb{R}^2)$ is

$$\mathrm{d}s^2 = E\,\mathrm{d}\sigma^2 + 2F\,\mathrm{d}\sigma\,\mathrm{d}\theta + G\,\mathrm{d}\theta^2$$

where the terms E, F and G are given by

$$\begin{aligned} E(\sigma) &= \rho'(\sigma)^2 + u_\varepsilon'(\sigma)^2, \\ F(\sigma) &= \frac{u_\varepsilon'(\sigma)}{H\sqrt{1-4H^2}\sqrt{4\tau^2+1}}\Big(\varepsilon H\sqrt{1-4H^2} - 2\tau\sqrt{H^2+\tau^2}\cosh(\sigma)\Big), \\ G(\sigma) &= \frac{H^2+\tau^2}{H^2(1-4H^2)}\cosh^2(\sigma), \end{aligned}$$

where $\rho(\sigma) = 2\operatorname{arctanh}(f(\sigma))$.

Since the coefficients of the first fundamental form of $X_\varepsilon(\mathbb{R}^2)$ depend only of σ, standard computations using the Christoffel symbols show that the intrinsic curvature K of $\mathrm{d}s^2$ is given by

$$K(\sigma) = \frac{1}{2(EG-F^2)^2}\left\{\frac{1}{2}(EG-F^2)_\sigma G_\sigma - (EG-F^2)G_{\sigma\sigma}\right\}. \tag{2.31}$$

We compute all terms involved in (2.31):

$$\begin{aligned} G_\sigma &= \frac{2(H^2+\tau^2)}{H^2(1-4H^2)}\cosh(\sigma)\sinh(\sigma),\\ G_{\sigma\sigma} &= \frac{2(H^2+\tau^2)}{H^2(1-4H^2)}\Big(2\cosh^2(\sigma)-1\Big),\\ EG-F^2 &= \frac{H^2+\tau^2}{H^2(1-4H^2)^2}\cosh^2(\sigma),\\ (EG-F^2)_\sigma &= \frac{2(H^2+\tau^2)}{H^2(1-4H^2)^2}\cosh(\sigma)\sinh(\sigma). \end{aligned}$$

A straightforward computation using these expressions and (2.31) shows that $X_\varepsilon(\mathbb{R}^2)$ has constant intrinsic curvature $K = 4H^2 - 1 < 0$.

We now study the behaviour of the generating curve $\Gamma_\varepsilon = \{X_\varepsilon(\sigma, 0) : \sigma \in \mathbb{R}\}$. First, note that $u_\varepsilon(\sigma) \to +\infty$ when $\sigma \to +\infty$ and u_ε is odd; hence $X_\varepsilon(\mathbb{R}^2)$ is a complete surface. We distinguish three cases in terms of $\varepsilon\tau$ and H (see figures 12, 13 and 14):

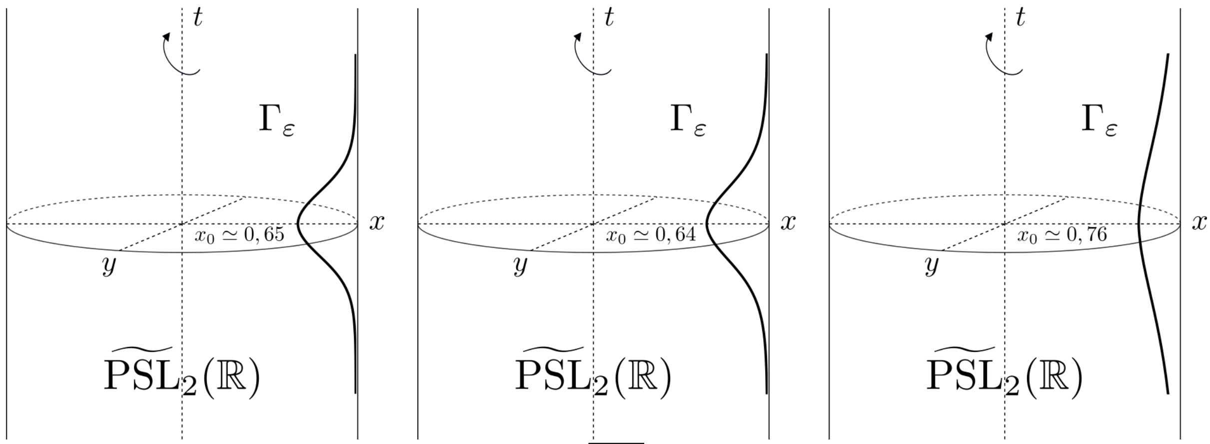
Type I If $\varepsilon\tau > 0$ or if $\varepsilon\tau < 0$ and $\tau^2(1-8H^2) < 4H^4$, then $C < 1$, and so $u'_\varepsilon > 0$ and u_ε is strictly increasing on $(0,+\infty)$. Moreover, f is even and u_ε is odd, so Γ_ε is invariant by the π-rotation around the geodesic $\{y = 0, t = 0\}$.

Type II If $\varepsilon\tau < 0$ and $\tau^2(1-8H^2) > 4H^4$, then $C > 1$, and so we can consider $\sigma_1 = \operatorname{arccosh}(C)$. We have $u'_\varepsilon < 0$ on $(0,\sigma_1)$ and $u'_\varepsilon > 0$ on $(\sigma_1,+\infty)$, so u_ε is strictly decreasing on $(0,\sigma_1)$ and u'_ε is strictly increasing on $(\sigma_1,+\infty)$. Moreover, f is even and u_ε is odd, so Γ_ε is invariant by the π-rotation around the geodesic $\{y = 0, t = 0\}$. Since u_ε changes of sign, Γ_ε is not an embedded curve.

Type III If $\varepsilon\tau < 0$ and $\tau^2(1-8H^2) = 4H^4$, then $u'_\varepsilon > 0$ and so u_ε is strictly increasing on $(0,+\infty)$. Moreover, f and u_ε are odd, so Γ_ε is invariant by the π-rotation around the geodesic $\{x = 0, t = 0\}$.

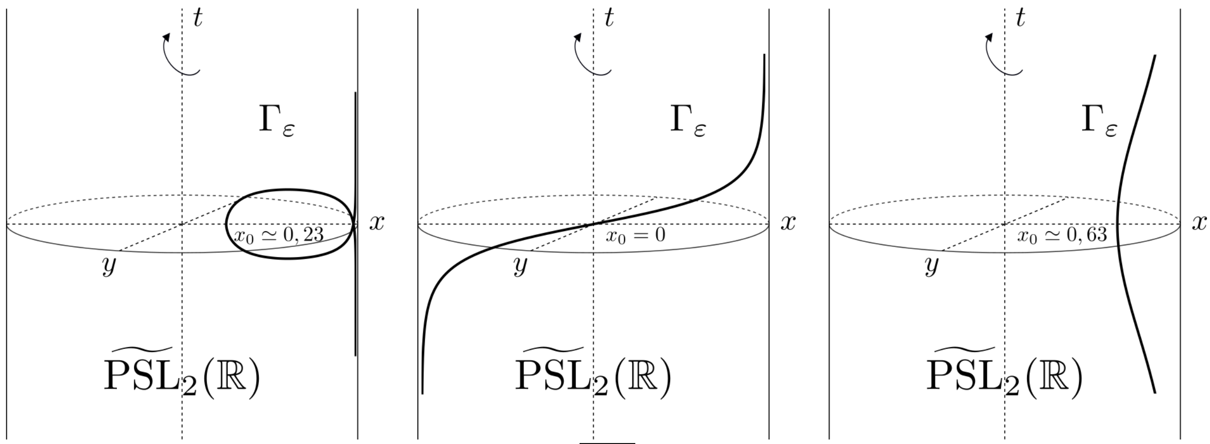
Therefore, given $H > 0$ satisfying $0 < 4H^2 < 1$, there are two complete H-constant mean curvature isometric immersions into $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$, with constant intrinsic curvature $K = 4H^2 - 1 < 0$: $X_1(\mathbb{R}^2)$ and $X_{-1}(\mathbb{R}^2)$ (these two surfaces are not congruent since they are invariant by screw motions with different pitches l).

Figure 12 – **Generating curve** Γ_ε **with** $\tau = \frac{1}{2}$ **and** $\varepsilon = 1$**:**



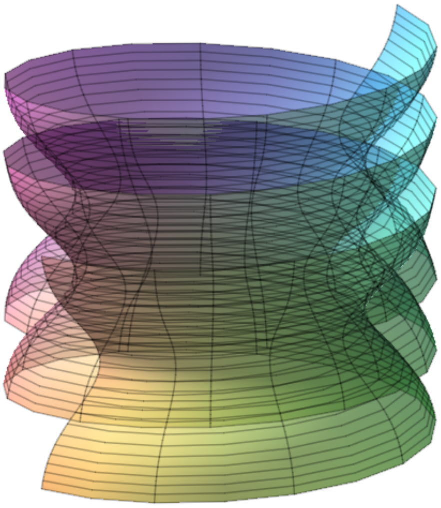
(a) $H = \frac{1}{4}$ (Type I). (b) $H = \frac{\sqrt{\sqrt{2}-1}}{2}$ (Type I). (c) $H = \frac{\sqrt{2}}{3}$ (Type I).

Figure 13 – **Generating curve** Γ_ε **with** $\tau = \frac{1}{2}$ **and** $\varepsilon = -1$**:**

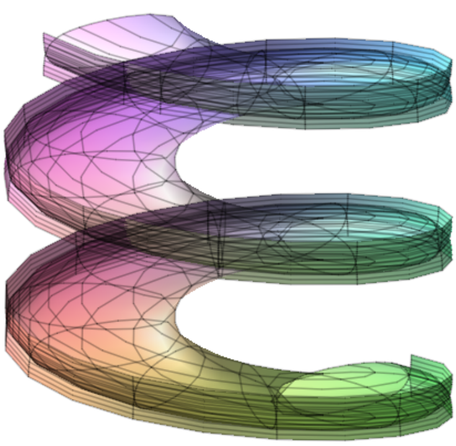


(a) $H = \frac{1}{4}$ (Type II). (b) $H = \frac{\sqrt{\sqrt{2}-1}}{2}$ (Type III). (c) $H = \frac{\sqrt{2}}{3}$ (Type I).

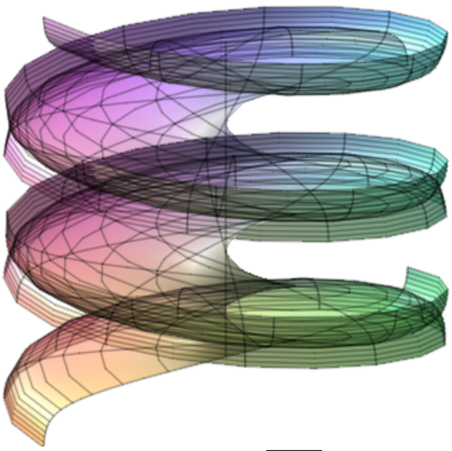
Figure 14 – **Complete screw motion surfaces in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$, with $\tau = \frac{1}{2}$, satisfying $K = 4H^2 - 1 < 0$:**



(a) $\varepsilon = 1$ and $H = \frac{1}{4}$ (Type I).



(b) $\varepsilon = -1$ and $H = \frac{1}{4}$ (Type II).



(c) $\varepsilon = -1$ and $H = \frac{\sqrt{\sqrt{2}-1}}{2}$ (Type III).

2.4.3 Classification

Theorem 2.21. *Let κ and τ be real numbers such that $\tau \neq 0$ and $\kappa - 4\tau^2 \neq 0$, and Σ be an H-constant mean curvature surface in $\mathbb{E}(\kappa,\tau)$ with constant intrinsic curvature K. Then one of the following holds:*

- *either $K = 0$ and Σ is part of a vertical cylinder over a curve $\gamma \subset \mathbb{M}^2_\kappa$ with geodesic curvature $2H$,*
- *or $\kappa < 0$, $K = 4H^2 + \kappa < 0$ and Σ is part of a generalised ARL-surface,*
- *or $\kappa < 0$, $H = 0$, $K = \kappa$ and Σ is part of a minimal surface of Example 2.19,*
- *or $\kappa < 0$, $H \neq 0$, $K = 4H^2 + \kappa < 0$ and Σ is part of one of twin helicoidal surfaces of Example 2.20.*

Proof. Let $\nu : \Sigma \to [-1, 1]$ be the angle function of Σ. If ν is a constant function, since $\tau \neq 0$ by [ER11, Theorem 2.2] we get that either $\nu = 0$, $K = 0$ and Σ is part of a vertical cylinder over a complete curve $\gamma \in \mathbb{M}^2_\kappa$ with geodesic curvature $2H$, or $0 < \nu^2 < 1$, $K = 4H^2 + \kappa < 0$ and Σ is part of a generalised ARL-surface.

From now on, suppose ν is not a constant function. Let $(\mathrm{d}s^2, S, T, \nu)$ be the Gauss-Codazzi data on Σ into $\mathbb{E}(\kappa,\tau)$. Consider $\overline{H}, \overline{\kappa} \in \mathbb{R}$ such that $\overline{H}^2 = H^2 + \tau^2$, $\overline{\kappa} = \kappa - 4\tau^2$ and $\overline{\tau} = 0$. By the sister surface correspondence [Dan07, Theorem 5.2] the 4-uple $(\mathrm{d}s^2, \overline{S}, \overline{T}, \nu)$ is the Gauss-Codazzi data of surface $\overline{\Sigma}$ in $\mathbb{E}(\overline{\kappa}, 0) = \mathbb{M}^2_{\overline{\kappa}} \times \mathbb{R}$, where

$$\overline{S} = e^{\theta J}(S - HI) + \overline{H} I \quad \text{and} \quad \overline{T} = e^{\theta J} T,$$

with $\theta \in \mathbb{R}$ given by $i\overline{H} = e^{i\theta}(\tau + iH)$ and therefore $\overline{\Sigma}$ has constant mean curvature $\overline{H}$ and is isometric to Σ. Since $\mathrm{d}s^2$ has constant intrinsic curvature and ν is not a constant function, by Theorem 2.14 we get that $\overline{\kappa} < 0$ and $K = 4\overline{H}^2 + \overline{\kappa} < 0$, that is, $\kappa - 4\tau^2 < 0$ and $K = 4H^2 + \kappa < 0$, and $\overline{\Sigma}$ is part of the helicoidal surface of Example 2.12 in $\mathbb{M}^2_{\kappa - 4\tau^2} \times \mathbb{R}$.

However, the helicoidal surface of Example 2.20 in $\mathbb{M}^2_{\kappa-4\tau^2} \times \mathbb{R}$ of constant mean curvature $\overline{H}$ has at most two constant mean curvature sister surfaces Σ_1 and Σ_2 in $\mathbb{E}(\kappa,\tau)$: denote by H_k and $(\mathrm{d}s^2, S_k, T_k, \nu)$, with $k = 1, 2$, their respective mean curvature and Gauss-Codazzi data, then $\overline{H}^2 = H_k^2 + \tau^2$, i.e., $H_1 = -H_2 = \pm H$, and

$$S_k = e^{\theta_k J}(\overline{S} - \overline{H}\mathrm{I}) + H_i \mathrm{I} \quad \text{and} \quad T_k = e^{\theta_k J}\overline{T},$$

with $\theta_k \in \mathbb{R}$ given by $\tau + iH_k = ie^{i\theta_k}\overline{H}$. Consequently, Σ is part of Σ_1 or of Σ_2.

Assume $H \neq 0$. Then Σ_1 and Σ_2 are twin surfaces [Dan07, Theorem 5.14]. On the other hand, up to an orientation reversing isometry, Example 2.20 provided two non-congruent H-constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$ with constant intrinsic curvature $K = 4H^2 + \kappa < 0$ and non-constant angle function. Then, these two surfaces coincide with Σ_1 and Σ_2, and we conclude that Σ is part of one of the surfaces of Example 2.20.

Assume $H = 0$. Then $\Sigma_1 = \Sigma_2$ (since they have the same Gauss-Codazzi data). On the other hand, Example 2.19 provided a minimal surface in $\mathbb{E}(\kappa, \tau)$ with constant intrinsic curvature $K = \kappa < 0$ and non-constant angle function. Then, this surface coincide with Σ_1, and we conclude that Σ is part of one of the surfaces of Example 2.19. □

Chapter 3

The Gauss map of minimal surfaces into $\mathbb{S}^2 \times \mathbb{R}$

This chapter is contained in the work:

The Gauss map of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$

Preprint: arxiv:2006.09995 [math.DG]

Abstract

In this chapter, we consider the model of $\mathbb{S}^2 \times \mathbb{R}$ isometric to $\mathbb{R}^3 \setminus \{0\}$, endowed with a metric conformally equivalent to the Euclidean metric of $\mathbb{R}^3$, and we define a Gauss map for surfaces in this model likewise in the Euclidean 3-space. We show as a main result that any two minimal conformal immersions in $\mathbb{S}^2 \times \mathbb{R}$ with the same non-constant Gauss map differ by only two types of ambient isometries: either $f = (\mathrm{Id}, T)$, where T is a translation on $\mathbb{R}$, or $f = (\mathcal{A}, T)$, where $\mathcal{A}$ denotes the antipodal map on $\mathbb{S}^2$. Moreover, if the Gauss map is singular, we show that it is necessarily constant, and then only vertical cylinders over geodesics of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$ appear with this assumption. We also study some particular cases, among them we prove that there is no minimal conformal immersion in $\mathbb{S}^2 \times \mathbb{R}$ which the Gauss map is a non-constant anti-holomorphic map.

3.1 Introduction

A fundamental contribution to the theory of minimal surfaces in $\mathbb{R}^3$ is due to K. Weierstrass: Given $g : \Omega \subset \mathbb{C} \to \bar{\mathbb{C}}$ a meromorphic function and η a holomorphic $1-$form defined on an open set $\Omega \subset \mathbb{C}$, such that whenever that g has a pole of order m at $z \in \Omega$ implies that η has a zero of order $2m$ at $z \in \Omega$, the Weierstrass Representation states that

$$X(z) = X(z_0) + \operatorname{Re} \int_{z_0}^{z} \left(\frac{1}{2}(1-g^2)\eta, \frac{i}{2}(1+g^2)\eta, g\eta \right)$$

define a conformal minimal surface in $\mathbb{R}^3$ with Gauss map g. In light of this, the relationship between the geometry of minimal surfaces and its Gauss map g has been studied in the last century, and some Weierstrass-type Representations has been established for minimal surfaces in others 3-dimensional ambient Riemannian manifolds.

In general, given an isometric (or conformal) immersion into some 3-dimensional ambient Riemannian manifold, the first problem is to define a Gauss map of a such surface into the unit 2-sphere $\mathbb{S}^2$, preserving some proprieties such as the Gauss map defined in $\mathbb{R}^3$, whenever possible. The second problem is how "good" is this definition, that is, if the Gauss map defined carries important informations about the surface. However, the harmonicity condition of this map is not satisfied generally for an arbitrary ambient manifold.

For simply connected homogeneous 3-manifolds with 4-dimensional isometry group, the topic of Gauss map theory is very active and several results were established in the last decade. For instance, by B. Daniel [Dan11] on the Heisenberg Group Nil_3, by B. Daniel, I. Fernandéz and P. Mira [DFM15] on $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$, by I. Fernandéz and P. Mira [FM07] on $\mathbb{H}^2 \times \mathbb{R}$ and by M. Leite and J. Ripoll [LR11] on $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

W. Meeks and J. Pérez studied surfaces with constant mean curvature in 3-dimensional simply connected metric Lie groups in the survey [MP12]. Due to the Lie group structure, they presented a *left invariant Gauss map* for an oriented surface in metric Lie groups. This map assumes values in the unit 2-sphere of the metric Lie algebra and it can be generalised to higher dimensions. In [MMPR13], W. Meeks, P. Mira, J. Pérez and A. Ros found a 2-order equation satisfied by the *left invariant Gauss Map*. They established, from this equation, a Weierstrass-type representation for surfaces with constant mean curvature in a 3-dimensional metric Lie groups. However, since $\mathbb{S}^2 \times \mathbb{R}$ is the only simply connected homogeneous Riemannian 3-manifold that is not isometric to a Lie group endowed with a left invariant metric, this ambient manifold was excluded from this list.

Motivated by the exceptional case of $\mathbb{S}^2 \times \mathbb{R}$ not discussed in [MMPR13, MP12], we define a new Gauss map for surfaces in $\mathbb{S}^2 \times \mathbb{R}$. We use the model of $\mathbb{S}^2 \times \mathbb{R}$ isometric to $\mathbb{R}^3 \setminus \{0\}$, endowed with a metric conformally equivalent to the Euclidean metric of $\mathbb{R}^3$.

Therefore, given a surface in $\mathbb{S}^2 \times \mathbb{R}$ we define the Gauss map likewise in $\mathbb{R}^3$, identifying each unit 2-sphere on each tangent plane to this surfaces with the unit 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

The goal of this chapter is to study minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ from this Gauss map. More specifically, we are interested to know when a such minimal immersion is determined by its conformal structure and its Gauss map. As a main result, we prove that:

Theorem 3.17. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion and g be its non-constant Gauss map. If $\hat{X} : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ is another minimal conformal immersion with the same Gauss map of X, then $\hat{X} = f \circ X$, with $f \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ given either by $f = (\mathrm{Id}, T)$ or $f = (\mathcal{A}, T)$, where $\mathcal{A}$ denotes the antipodal map on $\mathbb{S}^2$ and T is a translation on $\mathbb{R}$.*

Unlike what happens in $\mathbb{R}^3$, any deformation of a minimal conformal immersion preserving this Gauss map is in fact rigid, i.e., it is given by an isometry of $\mathbb{S}^2 \times \mathbb{R}$. However, we show that the zeroes of the curvature K of the metric induced by X are the singular points of g, hence either $K = 0$ or its zeroes are isolated, likewise what happens in $\mathbb{R}^3$.

Under hypothesis of anti-holomorphic non-constant Gauss map g, we prove that this condition is an obstruction to the existence of a minimal conformal immersion into $\mathbb{S}^2 \times \mathbb{R}$. For instance:

Proposition 3.9. *There is no minimal conformal immersion $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ which its Gauss map g is an anti-holomorphic non-constant map.*

In the case of minimal conformal immersion into $\mathbb{S}^2 \times \mathbb{R}$ with a singular Gauss map, we show that only vertical cylinders over geodesics of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$ appear, and then the Gauss map is necessarily constant.

Proposition 3.14. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion and g be its Gauss map. If g is a singular map then $X(\Sigma)$ is part of a vertical cylinder over a geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$. In particular, g is constant.*

It is worthwhile to point out our definition of Gauss map does not coincide with the *Twisted normal map* defined by M. Leite and J. Ripoll in [LR11]. In fact, the *Twisted normal map* assumes values into the unit 3-sphere $\mathbb{S}^3$ and the harmonicity of this map is equivalent to affirm that the surface has constant mean curvature in $\mathbb{S}^2 \times \mathbb{R}$. In our case, only totally geodesic 2-spheres in $\mathbb{S}^2 \times \mathbb{R}$ have non-constant harmonic Gauss map.

Corollary 3.15. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion and g be its Gauss map. If g is a non-constant harmonic map then $X(\Sigma)$ is part of a totally geodesic 2-sphere in $\mathbb{S}^2 \times \mathbb{R}$.*

3.2 The model of $\mathbb{S}^2 \times \mathbb{R}$

Let $\mathbb{S}^2$ be the unit 2-sphere in $\mathbb{R}^3$ endowed with the induced metric, and consider the Riemannian product manifold $\mathbb{S}^2 \times \mathbb{R}$ endowed with the product metric (referred to as the standard model of $\mathbb{S}^2 \times \mathbb{R}$). We denote by $\Pi_1 : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2$ and $\Pi_2 : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ the projections into the factors $\mathbb{S}^2$ and $\mathbb{R}$, respectively, given by $\Pi_1(y,t) = y$ and $\Pi_2(y,t) = t$.

We consider $\mathbb{R}^3 \setminus \{0\}$ endowed with the metric

$$\mathrm{d}\mu^2 = \frac{1}{x_1^2 + x_2^2 + x_3^2}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)$$

and the smooth map $\phi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}$, defined by $\phi(y,t) = e^t y$, with inverse map given by $\phi^{-1}(x) = (x/|x|, \log|x|)$, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}$. In addition, we define $\pi_j = \Pi_j \circ \phi^{-1}$ for $j = 1, 2$, i.e.,

$$x \mapsto \pi_1(x) = \frac{x}{|x|} \text{ and } x \mapsto \pi_2(x) = \log|x|.$$

A straightforward computation shows that ϕ is a global isometry between $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{R}^3 \setminus \{0\}$, therefore, we may consider $\mathbb{S}^2 \times \mathbb{R}$ as $(\mathbb{R}^3 \setminus \{0\}, \mathrm{d}\mu^2)$.

Let $\{\partial_{x_j}\}_{j=1,2,3}$ be the canonical orthonormal frame of $\mathbb{R}^3$ and we set $E_j(x) = |x|\partial_{x_j}$, for $x \in \mathbb{S}^2 \times \mathbb{R}$ and $j = 1, 2, 3$. The frame $\{E_j(x)\}_{j=1,2,3}$ is an orthonormal frame of $\mathbb{S}^2 \times \mathbb{R}$. If $\bar{\nabla}$ denotes the Riemannian connection of $\mathbb{S}^2 \times \mathbb{R}$, by a standard computation we get

$$\bar{\nabla}_{E_1} E_1 = \frac{1}{|x|}(x_2 E_2 + x_3 E_3), \quad \bar{\nabla}_{E_2} E_1 = -\frac{x_1}{|x|} E_2, \quad \bar{\nabla}_{E_3} E_1 = -\frac{x_1}{|x|} E_3,$$

$$\bar{\nabla}_{E_1} E_2 = -\frac{x_2}{|x|} E_1, \quad \bar{\nabla}_{E_2} E_2 = \frac{1}{|x|}(x_1 E_1 + x_3 E_3), \quad \bar{\nabla}_{E_3} E_2 = -\frac{x_2}{|x|} E_3,$$

$$\bar{\nabla}_{E_1} E_3 = -\frac{x_3}{|x|} E_1, \quad \bar{\nabla}_{E_2} E_3 = -\frac{x_3}{|x|} E_2, \quad \bar{\nabla}_{E_3} E_3 = \frac{1}{|x|}(x_1 E_1 + x_2 E_2).$$

Given a vector $V = v_1\partial_{x_1} + v_2\partial_{x_2} + v_3\partial_{x_3}$ tangent at the point $x = (x_1, x_2, x_3)$ of $\mathbb{S}^2 \times \mathbb{R}$, we will use brackets to express the coordinates of V in the frame $\{E_j(x)\}_{j=1,2,3}$, that is,

$$v_1\partial_{x_1} + v_2\partial_{x_2} + v_3\partial_{x_3} = \frac{1}{|x|}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The isometry group of $\mathbb{S}^2 \times \mathbb{R}$ is a 4-dimensional group isomorphic to $\mathrm{Iso}(\mathbb{S}^2) \times \mathrm{Iso}(\mathbb{R})$. In particular, it has 4 connected components: a given isometry can either preserves or reverses the orientation in each factor $\mathbb{S}^2$ and $\mathbb{R}$. Consider $f = (M, T) \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ an ambient isometry. If f preserves the orientation of $\mathbb{R}$, i.e., when T is given by $T(t) = s + t$, then $\phi^{-1} \circ f \circ \phi$ is the correspondent isometry of f in our model of $\mathbb{S}^2 \times \mathbb{R}$, given by

$$x \in \mathbb{R}^3 \setminus \{0\} \mapsto e^s M(x),$$

with $s \in \mathbb{R}$ and $M \in O(3)$, where $O(3)$ denotes the 3-dimensional Orthogonal Group. If f reverses the orientation of $\mathbb{R}$, i.e., when T is given by $T(t) = s - t$, then the correspondent isometry of f in our model of $\mathbb{S}^2 \times \mathbb{R}$ is given by

$$x \in \mathbb{R}^3 \setminus \{0\} \mapsto \frac{e^s}{|x|^2} M(x),$$

with $s \in \mathbb{R}$ and $M \in O(3)$. In order to simplify our notation, given an isometry $f \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$, we identify f and its correspondent $\phi^{-1} \circ f \circ \phi$ in our model of $\mathbb{S}^2 \times \mathbb{R}$.

We say that an isometry $f \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ is a vertical translation of $\mathbb{S}^2 \times \mathbb{R}$ when it is the identity on the factor $\mathbb{S}^2$ and it preserves the orientation of $\mathbb{R}$, that is, $f = (\mathrm{Id}, T)$ where $T(t) = s + t$ is a translation on $\mathbb{R}$, with $s \in \mathbb{R}$ fixed.

3.3 The Gauss map

In this section, we identify

$$\mathbb{S}^2 \times \mathbb{R} \ \text{ with } \ (\mathbb{R}^3 \setminus \{0\}, \mathrm{d}\mu^2).$$

In our model of $\mathbb{S}^2 \times \mathbb{R}$, we consider $\Pi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{C}$ the projection into $\mathbb{R}^2$ (identified with $\mathbb{C}$) and $\pi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ the projection into $\mathbb{R}$, respectively, given by $\Pi(x_1, x_2, x_3) = x_1 + ix_2$ and $\pi(x_1, x_2, x_3) = x_3$.

Let Σ be an oriented Riemannian surface and $z = u + iv$ be a conformal coordinate on Σ and consider $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ a conformal immersion. We define the projections of X on $\mathbb{C}$ and $\mathbb{R}$ by $F = \Pi \circ X : \Sigma \to \mathbb{C}$ and $h = \pi \circ X : \Sigma \to \mathbb{R}$, and then the conformal immersion X is written as $X = (F, h)$. We denote by $N : \Sigma \to \mathrm{U}(\mathbb{S}^2 \times \mathbb{R})$ the unit normal field to Σ, where $\mathrm{U}(\mathbb{S}^2 \times \mathbb{R})$ is the unit tangent bundle to $\mathbb{S}^2 \times \mathbb{R}$.

Since $\{E_j(x)\}_{j=1,2,3}$ is a global orthonormal frame of $\mathbb{S}^2 \times \mathbb{R}$, then for each $x \in \mathbb{S}^2 \times \mathbb{R}$, we identify the unit 2-sphere of $T_x(\mathbb{S}^2 \times \mathbb{R})$ with the unit 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, i.e., $v_1E_1 + v_2E_2 + v_3E_3 \in \mathbb{S}^2 \subset T_x(\mathbb{S}^2 \times \mathbb{R})$ is identified with $v_1\partial_{x_1} + v_2\partial_{x_2} + v_3\partial_{x_3} \in \mathbb{S}^2 \subset \mathbb{R}^3$. Therefore, we may consider the unit normal vector N into the 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ up to this identification.

Definition 3.4. The Gauss map of X is the map $g = \varphi \circ N : \Sigma \to \bar{\mathbb{C}}$, where $\varphi : \mathbb{S}^2 \to \bar{\mathbb{C}}$ is the stereographic projection with respect to the southern pole, that is, if $N = N_1E_1 + N_2E_2 + N_3E_3$ then

$$g = \frac{N_1 + iN_2}{1 + N_3} : \Sigma \to \bar{\mathbb{C}}$$

satisfying

$$N = \frac{1}{1+|g|^2}\begin{bmatrix} g + \bar{g} \\ i(\bar{g} - g) \\ 1 - |g|^2 \end{bmatrix}.$$

Remark 3.1. For the other choice of N, we replace g by $\widetilde{g} = -1/\bar{g}$.

Remark 3.2. Let $f = (M, T) \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$, where T is a translation by s on $\mathbb{R}$, and consider the conformal immersion $\hat{X} = f \circ X$. Then $\hat{N} = (\det M)M \circ N$. In fact, since $f : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R}$, given by $f(x) = e^s M(x)$, is a linear map then, for $V \in \mathrm{T}_x(\mathbb{S}^2 \times \mathbb{R})$, we have $\mathrm{d}f_x(V) = e^s M(V)$ and the unit normal $\hat{N}$ is given by

$$\hat{N} = \frac{\mathrm{d}f_{X(z)}(X_u) \times \mathrm{d}f_{X(z)}(X_v)}{\|\,\mathrm{d}f_{X(z)}(X_u) \times \mathrm{d}f_{X(z)}(X_v)\|}.$$

However,

$$\mathrm{d}f_{X(z)}(X_u) \times \mathrm{d}f_{X(z)}(X_v) = e^{2s}(\det M)M(X_u \times X_v)$$

and

$$\|\,\mathrm{d}f_{X(z)}(X_u) \times \mathrm{d}f_{X(z)}(X_v)\| = e^{2s}\|X_u \times X_v\|,$$

once $M \in O(3)$, thus $\hat{N} = (\det M)M \circ N$. Moreover, since $(\det M)M \in SO(3)$, there is $R_M \in SU(2)$ such that $(\det M)M(z_1) = z_2$ implies $R_M(\varphi(z_1)) = \varphi(z_2)$, for $z_1, z_2 \in \mathbb{S}^2$, where $SU(2)$ denotes the Special Unitary Group of degree 2. Consequently, we get that $\hat{g} = R_M \circ g$. In particular, when M is the antipodal map $\mathcal{A}$, then $\hat{N} = N$ and hence $\hat{g} = g$.

In the following computations, we adapt the method used by Daniel in [Dan11] to study the Gauss map of surfaces in the Heisenberg group Nil_3. For sake of clarity, we maintain the Daniel's notations.

We consider the complex function $\eta = 2h_z$. For a conformal immersion X, we have

$$X_z = \frac{1}{2|X|}\begin{bmatrix} (F + \bar{F})_z \\ i(\bar{F} - F)_z \\ \eta \end{bmatrix} \text{ and } X_{\bar{z}} = \frac{1}{2|X|}\begin{bmatrix} (F + \bar{F})_{\bar{z}} \\ i(\bar{F} - F)_{\bar{z}} \\ \bar{\eta} \end{bmatrix},$$

where $|X| = \sqrt{|F|^2 + h^2}$. We note that X is a conformal immersion if, and only if, $|X_z| = 0$, that is,

$$F_z\bar{F}_z = -\frac{\eta^2}{4}. \tag{3.1}$$

We compute

$$X_z \times X_{\bar{z}} = \frac{i}{2|X|^2} \begin{bmatrix} \mathrm{Re}(\eta F_{\bar{z}} - \bar{\eta} F_z) \\ \mathrm{Im}(\eta F_{\bar{z}} - \bar{\eta} F_z) \\ |F_z|^2 - |F_{\bar{z}}|^2 \end{bmatrix}$$

and we get

$$X_u \times X_v = -2iX_z \times X_{\bar{z}} = \frac{1}{|X|^2} \begin{bmatrix} \mathrm{Re}(\eta F_{\bar{z}} - \bar{\eta} F_z) \\ \mathrm{Im}(\eta F_{\bar{z}} - \bar{\eta} F_z) \\ |F_z|^2 - |F_{\bar{z}}|^2 \end{bmatrix}.$$

We note that $g = 0$ if, and only if, N is the northern pole, equivalently

$$\eta F_{\bar{z}} - \bar{\eta} F_z = 0,$$
$$|F_z|^2 - |F_{\bar{z}}|^2 > 0.$$

Then by equation (3.1) we get $\bar{\eta}(|F_z|^2 + |\eta|^2/4) = 0$ that implies $\eta = 0$, since $|F_z|^2 > 0$. Again, by (3.1) we get $F_z F_{\bar{z}} = 0$, that is, $F_{\bar{z}} = 0$. Analogously, $g = \infty$ if, and only if, N is the southern pole, equivalently, $\eta = 0$ and $F_z = 0$. Therefore $g = 0$ or ∞ if, and only if, $\eta = 0$. Moreover, $g = 0$ if, and only if, $F_{\bar{z}} = 0$ and $g = \infty$ if, and only if, $F_z = 0$.

We compute

$$\|X_u \times X_v\| = 2\langle X_z, X_{\bar{z}} \rangle = \frac{1}{|X|^2}\Big(|F_z|^2 + |F_{\bar{z}}|^2 + \frac{1}{2}|\eta|^2\Big).$$

Then, when $g \neq \infty$, we get

$$\frac{2g}{1+|g|^2} = \frac{\eta F_{\bar{z}} - \bar{\eta} F_z}{|F_z|^2 + |F_{\bar{z}}|^2 + \frac{1}{2}|\eta|^2} \quad \text{and} \quad \frac{1-|g|^2}{1+|g|^2} = \frac{|F_z|^2 - |F_{\bar{z}}|^2}{|F_z|^2 + |F_{\bar{z}}|^2 + \frac{1}{2}|\eta|^2},$$

therefore

$$g = \frac{\eta F_{\bar{z}} - \bar{\eta} F_z}{2|F_z|^2 + \frac{1}{2}|\eta|^2}.$$

Using (3.1) in the expression above, we get

$$\bar{g} F_z = -\frac{\eta}{2} \quad \text{and} \quad F_{\bar{z}} = \frac{g\bar{\eta}}{2}, \tag{3.2}$$

which implies that the functions $\eta/\bar{g}$ and $g\bar{\eta}$ can be extended smoothly to the points where $g = 0$ or ∞. Furthermore, at the points where $g \neq 0$ or ∞, the metric induced by X is given by

$$\mathrm{d}s^2 = \frac{(1+|g|^2)^2|\eta|^2}{4|g|^2|X|^2}|\,\mathrm{d}z|^2. \tag{3.3}$$

3.5 Minimal immersions into $\mathbb{S}^2 \times \mathbb{R}$

We devote this section to study minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ from the Gauss map defined in the previous section. We recall that, unless otherwise stated, we identify

$$\mathbb{S}^2 \times \mathbb{R} \text{ with } (\mathbb{R}^3 \setminus \{0\}, \mathrm{d}\mu^2).$$

From now on, we suppose that $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ is a minimal conformal immersion. As a first result, we translate the minimality condition of X in terms of its Gauss map g and its complex function η.

Lemma 3.3. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion. Then the Gauss map g and the complex function η satisfy*

$$\frac{1-|g|^2}{1+|g|^2}(\log \bar{g})_{\bar{z}} = (\log \eta)_{\bar{z}} \tag{3.4}$$

when $g \neq \infty$.

Proof. We consider $U \subset \Sigma$ an open set which $g \neq 0$. Firstly, note that by equations (3.2), we can write

$$X_z = \frac{\eta}{4|X|}\begin{bmatrix} \bar{g} - \bar{g}^{-1} \\ i(\bar{g} + \bar{g}^{-1}) \\ 2 \end{bmatrix} \text{ and } X_{\bar{z}} = \frac{\bar{\eta}}{4|X|}\begin{bmatrix} g - g^{-1} \\ i(g + g^{-1}) \\ 2 \end{bmatrix}.$$

Therefore, setting

$$A_1 = \frac{\eta}{4|X|}\Big(\bar{g} - \frac{1}{\bar{g}}\Big),\ A_2 = \frac{\eta}{4|X|}\Big(\bar{g} + \frac{1}{\bar{g}}\Big) \text{ and } A_3 = \frac{\eta}{2|X|},$$

we have that $X_z = \sum_{j=1}^{3} A_j E_j$ and $X_{\bar{z}} = \sum_{j=1}^{3} \bar{A}_j E_j$.

Given $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ a conformal immersion, a necessary and sufficient condition for X to be minimal is

$$\bar{\nabla}_{X_{\bar{z}}} X_z = 0,$$

that is,

$$\sum_{k=1}^{3} (A_k)_{\bar{z}} E_k + \sum_{\ell,j=1}^{3} \bar{A}_\ell A_j \bar{\nabla}_{E_\ell} E_j = 0. \tag{3.5}$$

A straightforward computation gives:

$$(A_1)_{\bar{z}} = \frac{1}{16|X|^3|g|^2\bar{g}}\Big\{4|X|^2|g|^2(\bar{g}^2-1)\eta_{\bar{z}} + 4|X|^2(\bar{g}^2+1)g\eta\bar{g}_{\bar{z}} \\ - |\eta|^2(\bar{g}^2-1)(|g|^2 g\bar{F} - \bar{g}F + 2h|g|^2)\Big\},$$

$$(A_2)_{\bar z} = \frac{i}{16|X|^3|g|^2\bar g}\Big\{4|X|^2|g|^2(\bar g^2+1)\eta_{\bar z} + 4|X|^2(\bar g^2-1)g\eta\bar g_{\bar z} - |\eta|^2(\bar g^2+1)(|g|^2 g\bar F - \bar g F + 2h|g|^2)\Big\}$$

and

$$(A_3)_{\bar z} = \frac{1}{8|X|^3 g}\Big\{4|X|^2 g\eta_{\bar z} - |\eta|^2(g^2\bar F - F + 2hg)\Big\}.$$

Thus

$$\begin{aligned}\sum_{j,k} \bar A_j A_k \bar\nabla_{E_j}E_k = \\ &= \frac{|\eta|^2}{16|X|^3|g|^2}\Big\{(|g|^4+g^2)\bar F + (\bar g^2+1)F - 2h\bar g(g^2-1) + 4|g|^2\mathrm{Re}F\Big\}E_1 \\ &+ \frac{|\eta|^2}{16|X|^3|g|^2}\Big\{i(|g|^4-g^2)\bar F + i(\bar g^2-1)F + 2ih\bar g(g^2+1) + 4|g|^2\mathrm{Im}F\Big\}E_2 \\ &+ \frac{|\eta|^2}{8|X|^3|g|^2}\Big\{(1+|g|^4)h - |g|^2\bar g F + g\bar F\Big\}E_3.\end{aligned}$$

The vertical part of (3.5) is equal to zero if, and only if,

$$4|X|^2|g|^2\eta_{\bar z} - |\eta|^2(|g|^2 g\bar F - \bar g F + 2|g|^2 h) + |\eta|^2((1+|g|^4)h - |g|^2\bar g F + g\bar F) = 0,$$

i.e.,

$$4|X|^2|g|^2\bar\eta_z = -|\eta|^2(1-|g|^2)\Omega,$$

where $\Omega = F\bar g + \bar F g + h(1-|g|^2) \in \mathbb{R}$.

The horizontal part of (3.5) is equal to zero if, and only if,

$$\begin{aligned}&-4|X|^2|g|^2\eta_{\bar z} + 4|X|^2\eta g\bar g_{\bar z} + |\eta|^2(|g|^2 g\bar F - \bar g F + 2h|g|^2) \\ &\qquad + |\eta|^2(|g|^2 g\bar F + \bar g F - 2|g|^4 h + 2|g|^2\bar g F) = 0,\end{aligned}$$

i.e.,

$$4|X|^2\eta g\bar g_{\bar z} - 4|X|^2|g|^2\eta_{\bar z} + 2|\eta|^2|g|^2\Omega = 0.$$

Using (3.7) in the equation above we get

$$4|X|^2 g\bar g_{\bar z} = -\bar\eta(1+|g|^2)\Omega.$$

By continuity, equations (3.7) and (3.6) also hold on a neighbourhood of a point which $g=0$. Therefore,

$$4|X|^2 g\bar g_{\bar z} = -\bar\eta(1+|g|^2)\Omega, \tag{3.6}$$
$$4|X|^2|g|^2\bar\eta_z = -|\eta|^2(1-|g|^2)\Omega \tag{3.7}$$

are necessary and sufficient conditions for X to be minimal, when $g \neq \infty$.

Suppose that X is a minimal conformal immersion. At a point which $g \neq 0$, we have that $\eta \neq 0$, then by (3.6) and (3.7) we have

$$-\frac{\bar{\eta}\Omega}{4|X|^2 g} = \frac{\bar{g}_{\bar{z}}}{1+|g|^2} \quad \text{and} \quad -\frac{\bar{\eta}(1-|g|^2)\Omega}{4|X|^2 g} = \frac{\bar{g}\bar{\eta}_z}{\eta}.$$

Multiplying the first equation by $(1-|g|^2)$ and substituting it in the second one, we get our assertion. Again by continuity, (3.3) holds at a point where $g = 0$. □

Example 3.4 (Horizontal surfaces and vertical cylinders)**.** The simplest examples of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ are the horizontal surfaces and the vertical cylinders over geodesics of $\mathbb{S}^2$. The totally geodesic 2-spheres $\mathbb{S}^2 \times \{a\}$ in the standard model of $\mathbb{S}^2 \times \mathbb{R}$, for $a \in \mathbb{R}$, are 2-spheres of radius e^a centered at the origin of $\mathbb{R}^3$ in our model of $\mathbb{S}^2 \times \mathbb{R}$. The vertical cylinders $\gamma \times \mathbb{R}$ over geodesics of $\mathbb{S}^2$ in the standard model of $\mathbb{S}^2 \times \mathbb{R}$ are planes passing through the origin of $\mathbb{R}^3$ in our model of $\mathbb{S}^2 \times \mathbb{R}$.

Proposition 3.5. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion and g be its Gauss map. Then g is constant if, and only if, $X(\Sigma)$ is part of a vertical cylinder over a geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$.*

Proof. Suppose that g is constant. Then the normal vector N is also constant on Σ, so $X(\Sigma)$ is contained in a plane P of $\mathbb{R}^3$. If P does not pass through the origin of $\mathbb{R}^3$, there is a 2-sphere centered at the origin of $\mathbb{R}^3$ which it is tangent to P at a point z_0. However, by the Tangency Principle, P must coincide with this 2-sphere on a neighbourhood of z_0, that is a contradiction. Therefore, P passes through the origin of $\mathbb{R}^3$ and then $X(\Sigma)$ is part of a vertical cylinder over a geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$.

Conversely, since vertical cylinders over geodesics of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$ are planes in our model of $\mathbb{S}^2 \times \mathbb{R}$, then the Gauss map g must be constant. □

Given $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ a minimal conformal immersion, we consider the smooth maps $p : \Sigma \to \bar{\mathbb{C}}$ and $r : \Sigma \to \mathbb{R}$ defined by $p(z) = \varphi \circ \pi_1 \circ X(z)$ and $r(z) = \pi_2 \circ X(z)$, respectively, that is,

$$z \mapsto p(z) = \frac{F(z)}{|X(z)| + h(z)} \quad \text{and} \quad z \mapsto r(z) = \log|X(z)|.$$

We observe that $p = 0$ or ∞ if, and only if, $F = 0$. Then p cannot be identically equal to 0 or ∞ on an open set, since X is an immersion. Moreover, p is a harmonic map into $(\bar{\mathbb{C}}, 4/(1+|w|^2)^2|\,dw|^2)$ and r is a harmonic function on $\mathbb{R}$. Indeed, if X is minimal then $\pi_j \circ X$ is a harmonic map, for $j = 1, 2$. Since the stereographic projection φ is an isometry between $\mathbb{S}^2$ and $\bar{\mathbb{C}}$ endowed with the metric $4/(1+|w|^2)^2|\,dw|^2$, we have that $\varphi \circ \pi_1 \circ X = p$ and $\pi_2 \circ X = r$ are harmonic maps.

Proposition 3.6. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion. Let $g : \Sigma \to \bar{\mathbb{C}}$ be the Gauss map of X and suppose that $g \neq \infty$. Then the following equations hold:*

$$(1 + |p|^2)p_{z\bar{z}} - 2\bar{p}p_z p_{\bar{z}} = 0, \tag{3.8}$$

$$(\bar{g} - \bar{p})^2 p_z + (1 + \bar{g}p)^2 \bar{p}_z = 0, \tag{3.9}$$

$$(1 + |p|^2)^2 g_z = (1 + g\bar{p})^2 p_z + (g - p)^2 \bar{p}_z. \tag{3.10}$$

These equations hold when $p \neq \infty$.

Proof. Indeed, equation (3.8) is the harmonicity condition for the map p.

We restrict ourselves on an open set of Σ which $p \neq 0$. Away from the points where $g = 0$ and ∞, by definition of p and (3.2), we compute

$$p_z = -\frac{\eta(|X| + h + \bar{g}F)^2}{4|X|(|X| + h)^2 \bar{g}}, \tag{3.11}$$

$$\bar{p}_z = \frac{\eta(\bar{g}|X| + h\bar{g} - \bar{F})^2}{4|X|(|X| + h)^2 \bar{g}}. \tag{3.12}$$

On the other hand, we have $|X| + h = F/p \in \mathbb{R}$,

$$(|X| + h + \bar{g}F)^2 = \frac{F^2}{p^2}(1 + \bar{g}p)^2 \text{ and } (|X| + h + \bar{g}F)^2 = \frac{F^2}{p^2}(1 + \bar{g}p)^2.$$

Then, by equations (3.11) and (3.13), we get

$$4|X|\bar{g}p_z = -\eta(1 + \bar{g}p)^2, \tag{3.13}$$

$$4|X|\bar{g}\bar{p}_z = \eta(\bar{g} - \bar{p})^2. \tag{3.14}$$

Therefore, $(\bar{g} - \bar{p})^2$(3.13) $+ (1 + \bar{g}p)^2$(3.14) $= 0$ that implies (3.9). By continuity, (3.9) holds at a point where $g = 0$. Analogously, equation (3.9) holds on a neighbourhood of a point which $p = 0$.

If $p = g$ or $p = -1/\bar{g}$ on an open set of Σ, then equation (3.10) is trivial. Suppose that $g \neq 0$ on an open set of Σ. Then, away from the points where $p = g$ and $p = -1/\bar{g}$, we observe that p_z is well defined by (3.13) and it does not vanish. Differentiating equation (3.9) with respect to $\bar{z}$ and multiplying by $(1 + |p|^2)(1 + \bar{g}p)$, we have

$$2(1 + |p|^2)(\bar{g} - \bar{p})(1 + \bar{g}p)(\bar{g}_{\bar{z}} - \bar{p}_{\bar{z}})p_z + (\bar{g} - \bar{p})^2(1 + \bar{g}p)\Big[(1 + |p|^2)p_{z\bar{z}}\Big] + 2(1 + |p|^2)(\bar{g}_{\bar{z}}p + \bar{g}p_{\bar{z}})\Big[(1 + \bar{g}p)^2\bar{p}_z\Big] + (1 + \bar{g}p)^3\Big[(1 + |p|^2)\overline{p_{z\bar{z}}}\Big] = 0.$$

From the equation above, using (3.8) and (3.9) into to the brackets terms, we get

$$2(\bar{g} - \bar{p})p_z\Big((1 + |p|^2)^2\bar{g}_{\bar{z}} - (1 + \bar{g}p)^2\bar{p}_{\bar{z}} - (\bar{g} - \bar{p})^2 p_{\bar{z}}\Big) = 0,$$

which implies (3.10). By continuity, (3.10) holds at a point where $p = g$ or $p = -1/\bar{g}$. Analogously, equation (3.10) holds on a neighbourhood of a point which $g = 0$. □

Remark 3.7. We notice that if $p = g$ (respec., $p = -1/\bar{g} = \tilde{g}$) on Σ, by the definitions of p and g, we have that $X/|X| = N$ (respec., $X/|X| = -N$), i.e., $X(\Sigma)$ is part of a totally geodesic 2-sphere in $\mathbb{S}^2 \times \mathbb{R}$. Moreover, in both cases, the Gauss map is holomorphic.

In the next result, we derive an important second order equation satisfied by the Gauss map g and the map p.

Proposition 3.8. *The Gauss map g and the map p satisfy*

$$\Big(|g-p|^2 - |1+\bar{g}p|^2\Big)(1+|g|^2)g_{z\bar{z}} + 2(g-p)(1+g\bar{p})|g_z|^2 + 2\Big(\bar{g}|1+\bar{g}p|^2 - (\bar{g}-\bar{p})(1+|g|^2)\Big)g_z g_{\bar{z}} = 0 \quad (3.15)$$

when $g \neq \infty$.

Proof. Firstly, note that if $p = g$ or $p = -1/\bar{g}$ on an open set of Σ, then (3.15) is equivalent to (3.8), and if $g_z = 0$ or $g = 0$ on an open set of Σ, then (3.15) is trivial.

Suppose that $g \neq 0$ on an open set of Σ. Then, away from the points where $p = g$ and $p = -1/\bar{g}$, p_z and $\bar{p}_z$ are well defined by (3.13) and (3.14) and they do not vanish. Differentiating equation (3.10) and using (3.8), we get

$$\begin{aligned}(1+|p|^2)^2 g_{z\bar{z}} &+ \frac{2\bar{p}p_{\bar{z}}}{1+|p|^2}\Big((1+|p|^2)^2 g_z - (1+g\bar{p})^2 p_z\Big) + 2(g-p)p_{\bar{z}}\bar{p}_z \\ &+ \frac{2p\bar{p}_{\bar{z}}}{1+|p|^2}\Big((1+|p|^2)^2 g_z - (g-p)^2\bar{p}_z\Big) - 2g(1+g\bar{p})p_z\bar{p}_{\bar{z}} \\ &- 2\bar{p}(1+g\bar{p})p_z g_{\bar{z}} - 2(g-p)\bar{p}_z g_{\bar{z}} = 0.\end{aligned}$$

By equation (3.10), we have

$$(1+|p|^2)^2 g_{z\bar{z}} + \frac{2(g-p)(1+g\bar{p})}{1+|p|^2}(|\bar{p}_z|^2 - |p_z|^2) - 2\bar{p}(1+g\bar{p})p_z g_{\bar{z}} - 2(g-p)\bar{p}_z g_{\bar{z}} = 0. \quad (3.16)$$

On the other hand, by equation (3.9), we have

$$p_z = -\frac{(1+\bar{g}p)^2}{(\bar{g}-\bar{p})^2}\bar{p}_z, \quad (3.17)$$

and then (3.10) implies

$$(1+|p|^2)^2 g_z = \frac{1}{(\bar{g}-\bar{p})^2}\Big(|g-p|^4 - |1+\bar{g}p|^4\Big)\bar{p}_z. \quad (3.18)$$

At a point where $g_z \neq 0$, we get $|g-p|^4 - |1+\bar{g}p|^4 \neq 0$. We also note that $|g-p|^4 - |1+\bar{g}p|^4 = (1+|g|^2)(1+|p|^2)(|g-p|^2 - |1+\bar{g}p|^2)$ and then we get

$$p_z = -\frac{(1+|p|^2)(1+\bar{g}p)^2}{(1+|g|^2)(|g-p|^2 - |1+\bar{g}p|^2)}g_z, \quad (3.19)$$

$$\bar{p}_z = \frac{(1+|p|^2)(\bar{g}-\bar{p})^2}{(1+|g|^2)(|g-p|^2 - |1+\bar{g}p|^2)}g_z. \quad (3.20)$$

Then, by (3.16), we have

$$(|g-p|^2 - |1+\bar{g}p|^2)(1+|p|^2)(1+|g|^2)g_{z\bar{z}} + 2(1+|p|^2)(g-p)(1+g\bar{p})|g_z|^2 \\ + 2\Big(\bar{p}(1+\bar{g}p)|1+\bar{g}p|^2 - (\bar{g}-\bar{p})|g-p|^2\Big)g_z g_{\bar{z}} = 0, \quad (3.21)$$

and since

$$\bar{p}(1+\bar{g}p)|1+\bar{g}p|^2 - (\bar{g}-\bar{p})|g-p|^2 = (1+|p|^2)\Big(\bar{g}|1+\bar{g}p|^2 - (\bar{g}-\bar{p})(1+|g|^2)\Big),$$

we get (3.15) dividing (3.21) by $(1+|p|^2)$. By continuity, (3.10) holds on a neighbourhood of a point which $g_z = 0$. If $p = g$ or $p = -1/\bar{g}$ at some point, again by continuity (3.10) holds. Analogously, equation (3.10) holds on a neighbourhood of a point which $g = 0$. □

Proposition 3.9. *There is no minimal conformal immersion $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ which its Gauss map g is an anti-holomorphic non-constant map.*

Proof. By contradiction, suppose that exists a such immersion which $g_z = 0$ on Σ. Away from the points where $p = g$ and $p = -1/\bar{g}$, by equation (3.18), we get that either $\bar{p}_z = 0$ or $|g-p|^2 - |1+\bar{g}p|^2 = 0$.

If $\bar{p}_z = 0$ on Σ, then, by equations (3.9) and (3.10), we get $p_z = 0$. Thus p is a constant function on Σ and, by definition of p, we have $\varphi^{-1}(\text{const}) = X/|X|$; however it is not possible since X is an immersion.

On the other hand, by the definition of p, $|g-p|^2 - |1+\bar{g}p|^2 = 0$ on Σ if, and only if,

$$F\bar{g} + \bar{F}g + h(1-|g|^2) = 0. \quad (3.22)$$

Differentiating equation (3.22) and using (3.2), we get $\bar{g}_z(F - hg) = 0$, that is, $F - hg = 0$ on Σ. Finally, differentiating $F - hg = 0$ and using (3.2), we get $-\eta(1+|g|^2)/\bar{g} = 0$ which holds if, and only if, $\eta = 0$, i.e., if $g = 0$ or ∞ on Σ. □

Remark 3.10. Suppose that g is a non-constant map. By equations (3.9) and (3.10), we get

$$(1+|p|^2)^3|g_z|^2 = (1+|g|^2)(|g-p|^2 - |1+\bar{g}p|^2)(|\bar{p}_z|^2 - |p_z|^2).$$

On the other hand, since (3.13) and (3.14), we have

$$16|X|^2|g|^2(|\bar{p}_z|^2 - |p_z|^2) = |\eta|^2(1+|g|^2)(1+|p|^2)(|g-p|^2 - |1+\bar{g}p|^2),$$

then

$$16|X|^2|g|^2(1+|p|^2)^2|g_z|^2 = |\eta|^2(1+|g|^2)^2(|g-p|^2 - |1+\bar{g}p|^2)^2.$$

Since g is not constant, η cannot vanish on an open set of Σ and, by Proposition 3.9, the same happens for g_z. At a point where $\eta \neq 0$ and $g_z \neq 0$, we have

$$\frac{(|g-p|^2 - |1+\bar{g}p|^2)^2}{|g_z|^2} = \frac{16|X|^2|g|^2(1+|p|^2)^2}{|\eta|^2(1+|g|^2)^2}.$$

Therefore, by continuity, the left side of the equation above is a function that do not vanish on Σ, when g is a non-constant map.

Corollary 3.11. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion. Then the metric induced by X is given by*

$$\mathrm{d}s^2 = \frac{4(1+|p|^2)^2|g_z|^2}{(|g-p|^2 - |1+\bar{g}p|^2)^2}|\,\mathrm{d}z|^2$$

when $g \neq \infty$.

Proof. Let $U \subset \Sigma$ be an open set which $p \neq g$. At a point where $g \neq 0$, by equations (3.3) and (3.14), we have that the metric induced by X, in the minimal case, is given by

$$\mathrm{d}s^2 = \frac{4(1+|g|^2)^2|\bar{p}_z|^2}{|g-p|^4}|\,\mathrm{d}z|^2.$$

However, using equation (3.20), we get

$$\mathrm{d}s^2 = \frac{4(1+|p|^2)^2|g_z|^2}{(|g-p|^2 - |1+\bar{g}p|^2)^2}|\,\mathrm{d}z|^2.$$

By continuity, this expression holds at a point where $g = 0$. Analogously, equation (3.10) holds on a neighbourhood of a point which $g = p$. □

Proposition 3.12. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion with a holomorphic non-constant Gauss map g. Then $X(\Sigma)$ is part of a totally geodesic 2-sphere in $\mathbb{S}^2 \times \mathbb{R}$.*

Proof. If g is a holomorphic non-constant function, by Proposition 3.8, we have $(g - p)(1+g\bar{p})|g_z|^2 = 0$, that is, either $p = g$ or $p = -1/\bar{g}$. Therefore, $X(\Sigma)$ is part of a totally geodesic 2-sphere in $\mathbb{S}^2 \times \mathbb{R}$. □

Lemma 3.13. *The curvature K of the metric induced by X is*

$$K = \frac{(|g-p|^2 - |1+\bar{g}p|^2)^2}{(1+|g|^2)^2(1+|p|^2)^2|g_z|^2}(|g_z|^2 - |\bar{g}_z|^2).$$

This formula holds when $g \neq \infty$. In particular, K vanishes at singular points of g.

Proof. Since X is a conformal immersion, we can compute the curvature K of the metric induced by X through $2K = -\Delta(\log\lambda)$, i.e., $K = -(2/\lambda)(\log\lambda)_{z\bar{z}}$, where λ is given by $\mathrm{d}s^2 = \lambda|\,\mathrm{d}z|^2$. In order to simplify our computations, we will use the expression (3.3) instead of expression obtained in Corollary 3.11. Then, λ is given by

$$\lambda = \frac{(1+|g|^2)^2|\eta|^2}{4|g|^2|X|^2}.$$

We restrict ourselves to a domain on which $g \neq 0$. We observe that, since

$$(\log\lambda)_{z\bar{z}} = 2(\log(1+|g|^2))_{z\bar{z}} + (\log\eta)_{z\bar{z}} + (\log\bar{\eta})_{z\bar{z}} - 2r_{z\bar{z}} - (\log g)_{z\bar{z}} - (\log\bar{g})_{z\bar{z}}$$

and $r_{z\bar{z}} = 0$, once X is a minimal immersion, by Lemma 3.3, we get

$$(\log \lambda)_{z\bar{z}} = \frac{2}{(1+|g|^2)^2}\Big((g_{z\bar{z}}\bar{g} + |g_z|^2 + |\bar{g}_z|^2 + g\bar{g}_{z\bar{z}})(1+|g|^2) - (g_z\bar{g} + g\bar{g}_z)(g_{\bar{z}}\bar{g} + g\bar{g}_{\bar{z}})\Big)$$
$$- \frac{2\bar{g}_{\bar{z}}}{(1+|g|^2)^2\bar{g}}(g_z\bar{g} + g\bar{g}_z) - \frac{2g_z}{(1+|g|^2)^2 g}(g_z\bar{g} + g\bar{g}_z)$$
$$- \frac{2g}{(1+|g|^2)\bar{g}}(\bar{g}_{z\bar{z}}\bar{g} - \bar{g}_z\bar{g}_{\bar{z}}) - \frac{2\bar{g}}{(1+|g|^2)g}(g_{z\bar{z}}g - g_z g_{\bar{z}}).$$

Follows from a direct computation that

$$(\log \lambda)_{z\bar{z}} = \frac{2}{(1+|g|^2)^2}(|\bar{g}_z|^2 - |g_z|^2).$$

Therefore, the curvature K is given by

$$K = \frac{16|g|^2|X|^2}{(1+|g|^2)^4|\eta|^2}(|g_z|^2 - |\bar{g}_z|^2),$$

and by continuity this formula holds at a point where $g = 0$.

Let $U \subset \Sigma$ be an open set on which $p \neq g$. By equation (3.14), we get

$$K = \frac{|g-p|^4}{(1+|g|^2)^4|\bar{p}_z|^2}(|g_z|^2 - |\bar{g}_z|^2).$$

However, by equation (3.20), we get

$$K = \frac{(|g-p|^2 - |1+\bar{g}p|^2)^2}{(1+|g|^2)^2(1+|p|^2)^2|g_z|^2}(|g_z|^2 - |\bar{g}_z|^2),$$

and by continuity, this formula holds at a point where $p = g$. Moreover, since $\det \mathrm{d}g = (|g_z|^2 - |\bar{g}_z|^2)$, by Remark 3.10, we have that K vanishes at singular points of g. □

Proposition 3.14. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion and g be its Gauss map. If g is a singular map then $X(\Sigma)$ is part of a vertical cylinder over a geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$. In particular, g is constant.*

Proof. Since $\det g = |g_z|^2 - |\bar{g}_z|^2$, if the Gauss map g is singular, then $|g_z| = |\bar{g}_z|$ on Σ, and by Lemma 3.13, we get that $K = 0$. Therefore, by [TU15, Torralbo-Urbano], $X(\Sigma)$ is part of a vertical cylinder over a geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$. Hence, by Proposition 3.5, g is constant. □

Corollary 3.15. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion and g be its Gauss map. If g is a non-constant harmonic map then $X(\Sigma)$ is part of a totally geodesic 2-sphere in $\mathbb{S}^2 \times \mathbb{R}$.*

Proof. If g is a non-constant harmonic map, then g satisfies

$$(1+|g|^2)g_{z\bar{z}} - 2\bar{g}g_z g_{\bar{z}} = 0,$$

and, by Proposition 3.8, we get that

$$|g - p|^2 |1 + \bar{g}p|^2 (|g_z|^2 - |g_{\bar{z}}|^2) = 0.$$

Since g is non-constant, then $(|g_z|^2 - |g_{\bar{z}}|^2) \neq 0$, otherwise, by Proposition 3.14, g must be constant. Therefore, $g = p$ or $g = -1/\bar{p}$ and then $X(\Sigma)$ is part of a totally geodesic 2-sphere in $\mathbb{S}^2 \times \mathbb{R}$ (Remark 3.7). □

Lemma 3.16. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion. Then the Gauss map g and the maps p and r satisfy*

$$r_z = \frac{2(\bar{g} - \bar{p})(1 + g\bar{p})}{(1 + |g|^2)(|g - p|^2 - |1 + \bar{g}p|^2)} g_z \tag{3.23}$$

when $g \neq \infty$.

Proof. Let $U \subset \Sigma$ be an open set on which $p \neq g$. Then, by equation (3.17), $\bar{p}_z$ does not vanish on U, otherwise p is a constant function. By Lemma 3.3 and equation (3.14), we get

$$\frac{1 - |g|^2}{1 + |g|^2} (\log \bar{g})_{\bar{z}} = \left(\log \frac{4|X|\bar{g}\bar{p}_z}{(\bar{g} - \bar{p})^2} \right)_{\bar{z}}.$$

Away from the points where $g = 0$ and ∞, we have

$$-\frac{2|g|^2}{1 + |g|^2} \frac{\bar{g}_{\bar{z}}}{\bar{g}} = r_{\bar{z}} + \frac{\bar{p}_{z\bar{z}}}{\bar{p}_z} - 2\frac{\bar{g}_{\bar{z}} - \bar{p}_{\bar{z}}}{\bar{g} - \bar{p}}$$

and, by equation (3.8), we obtain

$$r_{\bar{z}} = \frac{2(1 + g\bar{p})}{(\bar{g} - \bar{p})(1 + |g|^2)} \bar{g}_{\bar{z}} - \frac{2(1 + \bar{g}p)}{(\bar{g} - \bar{p})(1 + |p|^2)} \bar{p}_{\bar{z}}.$$

By continuity, this holds on a neighbourhood of a point which $g = 0$. Then, using equation (3.19), we get the conjugate of our assertion and, by continuity, it also holds at a point where $p = g$. □

In the following we establish our main result. As was seen before in the Proposition 3.14, vertical cylinders over geodesics of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$ are the only surfaces which the Gauss map is singular. Moreover, any two of them differ by an isometry of $\mathbb{S}^2 \times \mathbb{R}$. The next result prove that the same rigidity happens for any two minimal conformal immersions into $\mathbb{S}^2 \times \mathbb{R}$ with the same non-constant Gauss map.

Theorem 3.17. *Let $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion and g be its non-constant Gauss map. If $\hat{X} : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ is another minimal conformal immersion with the same Gauss map of X, then $\hat{X} = f \circ X$, with $f \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ given either by $f = (\mathrm{Id}, T)$ or $f = (\mathcal{A}, T)$, where $\mathcal{A}$ denotes the antipodal map on $\mathbb{S}^2$ and T is a translation on $\mathbb{R}$.*

Proof. Firstly, we can assume that g is a regular map. Otherwise, by Proposition 3.14, we get that $X(\Sigma)$ is a vertical cylinder over a geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$, and then g is constant.

Suppose that g is a regular map on Σ. Then $|g|$ cannot be constant on an open set $U \subset \Sigma$, otherwise $g_z\bar{g} = -g\bar{g}_z$ and follows that g is a singular map. So, we restrict ourselves to this open set $U \subset \Sigma$ on which $|g| \neq 1$.

By Proposition 3.8, for the minimal conformal immersion $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$, we get

$$A(1 - |p|^2) + Bp + C\bar{p} = 0, \tag{3.24}$$

where the coefficients A, B and C depend on $g, \bar{g}$ and theirs derivatives, given by

$$\begin{aligned} A &= -(1 - |g|^4)g_{z\bar{z}} + 2g|g_z|^2 - 2|g|^2\bar{g}g_zg_{\bar{z}}, \\ B &= -2\bar{g}(1 + |g|^2)g_{z\bar{z}} - 2|g_z|^2 + 2\bar{g}^2g_zg_{\bar{z}}, \\ C &= -2g(1 + |g|^2)g_{z\bar{z}} + 2g^2|g_z|^2 + 2(1 + 2|g|^2)g_zg_{\bar{z}}. \end{aligned}$$

If $A = 0$ at some $z_0 \in U$, at this point we have $(1 - |g|^2)Bp + (1 - |g|^2)C\bar{p} = 0$ and then

$$2(1 + |g|^2)g_z\Big\{(\bar{g}^2g_{\bar{z}} - \bar{g}_{\bar{z}})p + (g_{\bar{z}} - g^2\bar{g}_{\bar{z}})\bar{p}\Big\} = 0.$$

When $g_z(z_0) \neq 0$, the equation above has unique solution $p(z_0) = 0$ if, and only if, $|\bar{g}^2g_{\bar{z}} - \bar{g}_{\bar{z}}|^2 - |g_{\bar{z}} - g^2\bar{g}_{\bar{z}}|^2 \neq 0$, that is, $(1 - |g|^4)(|g_z|^2 - |\bar{g}_z|^2) \neq 0$ which holds in U. Moreover, A cannot vanish identically on an open set of Σ, since this would imply that X is not an immersion.

If $A \neq 0$ at $z_0 \in U$, then we consider the system

$$A(1 - |p|^2) + Bp + C\bar{p} = 0, \tag{3.25}$$

$$\bar{A}(1 - |p|^2) + \bar{C}p + \bar{B}\bar{p} = 0. \tag{3.26}$$

Since $\bar{A}$(3.25) + A(3.26) implies

$$(\bar{A}B - A\bar{C})p + (\bar{A}C - A\bar{B})\bar{p} = 0 \tag{3.27}$$

then, multiplying (3.25) by $(\bar{A}C - A\bar{B})$ and dividing by A, we get

$$(\bar{A}B - A\bar{C})p^2 + (|C|^2 - |B|^2)p + (\bar{A}C - A\bar{B}) = 0, \tag{3.28}$$

i.e., $\alpha p^2 + \beta p - \bar{\alpha} = 0$, with $\alpha = \bar{A}B - A\bar{C}$ and $\beta = |C|^2 - |B|^2$. Considering that the discriminant of the equation above is $\beta^2 + 4|\alpha|^2$, then equation (3.28) has two distinct solutions for $p(z_0)$ if, and only if, $\alpha \neq 0$, that is, $\bar{A}B - A\bar{C} \neq 0$ (once $\bar{A}B = A\bar{C}$ implies $|B|^2 = |C|^2$).

On the other hand, note that

$$2\bar{g}A - (1 - |g|^2)B = 2(1 + |g|^2)g_z D, \tag{3.29}$$

$$2gA - (1 - |g|^2)C = 2(1 + |g|^2)g_z E, \tag{3.30}$$

where $D = \bar{g}_{\bar{z}} - \bar{g}^2 g_{\bar{z}}$ and $E = g^2\bar{g}_{\bar{z}} - g_{\bar{z}}$. Then $\bar{A}(3.29) - A\overline{(3.30)}$ implies that

$$(1 - |g|^2)(A\bar{C} - \bar{A}B) = 2(1 + |g|^2)(g_z\bar{A}D - \bar{g}_{\bar{z}}A\bar{E}).$$

When $g_z(z_0) \neq 0$, then $\alpha = 0$ implies $|D|^2 = |E|^2$. By definition of D and E, $|D|^2 = |E|^2$ if, and only if, $(1-|g|^4)(|g_z|^2 - |\bar{g}_z|^2) = 0$ which does not happen on U. Therefore, equation (3.28) has at most two distinct solutions for each $p(z_0)$ on U, when g is not a constant map.

Let $\hat{X} : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a minimal conformal immersion with the same regular Gauss map g of X. In the following, we use the symbol ˆ to refer to objects related to $\hat{X}$.

If $p = \hat{p}$ then, by definition, we have that $X/|X| = \hat{X}/|\hat{X}|$, i.e., $\hat{X} = q(z)X$, where $q : \Sigma \to \mathbb{R}^+$ is a smooth function, given by $q(z) = |\hat{X}(z)|/|X(z)|$. However, we observe that the function q is constant. In fact, by Lemma 3.16, we get that $(\log q)_{\bar{z}} = 0$, since X and $\widehat{X}$ have the same Gauss map. Then q is a real holomorphic function, that is, $q(z) = q_0$ is constant and therefore $\hat{X} = f \circ X$, where $f = (\mathrm{Id}, T) \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ is a vertical translation of $\mathbb{S}^2 \times \mathbb{R}$ with $T(t) = t + \log q_0$.

If $p \neq \hat{p}$ then we claim that $\hat{p} = -1/\bar{p}$. Indeed, by a direct computation, we get that $-1/\bar{p}$ satisfies equation (3.15). Since (3.28) has exactly two distinct solutions when g is a regular map, then $\hat{p} = -1/\bar{p}$. Moreover, $\hat{p}\bar{p} = -1$ implies that $\varphi^{-1}(\hat{p})$ and $\varphi^{-1}(p)$ are antipodal points on $\mathbb{S}^2$, that is, $\hat{X} = q(z)\mathcal{A} \circ X$, where $\mathcal{A}$ is the antipodal map on $\mathbb{S}^2$ and $q(z) = |\hat{X}(z)|/|X(z)|$. However, by Lemma 3.16, we get $(\log q)_{\bar{z}} = 0$, since X and $\hat{X}$ have the same Gauss map and p and $\hat{p}$ are antipodal points. Then q is a real holomorphic function, that is, $q(z) = q_0$ is constant and therefore $\hat{X} = f \circ X$, where $f = (\mathcal{A}, T) \in \mathrm{Iso}(\mathbb{S}^2 \times \mathbb{R})$ with $T(t) = t + \log q_0$ a translation on $\mathbb{R}$. □

Remark 3.18. We notice that the minimal conformal immersion $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ can be recovered from of the Gauss map g, up to isometries described on Theorem 3.17, when g is a non-constant map. In fact, by definition of p, we have $X = |X|\varphi^{-1}(p)$. On the other hand, by Lemma 3.23, $|X|$ can be recovered from of g, up to a multiplicative constant, and the map p can be recovered from of g as the solution of equation (3.28), up to change of the antipodal map on $\mathbb{S}^2$.

Remark 3.19 (Chirstoffel's problem). In 1867, E. B. Christoffel [Chr67] considered the following problem: Given an immersion $X : \Sigma \to \mathbb{R}^3$ of an oriented Riemannian surface Σ, when does X possess a non-congruent immersion $\hat{X} : \Sigma \to \mathbb{R}^3$, such that the tangent planes at each point of X and $\hat{X}$ are parallels and the metrics induced by X and $\hat{X}$

are conformally equivalents? In the literature, this problem is known as Christoffel's problem (we may refer to [DT16], [JMN16, Chapter 9] and [Sam47]).

The local solution found by Chirstoffel depends of the orientation of X and $\hat{X}$. Suppose that there are X and $\hat{X}$ satisfying the Chirstoffel's problem. If X and $\hat{X}$ induce the same orientation at each point of Σ, then X is minimal and $\hat{X}$ is conformal associate of X. Otherwise, if X and $\hat{X}$ induce opposite orientations at each point of Σ, then X is isothermic and, in this case, the immersion $\hat{X}$ is called *Christoffel transform* of X.

Since we identify $\mathbb{S}^2 \times \mathbb{R}$ with $\mathbb{R}^3 \setminus \{0\}$, endowed with a metric conformally equivalent to the Euclidean metric of $\mathbb{R}^3$, a conformal immersion $X : \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ is a conformal immersion $X : \Sigma \to \mathbb{R}^3$ as well. Moreover, the definition of Gauss map in $\mathbb{S}^2 \times \mathbb{R}$ coincides with the definition in $\mathbb{R}^3$. Then given X and $\hat{X}$, the condition $g = \hat{g}$ implies that the tangent planes are parallels at each point of Σ and X and $\hat{X}$ induce the same orientation. By the Christoffel's Theorem, either X and $\hat{X}$ differ by an isometry of $\mathbb{R}^3$ or X and $\hat{X}$ are both minimal conformal immersions on $\mathbb{R}^3$. In the second case, we get that g is anti-holomorphic and, by Proposition 3.9, g must be a constant map. Then $X(\Sigma)$ and $\hat{X}(\Sigma)$ are parts of planes as surfaces in $\mathbb{R}^3$ with the same constant Gauss map. Therefore, $X(\Sigma)$ and $\hat{X}(\Sigma)$ are parts of the same vertical cylinder over a geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$.

3.6 Examples

As previously shown in the last section, minimal conformal immersions with constant Gauss map are vertical cylinders over geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$. In the case where the Gauss map g is a non-constant holomorphic map, we have totally geodesic 2-spheres in $\mathbb{S}^2 \times \mathbb{R}$. These ones are the simplest examples of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$.

Next, we describe briefly classical examples studied by Pedrosa and Ritoré [PR99], Rosenberg [Ros02] and Daniel [Dan09] using our model of $\mathbb{S}^2 \times \mathbb{R}$.

Example 3.20 (Helicoids)**.** Let $\beta \neq 0$. We consider the conformal immersion

$$X_\beta(u+iv) = \Big(e^v \sin\rho(u) e^{i\beta v}, e^v \cos\rho(u)\Big),$$

where the function ρ satisfies

$$\rho'^2(u) = 1 + \beta^2 \sin^2 \rho(u). \tag{3.31}$$

This conformal immersion was presented in [Dan09, Section 4.2] and corresponds to the minimal helicoid in $\mathbb{S}^2 \times \mathbb{R}$. Following Daniel's terminology, for $\beta > 0$ we say that X_β is a right helicoid and for $\beta < 0$ we say that X_β is a left helicoid.

We can assume that $\rho'(u) > 0$. We compute the normal vector N_β by

$$N_\beta(u+iv) = \frac{1}{\rho'(u)} \begin{bmatrix} \sin\beta v + \beta\sin^2\rho(u)\cos\beta v \\ -\cos\beta v + \beta\sin^2\rho(u)\sin\beta v \\ \beta\sin\rho(u)\cos\rho(u) \end{bmatrix}.$$

Then the Gauss map g_β is given by

$$g_\beta(u+iv) = \frac{(\beta\sin^2\rho(u) - i)e^{i\beta v}}{\rho'(u) + \beta\sin\rho(u)\cos\rho(u)}.$$

Moreover, the r_β is given by $r_\beta(u+iv) = v$ and p_β is given by

$$p_\beta(u+iv) = \frac{\sin\rho(u)}{1+\cos\rho(u)}e^{i\beta v}.$$

We note that when $\beta \to 0$ then $g_0(u+iv) = -i$, that is, X_0 is a vertical cylinder over a geodesic of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$.

In order to obtain an explicit parametrization of the minimal helicoid in $\mathbb{S}^2 \times \mathbb{R}$, we consider the following change of coordinates

$$u(s) = \int_0^s \frac{1}{\sqrt{1+\beta^2\sin^2(\sigma)}}\,\mathrm{d}\sigma,$$

and we note that $\rho \circ u(s) = s$ satisfies equation (3.31). In terms of these coordinates, we consider the immersion $\widetilde{X}_\beta : \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ given by

$$\widetilde{X}_\beta(s+iv) = \left(e^v\sin(s)e^{i\beta v}, e^v\cos(s)\right),$$

and, therefore, the surface $\widetilde{X}_\beta(\mathbb{C})$ is the analytic continuation of the minimal helicoid in $\mathbb{S}^2 \times \mathbb{R}$.

Figure 15 – **Part of minimal helicoid in** $\mathbb{S}^2 \times \mathbb{R}$ **with** $\beta = 4$**:**

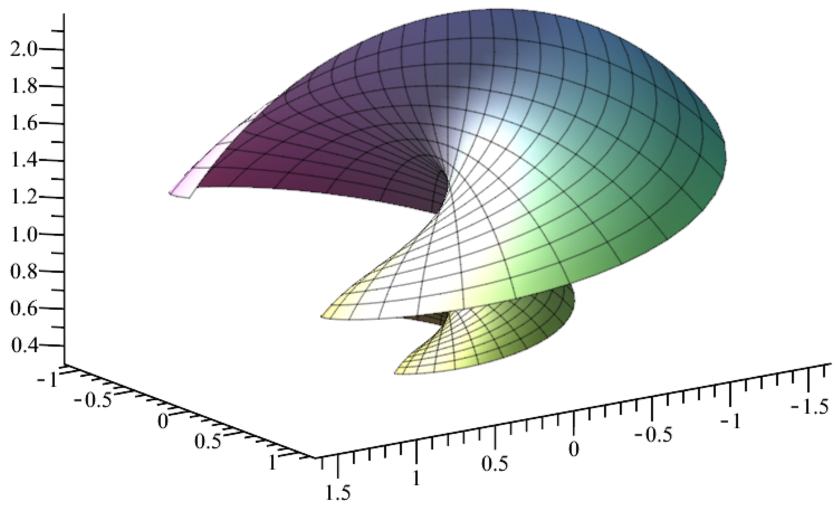
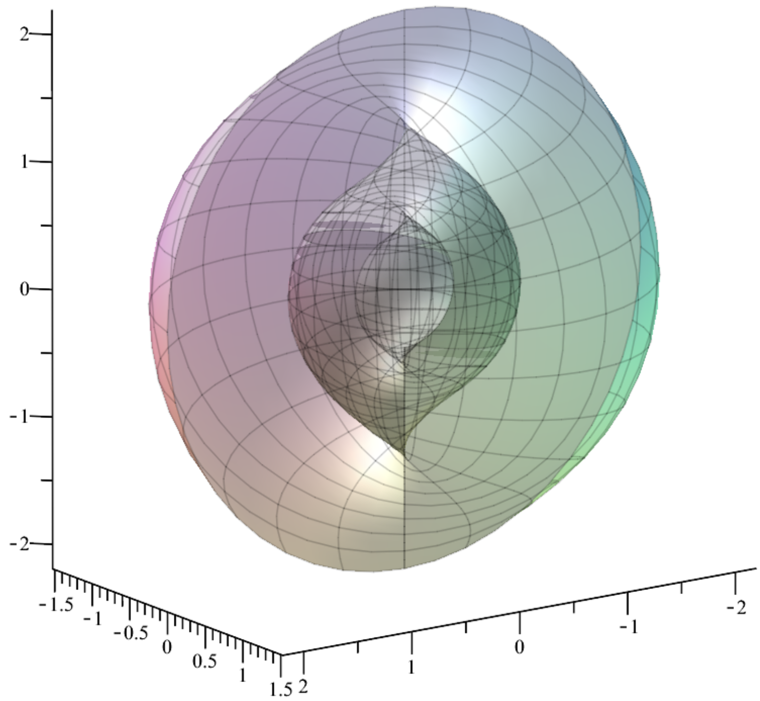


Figure 16 – **Minimal helicoid in** $\mathbb{S}^2 \times \mathbb{R}$ **with** $\beta = 4$**:**



Example 3.21 (Unduloids)**.** Let $\alpha \in \mathbb{R} \setminus [-1, 1]$. We consider the conformal immersion

$$X_\alpha(u + iv) = \Big(e^u \sin\rho(u) e^{i\alpha v}, e^u \cos\rho(u)\Big),$$

where the function ρ satisfies

$$\rho'^2(u) + 1 = \alpha^2 \sin^2 \rho(u). \tag{3.32}$$

This conformal immersion was presented in [Dan09, Section 4.2] and corresponds to the minimal unduloid in $\mathbb{S}^2 \times \mathbb{R}$.

We can assume that $\rho'(u) > 0$ and $\rho(u) \in (0, \pi)$. We compute the normal vector N_α by

$$N_\alpha(u + iv) = \frac{1}{\alpha \sin\rho(u)} \begin{bmatrix} (\rho'(u)\sin\rho(u) - \cos\rho(u))\cos\alpha v \\ (\rho'(u)\sin\rho(u) - \cos\rho(u))\sin\alpha v \\ \sin\rho(u) + \rho'(u)\cos\rho(u) \end{bmatrix}.$$

Then the Gauss map g_α is given by

$$g_\alpha(u + iv) = \frac{(\rho'(u)\sin\rho(u) - \cos\rho(u))e^{i\alpha v}}{(1+\alpha)\sin\rho(u) + \rho'(u)\cos\rho(u)}.$$

Moreover, the r_α is given by $r_\alpha(u + iv) = u$ and p_α is given by

$$p_\alpha(u + iv) = \frac{\sin\rho(u)}{1 + \cos\rho(u)} e^{i\alpha v}.$$

We note that when $\alpha \to \pm 1$ then $\rho'^2(u) + \cos^2\rho(u) = 0$, that is, $\rho(u) = \pi/2$. Since

$$N_{-1} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad N_{+1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

we have that $g_{-1}(u + iv) = \infty$ and $g_{+1}(u + iv) = 0$. Therefore, X_{-1} and X_{+1} are vertical cylinders over geodesics of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{R}$.

In order to obtain an explicit parametrization of the minimal unduloid in $\mathbb{S}^2 \times \mathbb{R}$, we consider the following change of coordinates

$$u(s) = \int_0^s \frac{1}{\sqrt{1 + (\alpha^2 - 1)\sin^2(\sigma)}}\, \mathrm{d}\sigma,$$

and we note that

$$\rho \circ u(s) = \arcsin\left(\frac{\sqrt{1 + (\alpha^2 - 1)\sin^2(s)}}{\alpha}\right)$$

satisfies equation (3.32). In terms of these coordinates, we consider the immersion $\widetilde{X}_\alpha : \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ given by

$$\widetilde{X}_\alpha(s + iv) = \frac{e^{u(s)}}{\alpha}\Big(\sqrt{1 + (\alpha^2 - 1)\sin^2(s)}\, e^{i\alpha v}, \sqrt{\alpha^2 - 1}\cos(s)\Big),$$

and, therefore, the surface $\widetilde{X}_\alpha(\mathbb{C})$ is the analytic continuation of the minimal unduloid in $\mathbb{S}^2 \times \mathbb{R}$.

Figure 17 – **Generating curve of minimal unduloid in $\mathbb{S}^2 \times \mathbb{R}$ with $\alpha = 8$:**

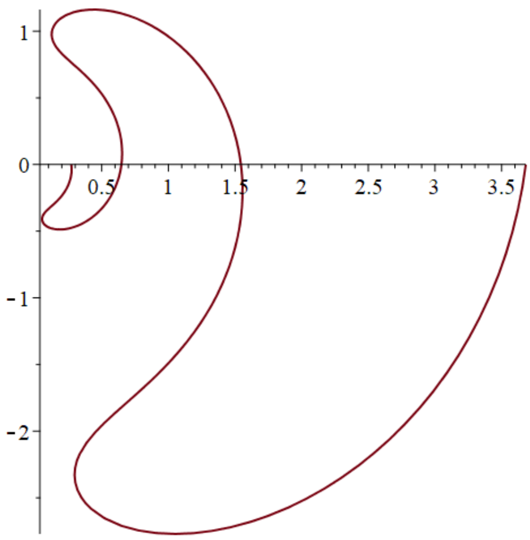
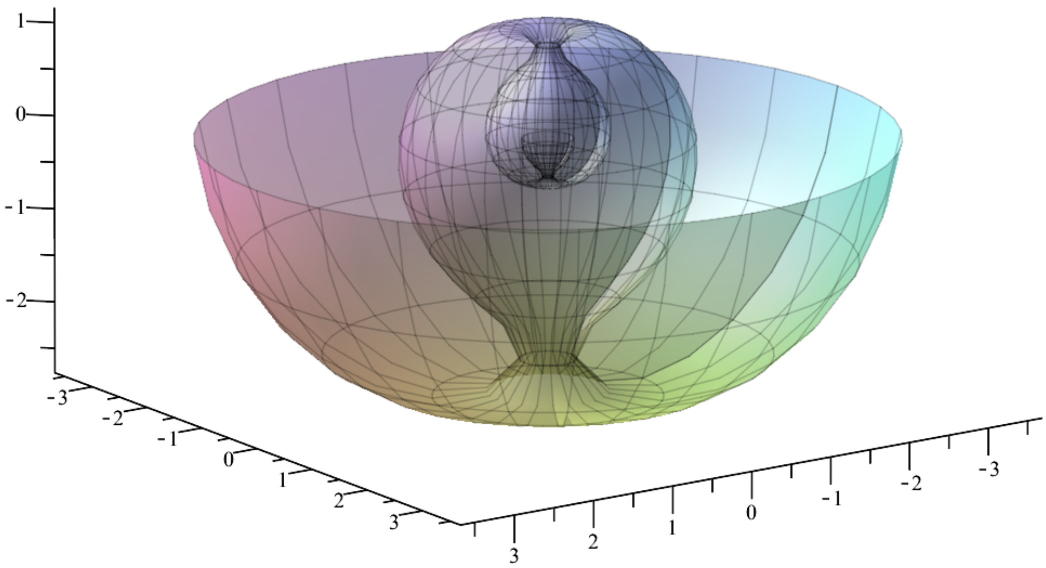


Figure 18 – **Minimal unduloid in $\mathbb{S}^2 \times \mathbb{R}$ with $\alpha = 8$:**



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